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On the Effective Boundary Conditions Between Different Flow Regions

Andro Mikelić*

Abstract. We give a review of recent mathematical results on the derivation of the effective constitutive laws at the interfaces separating different flow regions. Three cases are considered: (a) the effective transmission conditions at the boundary between two different porous media, (b) the effective boundary conditions for a viscous flow across a porous bed, and (c) the effective boundary conditions at the interface between a seabed and a sea. In all those situations the effective equations in the interior of every part are well-known, but their coupling at the interface requires studying boundary layers for the flow equations in a heterogeneous periodic porous medium.

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1. Introduction

Flows through heterogeneous structures, containing two or several subdomains with different micro-structures, arise in many applications. It is known that filtration of an incompressible viscous fluid through a rigid porous medium is described by the Darcy's law. For its derivation from the first principles, in the limit when the characteristic size of the pores ε tends to zero, the periodicity of the porous medium is required (see [32]). The periodicity condition can be relaxed to a kind of statistical homogeneity and ergodicity (see [4]), but clearly such assumptions break down close to the boundaries. Deviations from Darcy's law are expected only in thin layers near the interfaces. However, they can significantly change the structure of coefficients and even the effective constitutive law.

The first such problem of importance is finding a relationship between the filtration velocity and the pressure gradient for an incompressible viscous flow through a domain consisting of two different periodic porous media separated by an interface. The homogenization argument from [31], [1] or [28] is local and we get the Darcy's law in every porous part. However, due to the different geometric structures, the permeability tensors are different. Also, close to the interface the periodicity is lost.

*Analyse Numérique, UFR Mathématiques, CNRS UMR 5585, Bât. 101, Université Claude Bernard Lyon 1, 43, Bd. du onze novembre, 69622 Villeurbanne cedex, France,
e–mail: andro@maply.univ-lyon1.fr

The incompressibility of the flow implies the continuity of the normal components of the filtration velocities and it gives one of the required conditions. Another physically natural interface condition is the continuity of the effective pressure field. It is usually imposed without discussion as intuitively clear (see, e.g., Dagan [11]). These two conditions are called the *refraction conditions at a boundary between two porous bodies*. Nevertheless, one should not forget that Darcy's law is obtained after averaging the momentum equation and the orders of differential operators are changed. In general, the stress tensor completely changes in the weak limit and there are no *a priori* reasons to have the pressure continuity at the interfaces. However, the difference between the geometrical structure of the subdomains is not so big and we don't expect that the averaging procedure leads to drastic changes. Finally, there are reasons to believe in good understanding of the problem by engineers. The rigorous justification is in [19] and here we give the result in §2.

The next class of the problems involves contact between a porous medium and a continuum of a different nature. First, we can look for the effective behavior of the filtration velocity at an impervious boundary of the porous medium. The incompressibility and the no-slip boundary condition imply that the normal filtration velocity is zero. In the case of a flat impervious boundary it is possible to prove that such approximation leads to an error of order ε in the $H^{-1/2}$-norm. For the rigorous proof, which uses the appropriate boundary layers, we refer to [16]. In the same article the tangential component of the filtration velocity was determined and shown to be of order $O(1)$. Hence, the original no-slip condition was lost in the homogenization limit. The case of a general bounded domain was considered in [26] and an error estimate for the L^2-norm of the difference between Darcy's velocity and the rescaled physical velocity, of order $O(\varepsilon^{1/6})$ was justified.

The problems described above involve effective interface conditions which are obvious from modelling point of view. Finding the effective boundary conditions at a naturally permeable wall is a much more complicated problem. Namely, if a flow of a viscous fluid through a channel has a common boundary with a porous medium, then one wishes to have the effective boundary condition at such interface. For simplicity, we are considering a slow viscous and incompressible flow. Clearly, the effective flow through a porous medium is described by the Darcy's law and the filtration velocity is of order $O(\varepsilon^2)$. In the channel, the fluid flow remains governed by the Stokes system and the effective velocity field is of order 1. We note that this means that one should couple two systems of PDE's, one being a second order system for the velocity and a first order equation for the pressure, respectively, and the other being a scalar second order equation for the pressure and a first order system for the filtration velocity.

The coupling conditions should be imposed at the interface. One coupling condition is very simple. It is a consequence of the incompressibility and says that the normal velocity of the free fluid is of the order $O(\varepsilon^2)$ at the interface. This is not enough for determination of the effective flow, and one should specify more conditions. Classically, the tangential velocity of the free fluid was set to zero at the interface. This is a generalization of the no-slip condition at an impervious boundary, and it could not be justified, neither from mathematical nor from modelling or experimental point of view.

G. S. Beavers and D. D. Joseph concluded experimentally in [3] that the difference between the slip velocity of the free fluid and the tangential component of the seepage velocity at the interface, was proportional to the shear stress from the free fluid. This law was justified at a physical level of rigour by P. G. Saffman in [30], where it was observed that the seepage velocity contribution can be neglected. He used a statistical approach to extend Darcy's law to heterogeneous porous media. However, it should be noted that his argument is not entirely satisfactory, since he made an *ad hoc* hypothesis about the representation of the averaged interfacial forces as a linear integral functional of the velocity, with an unknown kernel. A similar argument is developed in [10], where Slattery's linear relationship between the pressure gradient and a combination of derivatives of the seepage velocity was assumed.

Neither paper [30] nor [10] contain construction of the boundary layers describing the flow behavior close to the interface. The Saffman's modification of the law by Beavers and Joseph is widely accepted and it reads

$$\sqrt{k^\varepsilon}\,\frac{\partial v_\tau}{\partial \nu} = \alpha v_\tau + O(k^\varepsilon). \tag{1}$$

Here α is a dimensionless parameter depending on the geometrical structure of the porous medium, ε is the characteristic pore size and $k^\varepsilon = \varepsilon^2 k$ is the (scalar) permeability. ν denotes the unit normal vector at the interface and v_τ is the slip velocity of the free fluid in the channel.

In the papers [12] and [23], H. Ene, Th. Levy and É. Sanchez–Palencia have undertaken the effort to find the effective interface laws by a formal asymptotic argument. They have considered two essentially different cases. The case of the flow in a cavity, lying inside of a porous matrix, was considered in [23]. By comparing the orders of the magnitude of characteristic quantities, it was found out that the effective pressure should be constant at the interface. This conclusion was rigorously justified in [17], after constructing the appropriate boundary layers.

The second case corresponds to the flow considered by Beavers and Joseph. In the paper [12] the continuity of the effective pressure was deduced, but without a rigorous argument or an asymptotic expansion. From modelling point of view this interface law is acceptable, since two pressures are of order $O(1)$. It could be considered as an alternative to (1), but the rigorous analysis and computations from [22] show that the continuity of the effective pressure is assured only for an isotropic porous medium. In the general case the effective pressure jump is proportional to the effective shear stress of the free fluid.

In §3 we are going to sketch the justification of the law (1) by the technique developed in [15] for Laplace's operator and then in [17] for the Stokes system. The detailed proof for the Navier–Stokes system and the boundary conditions for the pressure on the inlet and outlet boundaries is in [20]. Since the inertia effects and the outer boundary layers effects, due to the choice of the pressure boundary conditions, are not of the fundamental importance for the study of the interface boundary conditions, we'll make some non-essential simplifications. First, we neglect the inertial term. Anyhow, we are not able to find the boundary behavior for the turbulent free flow. The nonlinear stability results for the laminar Navier–Stokes system are in [20]. Second, we suppose that the boundary is sufficiently long and one can assume the periodic boundary conditions at the inlet/outlet boundary. The flow is then governed by a force coming from the pressure drop, equal to $((p_b - p_0)/b)\, e_1$.

In §4 we briefly address the derivation of the transmission condition at the interface between a seabed and a water flow. The fluid is supposed to be viscous and incompressible, and the solid part of the seabed elastic. The constitutive law for an elastic porous media was proposed by Biot in the fifties (see [5] and [6]). For its derivation using the formal asymptotic expansions we refer to [2], [7] and [31]. The first rigorous result is the paper [29], and the detailed theory of various cases is in the articles [13] and [8]. The question of the derivation of the conditions at the interface was addressed only for an inviscid fluid in [24], using the formal asymptotic matching. In §4 we announce the first mathematically rigorous results in this direction.

Let us also mention the derivation of the effective laws for flows through sieves and filters. We mention only the papers [9] and [18]. The paper [18] discusses the effective equations for a viscous incompressible flow through a filter of a finite thickness, and also uses the boundary layers developed in [17].

As general references on the homogenization and porous media we mention [14], [28] and [31].

2. Boundary conditions at the contact interface between two porous media

The goal of this section is to confirm rigorously the pressure continuity at the interface between two porous media with different micro-structures. In order to present the ideas we introduce a model problem.

We consider a slow viscous two-dimensional incompressible flow in a domain Ω^ε consisting of the porous media $\Omega_1 = (0, L) \times \mathbb{R}_+$ and $\Omega_2 = (0, L) \times \mathbb{R}_-$, and the interface $\Sigma = (0, L) \times \{0\}$ between them. We assume that the structure of the porous media is periodic. Ω_1 is generated by translations of a cell $Z^\varepsilon = \varepsilon Z$, where Z is the standard cell, $Z = (0, 1)^2$, consisting of an open set Z^*, $\partial Z^* \in C^\infty$, being strictly included in Z. Let $Z_F = Z \setminus \bar{Z}^*$ be connected and let χ_1 be the characteristic function of Z_F extended by periodicity to $\mathbb{R}^2$. We set $\chi_1^\varepsilon(x) = \chi_1(x/\varepsilon)$, $x \in \mathbb{R}^2$, and define Ω_1^ε by $\Omega_1^\varepsilon = \{x \mid x \in \Omega_1,\ \chi_1^\varepsilon(x) = 1\}$. Ω_2 is also generated by translations of a cell Z^ε, but this time we suppose that Z strictly includes the open set Y^*, $\partial Y^* \in C^\infty$, and that $Y_F = Z \setminus \bar{Y}^*$. Let χ_2 be the characteristic function of Y_F. We set $\chi_2^\varepsilon(x) = \chi_2(x/\varepsilon)$ on $\mathbb{R}^2$ and $\Omega_2^\varepsilon = \{x \mid x \in \Omega_2,\ \chi_2^\varepsilon(x) = 1\}$. Then $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Sigma \cup \Omega_2^\varepsilon$. It is supposed that $L/\varepsilon \in \mathbb{N}$ and $f \in C^\infty(\bar{\Omega})^2$, $\operatorname{supp} f$ is compact, f is L-periodic in x_1.

The flow through Ω^ε is described by the Stokes system

$$-\Delta u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in } \Omega^\varepsilon, \tag{2}$$
$$\operatorname{div} u^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \tag{3}$$
$$u^\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \setminus \partial(\Omega_1 \cup \Omega_2), \tag{4}$$
$$\{u^\varepsilon, p^\varepsilon\} \quad \text{is } L\text{-periodic in } x_1, \tag{5}$$
$$\nabla u^\varepsilon \in L^2(\Omega^\varepsilon)^4 \quad \text{and} \quad \nabla p^\varepsilon \in L^2(\Omega^\varepsilon)^2. \tag{6}$$

The choice of periodic boundary conditions in x_1 and unboundedness with respect to x_2, eliminates the effects of outer boundaries, which are of no importance for justifying the pressure continuity.

Before stating our results, we briefly discuss problem (2)–(6). We introduce the functional space W_ε by

$$W_\varepsilon = \big\{ z \in H^1(\Omega^\varepsilon)^2 \;\big|\; \operatorname{div} z = 0 \text{ a.e. in } \Omega^\varepsilon,\ z = 0 \text{ on } \partial\Omega^\varepsilon \setminus \partial(\Omega_1 \cup \Omega_2), \\ \text{and } z \text{ is } H^1\text{-periodic in } x_1 \big\}.$$

Then, using Poincaré's inequality, we easily get

Lemma 1. *Problem* (2)–(6) *has a unique solution* $u^\varepsilon \in W_\varepsilon$*. Furthermore, there exists* $p^\varepsilon \in L^2_{loc}(\Omega^\varepsilon)$*,* $\nabla p^\varepsilon \in L^2(\Omega^\varepsilon)^2$*, such that* (2) *holds in the sense of distributions. Finally,* $\{u^\varepsilon, p^\varepsilon\} \in C^\infty(\Omega^\varepsilon)^2 \times C^\infty(\Omega^\varepsilon)$*.*

Now we start with the description of the homogenized problem. First we introduce auxiliary problems defining the two permeabilities.

(i) We are looking for $\{w^j, \pi^j\}$, $j = 1, 2$, satisfying

$$\begin{cases} -\Delta_y w^j + \nabla_y \pi^j = e_j \quad \text{in } Z_F, \\ \operatorname{div}_y w^j = 0 \quad \text{in } Z_F, \quad \displaystyle\int_{Z_F} \pi^j(y)\, dy = 0, \\ w^j = 0 \quad \text{on } \partial Z^*, \quad \{w^j, \pi^j\} \quad \text{is 1-periodic.} \end{cases} \tag{7}$$

Problem (7) admits a unique solution $\{w^j, \pi^j\} \in C^\infty(\bar{Z}_F)^3$ and the matrix

$$K^w_{ij} = \int_{Z_F} w^i_j(y)\, dy, \quad i, j = 1, 2,$$

is symmetric and positive definite. Analogously, we consider the problem:

(ii) Find $\{v^j, \omega^j\} \in H^1(Y_F)^2 \times L^2(Y_F)$ such that

$$\begin{cases} -\Delta_y v^j + \nabla_y \omega^j = e_j \quad \text{in } Y_F, \\ \operatorname{div}_y v^j = 0 \quad \text{in } Y_F, \quad \displaystyle\int_{Y_F} \omega^j(y)\, dy = 0, \\ v^j = 0 \quad \text{on } \partial Y^*, \quad \{v^j, \omega^j\} \quad \text{is 1-periodic.} \end{cases}$$

The corresponding matrix is K^v, given by

$$K^v_{ij} = \int_{Y_F} v^i_j(y)\, dy, \quad i,j = 1,2.$$

Then the global permeability is defined by

$$K(x_2) = \begin{cases} K^w, & \text{for } x_2 > 0 \\ K^v, & \text{for } x_2 < 0, \end{cases}$$

and the homogenized problem reads:

Find $p^0 \in L^2_{loc}(\Omega)$, $\nabla p^0 \in L^2(\Omega_1 \cup \Omega_2)^2$ such that

$$\begin{cases} -\operatorname{div}\{K\nabla p^0\} = -\operatorname{div}\{Kf\} \quad \text{in } \Omega, \\ \qquad p^0 \text{ is } L\text{-periodic in } x_1, \end{cases} \tag{8}$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$.

Obviously, problem (8) admits a solution $p^0 \in H^1_{loc}(\Omega_1 \cup \Omega_2)$, unique up to a constant. It is important to establish the regularity properties of p^0. Due to the periodicity and the geometry, we immediately have

Lemma 2. *$p^0 \in C^\infty_{per}([0,L]; H^1_{loc}(\mathbb{R}_+ \cup \mathbb{R}_-))$, i.e., p^0 is C^∞ with respect to x_1.*

Unfortunately, there is a jump of derivatives with respect to x_2 on Σ. However, we have the following regularity result.

Proposition 1. *Let $f = 0$ on $[0,L] \times (-\delta,\delta)$ for some $\delta > 0$. Then $p^0 \in C^\infty([0,L] \times [0,+\infty)) \cap C^\infty([0,L] \times (-\infty,0])$ and $p^0(x_1,-0) = p^0(x_1,+0)$ on Σ.*

Proof. Proof is given in [19]. ■

Corollary 1. *For every $\alpha \in \mathbb{N}^2$ there exists $\delta_0(\alpha) > 0$ such that*

$$|D^\alpha p^0(x)| \le Ce^{-\delta_0(\alpha)|x_2|}$$

for $x_2 > x^*(\alpha)$.

Now we are able to give the main result. It is obtained in two steps, which can be considered as independent results. The first result is obtained under the hypothesis:

$$\begin{cases} f \in C^\infty_0(\Omega)^2, \quad f \text{ is } L\text{-periodic in } x_1, \\ \text{and } f = 0 \text{ on } [0,L] \times (-\delta,\delta) \text{ for some } \delta > 0, \end{cases} \tag{9}$$

and the second under the much weaker assumption:

$$f \in C^\infty_0(\Omega)^2, \quad f \text{ is } L\text{-periodic in } x_1. \tag{10}$$

Theorem 1. *Let us suppose* (9). *Then we have*

$$\left\| \frac{u^\varepsilon}{\varepsilon^2} - \sum_{k=1}^{2} \left(H(x_2)\, w^k\Big(\frac{x}{\varepsilon}\Big) + H(-x_2)\, v^k\Big(\frac{x}{\varepsilon}\Big) \right) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right) \right\|_{L^2(\Omega)^2} \le C\sqrt{\varepsilon},$$

and

$$\frac{u^\varepsilon}{\varepsilon^2} \rightharpoonup K^w(f - \nabla p^0)\, H(x_2) + K^v(f - \nabla p^0)\, H(-x_2), \tag{11}$$

weakly in $L^2(\Omega)^2$*. Furthermore, there exists an extension* $\tilde{p}^\varepsilon$ *of the pressure such that*

$$\left\| \frac{1}{1+|x_2|}\, (\tilde{p}^\varepsilon - p^0) \right\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}.$$

The next result corresponds to the assumption (10).

Theorem 2. *Let us suppose* (10). *Then we have*

$$\left\| \frac{u^\varepsilon}{\varepsilon^2} - \sum_{k=1}^{2} \left(H(x_2)\, w^k\Big(\frac{x}{\varepsilon}\Big) + H(-x_2)\, v^k\Big(\frac{x}{\varepsilon}\Big) \right) \left(f_k(x) - \frac{\partial p^0}{\partial x_k}(x) \right) \right\|_{L^2(\Omega)^2} \le C\varepsilon^{1/8},$$

and there exists an extension $\tilde{p}^\varepsilon$ *of the pressure such that*

$$\left\| \frac{1}{1+|x_2|}\, (\tilde{p}^\varepsilon - p^0) \right\|_{L^2(\Omega)} \le C\varepsilon^{1/8}.$$

Finally, the convergence (11) *holds true.*

Corollary 2.

$$e_2 \left.\frac{u^\varepsilon}{\varepsilon^2}\right|_\Sigma \rightharpoonup K^w(f - \nabla p^0)e_2 = K^v(f - \nabla p^0)e_2,$$

weakly in $L^2(\Sigma)$.

For the proofs we refer to [19]. It should be noted that the proofs involve correcting the compressibility results caused by the two-scale approximation for u^ε and the corresponding boundary layers. However, those corrections have their L^2-norm smaller than $O(\varepsilon)$ and we are not obliged to keep them in the final error estimate.

3. Boundary conditions at the contact interface between a porous medium and a channel flow

We consider the laminar viscous two-dimensional incompressible flow through a domain Ω consisting of the porous medium $\Omega_2 = (0,b) \times (-L,0)$, the channel $\Omega_1 = (0,b) \times (0,h)$, and the permeable interface $\Sigma = (0,b) \times \{0\}$ between them. We assume that the structure of the porous medium is periodic and generated by translations of a cell $Y^\varepsilon = \varepsilon Y$, where Y is the standard cell, $Y = (0,1)^2$, containing an open set Z^*, $\partial Z^* \in C^\infty$, strictly included in Y. Let $Y_F = Y \setminus \bar{Z}^*$ and let χ be the characteristic function of Y_F, extended by periodicity to $\mathbb{R}^2$. We set $\chi^\varepsilon(x) = \chi(x/\varepsilon)$, $x \in \mathbb{R}^2$, and define Ω_2^ε by $\Omega_2^\varepsilon = \{x \mid x \in \Omega_2,\ \chi^\varepsilon(x) = 1\}$. Furthermore, $\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon$ is the fluid part of Ω. It is supposed that $(b/\varepsilon, L/\varepsilon) \in \mathbb{N}^2$.

First, we give the basic conservation laws describing the viscous incompressible flow in the domain Ω^ε, with the rigid porous part Ω_2:

$$-\mu\Delta v^\varepsilon + \nabla p^\varepsilon = -\frac{p_b - p_0}{b}\, e_1 \quad \text{in } \Omega^\varepsilon, \tag{12}$$

$$\operatorname{div} v^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \tag{13}$$

$$v^\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \setminus \partial\Omega \text{ and on } (0,b) \times (\{-L\} \cup \{h\}), \tag{14}$$

$$\{v^\varepsilon, p^\varepsilon\} \quad \text{is } b\text{-periodic in } x_1, \tag{15}$$

where $\mu > 0$ is the viscosity, and p_0 and p_b are given constants. $\varepsilon > 0$ is the characteristic pore size, v^ε is the velocity and p^ε is the pressure field. Problem (12)–(15) has a unique solution $\{v^\varepsilon, p^\varepsilon\} \in H^1(\Omega^\varepsilon)^2 \times L_0^2(\Omega^\varepsilon)$.

Now one would like to study of the effective behavior of the velocities v^ε and pressures p^ε as $\varepsilon \to 0$. We follow the decomposition approach from [20]. First, we observe that the classic Poiseuille flow in Ω_1, satisfying the no-slip conditions at Σ, is given by

$$v^0 = \frac{p_b - p_0}{2b\mu}\, x_2(x_2 - h)\, e_1, \quad \pi^0 = 0.$$

We extend this solution to Ω_2 by setting $v^0 = 0$ for $-b \le x_2 \le 0$ and keeping the same form of π^0. Now, the idea is to construct the solution to (12)–(15) as a small perturbation to the Poiseuille flow. We need the following simple auxiliary result:

Lemma 3. *Let $\varphi \in H^1(\Omega_2^\varepsilon)$ be such that $\varphi = 0$ on $\partial\Omega_2^\varepsilon \setminus \partial\Omega_2$. Then we have*

$$\begin{aligned} \|\varphi\|_{L^2(\Omega_2^\varepsilon)} &\le C\varepsilon \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2} \\ \|\varphi\|_{L^2(\Sigma)} &\le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2} \\ \|\varphi(0,\cdot)\|_{H^{1/2}(-L,0)} &\le C\|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2}. \end{aligned}$$

It implies the following stability result (see [20] or [28]).

Proposition 2. *Let $\{v^\varepsilon, p^\varepsilon\}$ be the solution for* (12)–(14) *and v^0 the Poiseuille velocity. Then we have*

$$\begin{aligned} \|\nabla(v^\varepsilon - v^0)\|_{L^2(\Omega^\varepsilon)^4} &\le C\sqrt{\varepsilon} \\ \|v^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} &\le C\varepsilon\sqrt{\varepsilon} \\ \|v^\varepsilon\|_{L^2(\Sigma)} &\le C\varepsilon \\ \|v^\varepsilon - v^0\|_{L^2(\Omega_1)^2} &\le C\varepsilon \\ \|p^\varepsilon\|_{L^2(\Omega_1)} &\le C\sqrt{\varepsilon}. \end{aligned} \tag{16}$$

Therefore, we have obtained the uniform *a priori* estimates for $\{v^\varepsilon, p^\varepsilon\}$. Moreover, we have found that Poiseuille's flow in Ω_1 is an $O(\varepsilon)$ L^2-approximation for v^ε. Beavers and Joseph's law should correspond to the next order velocity correction.

The leading contribution for the estimate (16) was the interface integral term $\int_\Sigma \varphi_1$. Following the approach from [17], we eliminate it by constructing the *Navier's boundary layer*:

We introduce $S = (0,1) \times \{0\}$, $Z^+ = (0,1) \times (0, +\infty)$ and the semi-infinite porous slab $Z^- = \cup_{k=1}^{\infty} (Y_F - \{0,k\})$. The flow region is then $Z_{BL} = Z^+ \cup S \cup Z^-$ and the union of the solid boundaries $S_\infty = \cup_{k=1}^{\infty} (\partial Z^* - \{0,k\})$. We consider the following problem:

Find $\{\beta^{bl}, \omega^{bl}\}$ with square-integrable gradients, satisfying

$$-\Delta_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \quad \text{in } Z^+ \cup Z^-, \tag{17}$$

$$\operatorname{div}_y \beta^{bl} = 0 \quad \text{in } Z^+ \cup Z^-, \tag{18}$$

$$\big[\beta^{bl}\big]_S(\cdot, 0) = 0 \quad \text{on } S, \tag{19}$$

$$\big[\{\nabla_y \beta^{bl} - \omega^{bl} I\} e_2\big]_S(\cdot, 0) = e_1 \quad \text{on } S, \tag{20}$$

$$\beta^{bl} = 0 \quad \text{on } S_\infty, \quad \{\beta^{bl}, \omega^{bl}\} \quad \text{is } y_1\text{-periodic}. \tag{21}$$

Let

$$\begin{aligned} V = \big\{ z \in L^2_{loc}(Z_{BL})^2 \;\big|\; & \nabla_y z \in L^2(Z_{BL})^4,\ z \in L^2(Z^-)^2, \\ & z = 0 \text{ on } \cup_{k=1}^{\infty} (\partial Z^* - \{0,k\}), \\ & \operatorname{div}_y z = 0 \text{ in } Z_{BL} \text{ and } z \text{ is } y_1\text{-periodic} \big\}. \end{aligned}$$

Then, by Lax–Milgram lemma, there is a unique $\beta^{bl} \in V$ satisfying

$$\int_{Z_{BL}} \nabla \beta^{bl} \nabla \varphi \, dy = - \int_S \varphi_1 \, dy, \quad \forall \varphi \in V.$$

By using De Rham's theorem we obtain a function $\omega^{bl} \in L^2_{loc}(Z^+ \cup Z^-)$, unique up to a constant and satisfying (17). By the elliptic theory, $\{\beta^{bl}, \omega^{bl}\} \in C^\infty(Z^+ \cup Z^-)^3$ and $\forall R > 0$

$$\begin{aligned} \{\beta^{bl}, \omega^{bl}\} \in H^2\Big(\big(Z^+ \cap \{0 < y_2 < R\}\big) \cup \big(Z^- \cap \{-R < y_2 < 0\}\big) \Big)^2 \\ \times H^1\Big(\big(Z^+ \cap \{0 < y_2 < R\}\big) \cup \big(Z^- \cap \{-R < y_2 < 0\}\big) \Big). \end{aligned}$$

For a neighbourhood $\mathcal{O}$ of S, which does not include any solid boundary, we obtain $\beta^{bl} - \big((y_2 - y_2^2/2) e^{-y_2} H(y_2), 0 \big) \in W^{2,q}(\mathcal{O})^2$ and $\omega^{bl} \in W^{1,q}(\mathcal{O})$, $\forall q \in [1, \infty)$.

The goal is to prove that the system (17)–(21) describes a boundary layer, i.e., that β^{bl} and ω^{bl} stabilize exponentially towards constants, when $|y_2| \to \infty$. The proofs are in [20] and [28], and we give here only the statements with some comments.

Since we are studying an incompressible flow, it is useful to prove properties of the conserved averages. We have the following result:

Lemma 4. *Any solution* $\{\beta^{bl}, \omega^{bl}\}$ *satisfies*

$$\int_0^1 \beta_2^{bl}(y_1, b)\, dy_1 = 0, \quad \forall b \in \mathbb{R},$$
$$\int_0^1 \omega^{bl}(y_1, b_1)\, dy_1 = \int_0^1 \omega^{bl}(y_1, b_2)\, dy_1, \quad \forall b_1 > b_2 \geq 0,$$
$$\int_0^1 \beta_1^{bl}(y_1, b_1)\, dy_1 = \int_0^1 \beta_1^{bl}(y_1, b_2)\, dy_1, \quad \forall b_1 > b_2 \geq 0,$$
$$\int_0^1 \beta_1^{bl}(y_1, 0)\, dy_1 = -\int_{Z_{BL}} |\nabla \beta^{bl}(y)|^2\, dy.$$

Now we are able to formulate the result on the decays:

Proposition 3. *Let*

$$C_1^{bl} = \int_0^1 \beta_1^{bl}(y_1, 0)\, dy_1 < 0 \tag{22}$$

and

$$C_\omega^{bl} = \int_0^1 \omega^{bl}(y_1, 0)\, dy_1.$$

Then for every $y_2 \geq 0$ *and* $y_1 \in (0,1)$

$$|\beta^{bl}(y_1, y_2) - (C_1^{bl}, 0)| + |\omega^{bl}(y_1, y_2) - C_\omega^{bl}| \leq C e^{-\delta y_2}, \quad \forall \delta < 2\pi.$$

Furthermore, let k *be a negative integer and* $Z^-(k) = Z^- \cap (\,]0,1[\, \times\,]-\infty, k[\,)$*. Then there exist positive constants* C*,* γ_0 *and a constant* κ_∞*, independent of* k*, such that*

$$\|\beta^{bl}\|_{H^1(Z^-(k))} + \|\beta^{bl}\|_{L^\infty(Z^-(k))} + \|\omega^{bl} - \kappa_\infty\|_{L^\infty(Z^-(k))} \leq C e^{\gamma_0 k}.$$

Since ω^{bl} is unique up to a constant, we fix it by setting $\kappa_\infty = 0$. Hence, C_ω^{bl} is the pressure drop between $-\infty$ and ∞.

Remark 1. *If the geometry of* Z^- *is axisymmetric with respect to reflections around the axis* $y_1 = 1/2$*, then* $C_\omega^{bl} = 0$*. For the proof we refer to* [22]*. In* [22]*, a detailed numerical analysis of the problem* (17)–(21) *is given. Through numerical experiments it is shown that* $C_\omega^{bl} \neq 0$ *for a general geometry of* Z^-*.*

We now define our boundary layer velocities and pressures in Ω^ε:

$$\beta^{bl,\varepsilon}(x) = \varepsilon \beta^{bl}\Big(\frac{x}{\varepsilon}\Big) \quad \text{and} \quad \omega^{bl,\varepsilon}(x) = \omega^{bl}\Big(\frac{x}{\varepsilon}\Big), \quad x \in \Omega^\varepsilon. \tag{23}$$

We extend $\beta^{bl,\varepsilon}$ by zero to $\Omega \setminus \Omega^\varepsilon$. Then we have

$$\big\|\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl}, 0) H(x_2)\big\|_{L^q(\Omega)^2} = C\varepsilon^{1+1/q}, \quad \forall q \geq 1,$$
$$\big\|\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\big\|_{L^q(\Omega^\varepsilon)} + \big\|\nabla \beta^{bl,\varepsilon}\big\|_{L^q(\Omega_1 \cup \Sigma \cup \Omega_2)^4} = C\varepsilon^{1/q}, \quad \forall q \geq 1,$$

and

$$\left\|\omega^{bl,\varepsilon}(0,\cdot) - C^{bl}_{\omega} H(\cdot)\right\|_{H^{-1/2}(\mathbb{R})} + \sqrt{\varepsilon}\left\|\omega^{bl,\varepsilon}(0,\cdot) - C^{bl}_{\omega} H(\cdot)\right\|_{L^2(\mathbb{R})} = C\varepsilon,$$

$$\varepsilon^{-1/2}\left\|\beta^{bl,\varepsilon}(0,\cdot) - \varepsilon(C^{bl}_1, 0)H(\cdot)\right\|_{L^2(\mathbb{R})^2} + \left\|\frac{\partial \beta^{bl,\varepsilon}}{\partial x_2}(0,\cdot)\right\|_{H^{-1/2}(\mathbb{R})^2} = C\varepsilon.$$

As in [17], stabilization of $\beta^{bl,\varepsilon}$ towards a nonzero constant velocity $\varepsilon(C^{bl}_1, 0)$, at the upper boundary, generates a counterflow. It has the form of 2D Couette flow $d = (1 - x_2/h)\, e_1$.

Now, we introduce the correctors for the velocity and for the pressure. We have the following result (see [28] or [20]):

Theorem 3. *Let*

$$\mathcal{U}^{\varepsilon}(x) = v^{\varepsilon} - v^0 + \beta^{bl,\varepsilon}\frac{\partial v^0_1}{\partial x_2}(0) - \varepsilon C^{bl}_1 \frac{\partial v^0_1}{\partial x_2}(0)\, H(x_2)\, \frac{x_2}{h}\, e_1,$$

$$\mathcal{P}^{\varepsilon} = p^{\varepsilon} + \Big(\omega^{bl,\varepsilon} - C^{bl}_{\omega}\Big)\mu\, \frac{\partial v^0_1}{\partial x_2}(0),$$

where v^0 is the Poiseuille velocity and $\{\beta^{bl,\varepsilon}, \omega^{bl,\varepsilon}\}$ are defined by (23)*. Then we have the following estimates*

$$\|\nabla \mathcal{U}^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})^4} \le C\varepsilon, \tag{24}$$

$$\|\mathcal{U}^{\varepsilon}\|_{L^2(\Omega^{\varepsilon}_2)^2} \le C\varepsilon^2, \tag{25}$$

$$\|\mathcal{U}^{\varepsilon}\|_{L^2(\Sigma)^2} + \|\mathcal{U}^{\varepsilon}\|_{L^2(\Omega_1)^2} \le C\varepsilon^{3/2}, \tag{26}$$

$$\|\mathcal{P}^{\varepsilon}\|_{L^2(\Omega_1)} \le C\varepsilon. \tag{27}$$

We note that $\mathcal{U}^{\varepsilon}$ is not zero on the boundary $(0, b) \times \{h\}$, but it satisfies the inequality $|\mathcal{U}^{\varepsilon}(x)| \le C\exp(-C_0/\varepsilon)$ for some positive constants C and C_0. Consequently, we consider it being zero, without loosing generality.

The estimates (24)–(27) allow us to justify Saffman's modification (1) of the Beavers and Joseph's law. Let $H^{1/2}_{per}(\Sigma) = [H^1_{per}(\Sigma), L^2(\Sigma)]_{1/2}$. Then we have (see [28] and [20]):

Theorem 4. *Let v^{ε} be the velocity field determined by* (12)–(15) *and let the boundary layer tangential velocity at infinity C^{bl}_1 be given by* (22)*. Then we have*

$$\left\|v^{\varepsilon}_1 + \varepsilon C^{bl}_1 \frac{\partial v^{\varepsilon}_1}{\partial x_2}\right\|_{\left(H^{1/2}_{per}(\Sigma)\right)'} \le C\varepsilon^{3/2}.$$

Now we introduce the effective flow equations in Ω_1 through the following boundary value problem, with the velocity satisfying the law of Beavers and Joseph:

Find a velocity field u^{eff} and a pressure field p^{eff}, such that

$$-\mu\Delta u^{\mathit{eff}} + \nabla p^{\mathit{eff}} = -\frac{p_b - p_0}{b}\, e_1 \quad \text{in } \Omega_1, \tag{28}$$

$$\operatorname{div} u^{\mathit{eff}} = 0 \quad \text{in } \Omega_1, \tag{29}$$

$$u^{\mathit{eff}} = 0 \quad \text{on } (0,b)\times\{h\}, \tag{30}$$

$$u^{\mathit{eff}} \text{ and } p^{\mathit{eff}} \quad \text{are } b\text{-periodic}, \tag{31}$$

$$u_2^{\mathit{eff}} = 0 \quad \text{and} \quad u_1^{\mathit{eff}} + \varepsilon C_1^{bl}\,\frac{\partial u_1^{\mathit{eff}}}{\partial x_2} = 0 \quad \text{on } \Sigma. \tag{32}$$

Problem (28)–(32) has a unique solution

$$u^{\mathit{eff}} = \left(\frac{p_b - p_0}{2b\mu}\left(x_2 - \frac{\varepsilon C_1^{bl} h}{h - \varepsilon C_1^{bl}}\right)(x_2 - h), 0\right), \quad \text{for } 0 \le x_2 \le h,$$

$$p^{\mathit{eff}} = 0, \quad \text{for } 0 \le x_1 \le b.$$

The effective mass flow rate through the channel is then

$$M^{\mathit{eff}} = b\int_0^h u_1^{\mathit{eff}}(x_2)\,dx_2 = -\frac{p_b - p_0}{12\mu}\,h^3\,\frac{h - 4\varepsilon C_1^{bl}}{h - \varepsilon C_1^{bl}},$$

where $C_1^{bl} < 0$.

Let us estimate the error made when replacing $\{v^\varepsilon, p^\varepsilon, M^\varepsilon\}$ by $\{u^{\mathit{eff}}, p^{\mathit{eff}}, M^{\mathit{eff}}\}$. We have:

Proposition 4. *(Justification of the law by Beavers and Joseph.)*

$$\|\nabla(v^\varepsilon - u^{\mathit{eff}})\|_{L^1(\Omega_1)^4} \le C\varepsilon,$$

$$\|v^\varepsilon - u^{\mathit{eff}}\|_{L^2(\Omega_1)^2} \le C\varepsilon^{3/2},$$

$$|M^\varepsilon - M^{\mathit{eff}}| \le C\varepsilon^{3/2}.$$

For the proofs we refer to [28], [20] and [22]. The result implies that there is a significant gain in precision if we use the law by Beavers and Joseph, instead of imposing the no-slip condition at the interface.

We note that by using the similar technique it is possible to obtain the effective boundary conditions for viscous incompressible flows over rough surfaces (see [21]).

4. Boundary conditions at the interface between an elastic porous medium and a free viscous flow

The goal of this section is to determine rigorously the effective conditions at the interface between a seabed (i.e., an elastic porous medium) and a water flow in an ocean (i.e., a viscous incompressible fluid). In order to present the ideas we introduce a model problem in 2D.

We consider a slow viscous two-dimensional incompressible flow in a domain Ω^ε consisting of the free fluid domain $\Omega_1 = (0,L) \times \mathbb{R}_+$, the elastic porous part $\Omega_2 = (0,L) \times \mathbb{R}_-$, and the interface $\Sigma = (0,L) \times \{0\}$ between them. We assume that the structure of the porous media is periodic. Ω_2 is generated by translations of a cell $\mathcal{Y}^\varepsilon = \varepsilon\mathcal{Y}$, where $\mathcal{Y}$ is the standard cell, $\mathcal{Y} = (0,1)^2$, consisting of an open set $\mathcal{Y}_s$, $\partial\mathcal{Y}_s \in C^\infty$, being strictly included in $\mathcal{Y}$. Let $\mathcal{Y}_f = \mathcal{Y}\backslash\bar{\mathcal{Y}}_s$ be connected and let χ_1 be the characteristic function of $\mathcal{Y}_f$ extended by periodicity to $\mathbb{R}^2$. We set $\chi_1^\varepsilon(x) = \chi_1(x/\varepsilon)$, $x \in \mathbb{R}^2$, and define Ω_2^ε by $\Omega_2^\varepsilon = \{x \mid x \in \Omega_2,\ \chi_1^\varepsilon(x) = 1\}$. Then $\Omega_f^\varepsilon = \Omega_2^\varepsilon \cup \Sigma \cup \Omega_1$ and $\Omega_s^\varepsilon = \Omega_2 \setminus \Omega_2^\varepsilon$. It is supposed that $L/\varepsilon \in \mathbb{N}$ and $F \in C^\infty(\bar{\Omega})^2$, $\operatorname{supp} F$ is compact, F is L-periodic in x_1. The linearized interface between the fluid and the solid parts is $\Gamma_\varepsilon = \partial\Omega_s^\varepsilon \setminus \Omega_2$.

The flow through Ω^ε is described by the following coupling between the non-stationary Stokes system and the non-stationary equations of linear elasticity. The meaning of the factor ε^2 in the viscosity coefficient is explained in [8].

$$\rho_s \frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(\sigma^{s,\varepsilon}) = F\rho_s \quad \text{in } \Omega_s^\varepsilon \times]0,T[\tag{33}$$

$$\sigma^{s,\varepsilon} = AD(u^\varepsilon) \quad \text{in } \Omega_s^\varepsilon \times]0,T[\tag{34}$$

$$\rho_f \frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}(\sigma^{f,\varepsilon}) = F\rho_f \quad \text{in } \Omega_f^\varepsilon \times]0,T[\tag{35}$$

$$\sigma^{f,\varepsilon} = -p^\varepsilon I + 2\mu\varepsilon^2 D\left(\frac{\partial u^\varepsilon}{\partial t}\right) \quad \text{in } \Omega_f^\varepsilon \times]0,T[\tag{36}$$

$$\operatorname{div} \frac{\partial u^\varepsilon}{\partial t} = 0 \quad \text{in } \Omega_f^\varepsilon \times]0,T[, \tag{37}$$

where u^ε is the solid displacement in Ω_s^ε, $\frac{\partial u^\varepsilon}{\partial t}$ is the fluid velocity in Ω_f^ε, and p^ε is the fluid pressure. D denotes the symmetrized gradient, and the fourth order symmetric tensor A contains the elasticity coefficients. At the interface between fluid and solid parts we have

$$[u^\varepsilon] = 0, \quad \sigma^{s,\varepsilon} \cdot \nu = \sigma^{f,\varepsilon} \cdot \nu, \quad \text{on } \Gamma_\varepsilon \times]0,T[. \tag{38}$$

At the outer boundary we suppose integrability and L-periodicity in x_1, i.e.,

$$\nabla u^\varepsilon \in L^2(\Omega)^4, \quad \nabla p^\varepsilon \in L^2(\Omega_f^\varepsilon)^2, \tag{39}$$

$$\{u^\varepsilon, p^\varepsilon\} \quad \text{are } L\text{-periodic in } x_1. \tag{40}$$

For simplicity, we suppose that initially there was no flow and no deformations at $t = 0$, i.e.,

$$u^\varepsilon(x,0) = 0, \quad \frac{\partial u^\varepsilon}{\partial t}(x,0) = 0, \quad \text{in } \Omega.$$

Then it is easy to prove that there is a unique solution $u^\varepsilon \in H^1(]0,T[\times\Omega)^n$ with $\frac{d^2u^\varepsilon}{dt^2} \in L^2(]0,T[\times\Omega)^n$ and $p^\varepsilon \in L^2(]0,T[\times\Omega_f^\varepsilon)$, satisfying additional regularity properties in time. We refer to [8] and [13] for more details.

Considering only the flow in the seabed, it is known that the effective behavior is described by the law of Biot (see [5] and [6]). Its derivation by formal asymptotic expansions is given in [7], [2] and [31]. The rigorous justification of the regularized slightly compressible model is in [29]. The detailed theory for the system (33)–(40) can be found in [8]. For presenting the result we first need to solve the auxiliary problems.

We start with the perturbation of the gradient, due to the elastic solid structure. The auxiliary functions $w^{ij} \in H^1(\mathcal{Y}_s)^2 \cap C^\infty(\mathcal{Y}_s)^2$, $\int_{\mathcal{Y}_s} w^{ij} = 0$, and $w^0 \in H^1(\mathcal{Y}_s)^2 \cap C^\infty(\mathcal{Y}_s)^2$ are given by

$$\begin{aligned} \operatorname{div}_y \Big\{ A\Big(\frac{e_i\otimes e_j + e_j\otimes e_i}{2} + D_y(w^{ij})\Big)\Big\} &= 0 \quad \text{in } \mathcal{Y}_s, \\ A\Big(\frac{e_i\otimes e_j + e_j\otimes e_i}{2} + D_y(w^{ij})\Big)\nu &= 0 \quad \text{on } \partial\mathcal{Y}_s, \end{aligned}$$

and

$$\begin{aligned} -\operatorname{div}_y \big\{ A D_y(w^0)\big\} &= 0 \quad \text{in } \mathcal{Y}_s, \\ A D_y(w^0)\nu &= \nu \quad \text{on } \partial\mathcal{Y}_s. \end{aligned}$$

Furthermore, the tensors

$$A^H_{klij} = \Big(\int_{\mathcal{Y}_s} A\Big(\frac{e_i\otimes e_j + e_j\otimes e_i}{2} + D_y(w^{ij})\Big)\Big)_{kl}, \qquad \mathcal{B}^H = \int_{\mathcal{Y}_s} A D_y(w^0),$$

are positive definite. Let $\mathcal{C}^H_{ij} = \int_{\mathcal{Y}_s} \operatorname{div}_y w^{ij}(y)\,dy$.

Hence the oscillation of the gradient due to the elastic solid part is $\nabla_y u^1$, with u^1 having the form

$$u^1(x,y,t) = p^0(x,t)\,w^0(y) + \sum_{i,j} \big(D_x(u^0(x,t))\big)_{ij}\, w^{ij}(y).$$

We construct the oscillations v due to the fluid flow by first solving

$$\begin{cases} \rho_f \dfrac{\partial w^i}{\partial t} - \mu\Delta w^i + \nabla\pi^i = 0, \\ \operatorname{div}_y w^i = 0, \quad w^i(y,0) = e_i, \\ w^i\,|_{\partial\mathcal{Y}_f} = 0, \quad \{w^i,\pi^i\} \quad \text{is 1-periodic.} \end{cases}$$

This problem is studied in [27], and it is proved that the matrix $\mathcal{A}_{ij}(t) = \int_{\mathcal{Y}_f} w_i^j(y,t)\,dy$ is positive definite and tends exponentially to zero when $t \to +\infty$. We note that $\mathcal{A}_{ij}(0) = \int_{\mathcal{Y}_f} w_i^j(y,0)\,dy = |\mathcal{Y}_f|\,\delta_{ij}$.

Let u^0 denote the non-oscillatory part of the limit displacement and p^0 the effective pressure. They are determined by solving the following Biot's system in Ω_2

$$\begin{aligned}\operatorname{div}_x \Big\{ |\mathcal{Y}_f|\, \rho_f \frac{\partial u^0}{\partial t} + \sum_{i,j} e_i \int_0^t \mathcal{A}_{ij}(t-\tau) \Big[\rho_f F_j(x,\tau) - \frac{\partial p^0}{\partial x_j}(x,\tau) \Big]\, d\tau \\ - \sum_{i,j} e_i \int_0^t \mathcal{A}_{ij}(t-\tau)\, \rho_f \frac{\partial^2 u_j^0}{\partial \tau^2}(x,\tau)\, d\tau \Big\} \\ = \mathcal{C}^H D\Big(\frac{\partial u^0}{\partial t}\Big) + \frac{\partial p^0}{\partial t} \int_{\mathcal{Y}_s} \operatorname{div}_y w^0\, dy. \end{aligned} \tag{41}$$

$$\begin{aligned}\bar\rho \frac{\partial^2 u^0}{\partial t^2} - \sum_{i,j} e_i \frac{d}{dt} \int_0^t \mathcal{A}_{ij}(t-\tau) \Big[\frac{\partial p^0}{\partial x_j}(x,\tau) + \rho_f \frac{\partial^2 u^0}{\partial \tau^2}(x,\tau) \Big]\, d\tau \\ - \operatorname{div}_x \big\{ A^H D_x(u^0) \big\} - \operatorname{div}_x \big\{ p^0 \mathcal{B}^H \big\} + |\mathcal{Y}_f|\, \nabla_x p^0 \\ = \bar\rho F - \sum_{i,j} e_i \frac{d}{dt} \int_0^t \mathcal{A}_{ij}(t-\tau)\, \rho_f F_j(x,\tau)\, d\tau, \end{aligned} \tag{42}$$

$$u^0 \text{ and } p^0 \quad \text{are } L\text{-periodic in } x_1 \quad \text{and are zero initially.} \tag{43}$$

Then the oscillations of the velocity are given by

$$\frac{\partial v_i}{\partial t} = \sum_j \int_0^t w_i^j(y, t-\tau) \Big(F_i(x,\tau) - \frac{1}{\rho_f} \frac{\partial p^0}{\partial x_i}(x,\tau) - \frac{\partial^2 u^0}{\partial t^2}(x,\tau) \Big)\, d\tau.$$

For the detailed analysis of the system (41)–(43) we refer to [8]. We repeat here only the simplified version of the convergence result (see [8]):

Theorem 5. *Let u^0, p^0, u^1 and v be defined as above. Let $\Phi^\varepsilon = \int_{\Omega_2{}^\varepsilon} 1/(1-x_2)$. Then we have*

$$u^\varepsilon(x,t) - u^0(x,t) - \chi_{\Omega_2^\varepsilon} v\Big(x, \frac{x}{\varepsilon}, t\Big) \to 0 \quad \text{in } L^2_{loc}(\Omega_2)^2, \tag{44}$$

$$\chi_{\Omega_s^\varepsilon} \Big(\nabla u^\varepsilon - \nabla_x u^0(x,t) - \nabla_y u^1\Big(x, \frac{x}{\varepsilon}, t\Big) \Big) \to 0 \quad \text{in } L^2_{loc}(\Omega_2)^4, \tag{45}$$

$$\chi_{\Omega_2^\varepsilon} p^\varepsilon(x,t) - \frac{1}{\Phi^\varepsilon} \int_{\Omega_2^\varepsilon} \frac{p^\varepsilon}{1-x_2} - \chi_{\Omega_2^\varepsilon} p^0(x,t) \to 0 \quad \text{in } L^2_{loc}(\Omega_2), \tag{46}$$

as $\varepsilon \to 0$.

The interaction with the fluid in Ω_1 doesn't change the equations. In Ω_2 the system (41)–(42) remains valid. On the other hand, the non-stationary Stokes system in Ω_1 converges towards the linearized incompressible Euler equations. The challenging difficulty is finding the boundary conditions at the interface Σ. After applying the technique from [17] and constructing the appropriate boundary layers, we get the following result:

Theorem 6. *Let* u^0 *and* p^0 *satisfy* (41)–(42) *in* Ω_2, (43) *in* $\Omega_1 \cup \Sigma \cup \Omega_2$, *and the following equations in* $\Sigma \cup \Omega_1$

$$\begin{aligned}
&\rho_f \frac{\partial^2 u^0}{\partial t^2} + \nabla p^0 = \rho F \quad \text{in } \Omega_1 \times]0,T[\\
&\operatorname{div} u^0 = 0 \quad \text{in } \Omega_1 \times]0,T[\\
&\big[p^0\big] = 0 \quad \text{on } \Sigma \times]0,T[\\
&\frac{\partial u_2^0}{\partial t}(x_1, 0-, t) + \sum_j e_2 \int_0^t \mathcal{A}_{2j}(t-\tau) \bigg[F_j(x_1, 0-, \tau) - \frac{1}{\rho_f} \frac{\partial p^0}{\partial x_j}(x_1, 0-, \tau) \\
&\qquad - \frac{\partial^2 u_j^0}{\partial \tau^2}(x_1, 0-, t) \bigg] d\tau = \frac{\partial u_2^0}{\partial t}(x_1, 0+, t) \quad \text{on } \Sigma \times]0,T[.
\end{aligned}$$

Then the convergences (44)–(46) *take place as* $\varepsilon \to 0$, *with* Ω_2 *replaced by* Ω, *and* Ω_2^ε *by* Ω_f^ε, *respectively.*

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