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# Matrix Approach to the Calculation of Indirect Quotas

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**Abstract.** Calculation and publication of the so-called indirect quotas is for banks and bigger companies in Germany, USA, and several other countries of the western world, prescribed by the law. This calculation becomes more and more complex due to the complexity of ownership relations among companies.

Until now, this problem was solved by using tools of the graph theory. By new matrix approach this problem can be solved in a much more simplified and more efficient way.

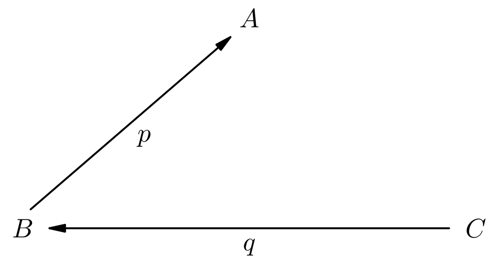
**AMS subject classification:** 91-08, 91B28, 91B60

**Key words:** indirect quotas, ownership, matrix approach

## 1. Introduction

To avoid insider profits and also to improve security of credit institutions several countries in the western world prescribed that the so-called significant shares in one firm owned by another firm have to be reported or published. In Germany, for example, for the banks and investment companies every crossing of the 10 percent barriers (i.e., 10%, 20%, etc.) must be reported[1]. These shares are to be calculated indirectly[2].

**Example 1.** *If company $A$ owns share $p$ of company $B$ and company $B$ owns share $q$ of company $C$, with $0 \leq p, q \leq 1$, then $A$ also owns $p \cdot q$ of company $C$.*

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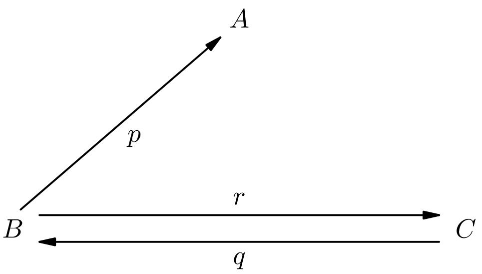
[1]Kreditwesengesetz, §24, Abs. 1

[2]Kreditwesengesetz, §10a, Abs. 3

**Example 2.** *Additional problem arises if, for example, company $C$ also owns $r$ of the company $B$.*




*Let $P$ be the profit gained by the company $B$. At the beginning, the profit $P$ is divided among companies $A$ and $C$ as follows: $pP$ and $rP$, respectively. The remainder $(1-p-r)P$ belongs to some other shareholders not included in the graph.*

*In the next round (call it the first), the part $qrP$ of the income $rP$ comes from $C$ to $B$. This part is in the next round divided among $A$, $C$ and other shareholders as: $pqrP$, $qr^2P$ and $(1-p-r)qrP$, respectively. In the next round $B$ gets $q^2r^2P$, and from this amount $A$ and $C$ get $pq^2r^2P$ and $q^2r^3P$, respectively. Summing up all the terms ending at $A$, we get*

$$o = (1 + qr + q^2r^2 + q^3r^3 + \cdots)pP = \frac{pP}{1-qr}, \tag{1}$$

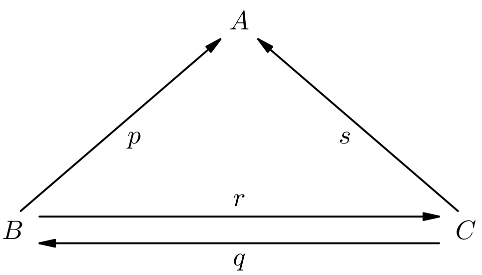
*so the indirect share of the company $A$ over $B$ is*

$$\frac{p}{1-qr}.$$

*Because of $p + r \le 1$ and $q \le 1$ we have*

$$\frac{p}{1-qr} \le \frac{1-r}{1-qr} \le 1.$$

**Example 3.** *When, in addition, the company $A$ owns $s$ of the company $C$*



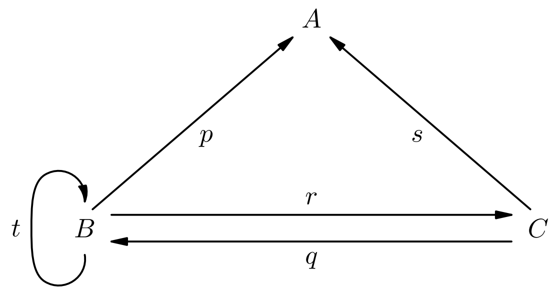

*the ownership of the company $A$ over the company $B$ is*

$$o = (p + rs)(1 + qr + q^2r^2 + q^3r^3 + \cdots) = \frac{p+rs}{1-qr}.$$

*Because* $p + r \leq 1$, $q + s \leq 1$, *we have*

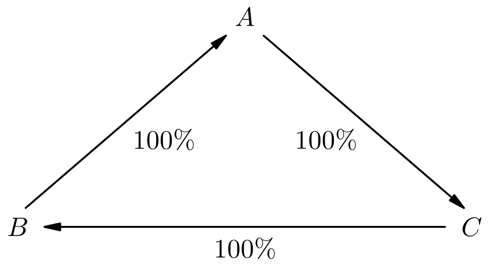
$$\frac{p + rs}{1 - qr} \leq \frac{1 - r + r(1 - q)}{1 - qr} = 1.$$

**Example 4.** *Even the following constructions of the ownership are possible.*




*The knot around* $B$ *means that the company* $B$ *owns a share of itself and this is allowed up to some percentage (e.g., in Germany up to* 10%*).*

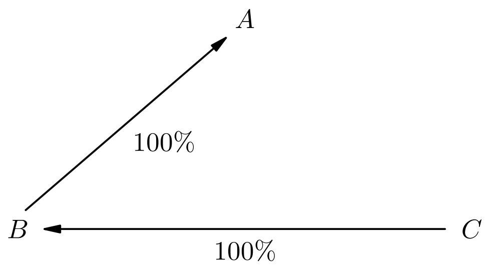
**Example 5.** *If* $A$ *owns* 100% *of* $B$, $B$ *owns* 100% *of* $C$ *and* $C$ *owns* 100% *of* $A$,




*the ownership of the company* $A$ *over* $B$ *would be*

$$o = 1 + 1 + 1 + 1 + \cdots,$$

*and this series is divergent. The reason for this is that here the profit of the company* $B$ *is calculated once more each time we cross the company* $A$ *(going around the loop of ownerships). So the ownership of the company* $A$ *should not be taken into account.*



*Such situations are, strictly speaking, forbidden (by law), i.e., in such cases all three companies must join into one company. Nevertheless, the conclusion that ownerships over $A$ should not be taken into account to avoid summing the same money several times is right.*

Most of the banks and investment companies try to disperse their ownerships as much as possible to lower the standard deviation of the expected profit, without significantly lowering the expected profit itself. This is also the known result of the portfolio theory. So, the web of ownership relations has become very complex and efficient calculation of these quotas arises as a mathematical problem.

Traditional approach is to find all paths from $A$ to $B$, to replace double relationships with formula (1), etc. For the web of, for example, 5000 knots, this is a very time consuming problem even for very powerful computers.

In this article we use tools of the matrix theory and solve the problem in a more efficient way. Moreover, we calculate the indirect ownerships of one company over all other companies in one step. In the next section we will find the solution and in the third section we will prove the consistency of the model.

## 2. Solution of the problem

Let $A_1$, $A_2$, $A_3$, $\ldots$, $A_n$ be some companies, and let $a_{ij}$ be the ownership of the company $A_i$ over the company $A_j$. We seek the indirect ownerships of the company $A_1$ over all other companies $A_2$, $A_3$, $\ldots$, $A_n$. As no ownership can be less than zero or greater than one, we have

$$0 \le a_{ij} \le 1. \tag{2}$$

The sum of all ownerships over one company is one, so

$$\sum_{i=1}^{n} a_{ij} \le 1. \tag{3}$$

We state here "less or equal" than one, because not all owners of all companies need to participate in the graph[3]. Since the ownerships over the company $A_1$ do not count, we set $a_{i1} = 0$ and get the matrix of ownerships

$$A = (a_{ij}) = \begin{bmatrix} 0 & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}.$$

First, we have to take care of the special case, when there exists a set of indices $S = \{j_1, j_2, \ldots, j_k\}$, $j_l \ne 1$, with the following properties:

$$a_{ij} = 0, \quad i \notin S, \quad j \in S, \tag{4}$$

[3] If the company $A_1$ is neither the direct nor the indirect owner of these companies, they do not interest us.

and

$$\sum_{i=1}^{n} a_{ij} = 1, \quad j \in S. \tag{5}$$

In this case, companies $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$ build an isolated part in the web of all ownerships. Moreover, each profit gained by these companies remains always in this isolated part of the web, and the company $A_1$ cannot get any money from them, so these rows and columns should be deleted from the matrix $A$ of ownerships[4]. We can more easily find such a set of indices by letting

- $A_2, A_3, \ldots, A_k$ be all the companies owned in part by the company $A_1$ (i.e., we choose $A_2, A_3, \ldots, A_k$ in such a way that $a_{1i} > 0$, $2 \le i \le k$, $a_{1i} = 0$, $i > k$),
- $A_{k+1}, A_{k+2}, \ldots, A_l$ be all other companies which are owned in part by some of the companies $A_2, \ldots, A_k$ (i.e., $a_{ij} = 0$ for $2 \le i \le k$, $j > l$),
- $A_{l+1}, A_{l+2}, \ldots, A_m$ be all other companies which are owned in part by some of the companies $A_{k+1}, \ldots, A_l$ (i.e., $a_{ij} = 0$ for $k+1 \le i \le l$, $j > m$),

and so on.

We proceed as in the Example 2. Let $p_j$ be the "clean" profit gained by $A_j$ (i.e., profit without profits of sharings by other companies $A_i$, $i \ne j$). We look how this profit is divided between the companies in each round. For each company $A_i$ we get the following table:

| Round | Gained profit from $p_j$ |
|---|---|
| 1 | $p_j a_{ij}$ |
| 2 | $p_j \sum_{k=1}^{n} a_{ik} a_{kj} = [A^2]_{ij} p_j$ |
| 3 | $p_j \sum_{k,l=1}^{n} a_{ik} a_{kl} a_{lj} = [A^3]_{ij} p_j$ |
| $\vdots$ | $\vdots$ |

Thus, the total amount that the company $A_1$ gets from the company $A_j$, $j > 1$, is

$$[A + A^2 + A^3 + \cdots]_{1j} p_j = [I + A + A^2 + A^3 + \cdots]_{1j} p_j = [(I - A)^{-1}]_{1j} p_j.$$

Therefore, the indirect ownership of the company $A_1$ over $A_j$ is

$$o_j = [(I - A)^{-1}]_{1j}.$$

[4]This situation is almost impossible in the real world and also forbidden in most countries. All the money gained by the companies must end either at the state, or at private investors, or at the so-called non-profit institutions. The state and non-profit institutions should not be taken into account from the very beginning.

## 3. Consistency of the matrix model

For the consistency of the proposed matrix model, we should prove the following facts:

1. If we have $A = (a_{ij})$ and $a_{i1} = 0$, then we also have $[A^n]_{i1} = 0$, for all $n > 0$.
2. The sum $I + A + A^2 + \cdots$ is convergent.
3. Matrix $I - A$ is invertible.
4. We have $0 \le [(I - A)^{-1}]_{1j} \le 1$, for $j > 1$.

We will prove this under the assumptions that the matrix $A$ satisfies (2), (3), $a_{i1} = 0$, $1 \le i \le n$, and there is no set of indices $S = \{j_1, j_2, \ldots, j_k\}$, $j_l \ne 1$, for which the relations (4) and (5) hold.

**Proof.**

Ad 1: $A^n e_1 = A^{n-1}(Ae_1) = A^{n-1}0 = 0$.

Ad 2: It is sufficient to show that all eigenvalues of the matrix $A^T$ have the absolute value less than one. Let $\lambda$ be an eigenvalue of $A^T$ and suppose that $|\lambda| \ge 1$. Let $b = (b_1, b_2, \ldots, b_n)^T$ be the corresponding eigenvector. From $A^T b = \lambda b$, we have

$$0 = b_1,$$
$$\sum_{i=1}^{n} a_{ij} b_i = \lambda b_j, \quad j > 1.$$

Let $S = \{i \mid |b_i| = \|b\|_\infty\}$. Since $b \ne 0$, we have $S \ne \emptyset$ and $1 \notin S$. However, the conditions (2) and (3), together with the inequality

$$\left| \sum_{i=1}^{n} a_{ij} b_i \right| \le \sum_{i=1}^{n} a_{ij} |b_i|,$$

imply that

$$\sum_{i=1}^{n} a_{ij} b_i = \lambda b_j, \quad j \in S,$$

can hold only if $|\lambda| = 1$,

$$\sum_{i=1}^{n} a_{ij} = 1, \quad j \in S,$$

and

$$a_{ij} = 0, \quad i \notin S, \quad j \in S. \tag{6}$$

However, (6) cannot hold, since existence of a set $S$ with such properties is by assumption precluded. Hence, we obtained a contradiction. Since $|\lambda| \ge 1$ yields contradiction, we have $|\lambda| < 1$ for all $\lambda$.

Ad 3: The proof is obvious, since all eigenvalues of $A$ are in the open unit ball.

Ad 4: Because of

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots,$$

all components of $B = (I-A)^{-1}$ are nonnegative. The components $b_{ij} = [(I-A)^{-1}]_{ij}$, for $j > 1$, fulfill the system of equations

$$\begin{bmatrix} 1 & -a_{12} & -a_{13} & \dots & \dots & \dots & -a_{1n} \\ 0 & 1-a_{22} & -a_{23} & \dots & \dots & \dots & -a_{2n} \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & -a_{n2} & -a_{n3} & \dots & \dots & \dots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

From the first equation we have

$$b_{1j} = \sum_{k=2}^{n} a_{1k} b_{kj},$$

and from the sum of all other equations we get

$$1 = \sum_{i=2}^{n} b_{ij} - \sum_{k=2}^{n} \sum_{i=2}^{n} a_{ki} b_{ij} = \sum_{i=2}^{n} \left(1 - \sum_{k=2}^{n} a_{ki}\right) b_{ij}.$$

Now, because of

$$\sum_{k=1}^{n} a_{ki} \leq 1, \quad b_{ij} \geq 0,$$

and (3), we obtain

$$b_{1j} = \sum_{i=2}^{n} a_{1i} b_{ij} \leq \sum_{i=2}^{n} \left(1 - \sum_{k=2}^{n} a_{ki}\right) b_{ij} = 1.$$

■

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