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Dynamic Model of Curved Rods

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Abstract. The dynamic model of curved rod is obtained as an approximation of oscillations of three-dimensional curved rod-like linearized elastic body. The corresponding convergence result is stated.

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1. Introduction

In this paper we derive the evolution model of curved rod. The corresponding convergence result is stated. A similar problem in case of plates has been considered by Raoult [6], while Xiao Li-ming [5] derived and justified the evolution model of shells. The method used is a version of the asymptotic expansion method already applied to derivation of equilibrium models of curved rod [2, 3]. Problems and results are stated precisely but the proofs are mainly omitted; details can be found in [8].

By $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ we denote the canonical basis in $\mathbb{R}^3$. Vectors, vector-valued functions, matrices and matrix-valued functions are denoted by boldface letters. Euclidean inner product and norm of vectors in $\mathbb{R}^n$ is denoted by $\cdot$ and $| \, |$, respectively. Repeated index convention is accepted. The dual space of a Hilbert space H is denoted by H'. The same symbol $'$ stands for the derivative with respect to the longitudinal variable; partial derivatives with respect to space and time variables are denoted by ∂_i, $i = 1, 2, 3$ and ∂_t, ∂_{tt}, respectively.

2. The curved rod

Let $\mathcal{C}$ be a simple regular curve in $\mathbb{R}^3$ defined by its natural parametrization $\boldsymbol{\Phi} \in C^3([0, \ell]; \mathbb{R}^3)$. The tangent on $\mathcal{C}$ at $\boldsymbol{\Phi}(y^1)$ is defined by $\boldsymbol{t}(y^1) = \boldsymbol{\Phi}'(y^1)$; obviously $\boldsymbol{t} \in C^2([0, \ell]; \mathbb{R}^3)$. In [4] it is proved that there exists a matrix-valued function $\mathbf{Q} \in C^2([0, \ell]; \mathrm{SO}(3))$ such that

$$\mathbf{Q}\boldsymbol{e}_1 = \boldsymbol{t} \quad \text{on } [0, \ell].$$

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Then the local basis $\{\boldsymbol{t}(y^1), \boldsymbol{n}(y^1), \boldsymbol{b}(y^1)\}$ on $\mathcal{C}$ at $\boldsymbol{\Phi}(y^1)$ can be defined by

$$\boldsymbol{t}(y^1) = \mathbf{Q}(y^1)\boldsymbol{e}_1, \quad \boldsymbol{n}(y^1) = \mathbf{Q}(y^1)\boldsymbol{e}_2, \quad \boldsymbol{b}(y^1) = \mathbf{Q}(y^1)\boldsymbol{e}_3, \quad y^1 \in [0,\ell].$$

Smoothness of $\mathbf{Q}$ implies that $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b} \in C^2([0,\ell];\mathbb{R}^3)$. Note that this basis exists even though the smooth Frenet frame does not exist globally. Let

$$\mathbf{R}(y^1) = \left(\frac{d}{dy^1}\mathbf{Q}^T(y^1)\right)\mathbf{Q}(y^1), \quad y^1 \in [0,\ell].$$

Matrix $\mathbf{R}(y^1)$ is antisymmetric for all $y^1 \in [0,\ell]$. Moreover, the following generalization of the Frenet equations holds:

$$\frac{d}{dy^1}\mathbf{Q}^T = \mathbf{R}\mathbf{Q}^T.$$

In case of generic curve ($|\boldsymbol{\Phi}''| > 0$), matrix-valued function $\mathbf{Q}$ can be chosen so that $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ and the Frenet frame coincide. In this case it holds

$$\mathbf{R} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix},$$

where κ is the curvature and τ is the torsion of the curve $\mathcal{C}$.

Now we define the domain in $\mathbb{R}^3$ which represents the curved rod. Let $\varepsilon > 0$ be a small parameter and let $S \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary; points in S are denoted by (z^2, z^3). The coordinates z^2 and z^3 can be chosen so that $(0,0)$ is the center of mass of S, i.e.,

$$\int_S z^2\, dz^2\, dz^3 = \int_S z^3\, dz^2\, dz^3 = 0. \tag{1}$$

Let

$$\Omega^\varepsilon = (0,\ell) \times \varepsilon S, \quad \Gamma^\varepsilon = (0,\ell) \times \varepsilon\partial S, \quad B^\varepsilon_{y^1} = \{y^1\} \times \varepsilon S.$$

Γ^ε is the lateral boundary, ℓ is the length, $B^\varepsilon_{y^1}$ is the cross-section at $y^1 \in [0,\ell]$, while ε is the thickness of the cylinder Ω^ε. B^ε_0 and B^ε_ℓ are called the bases of the rod. Let $\boldsymbol{P} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ be defined by

$$\boldsymbol{P}(y) = \boldsymbol{\Phi}(y^1) + y^2\, \boldsymbol{n}(y^1) + y^3\, \boldsymbol{b}(y^1), \quad y^1 \in [0,\ell],\ (y^2, y^3) \in \overline{\varepsilon S}.$$

Since $\mathcal{C}$ is simple and regular curve, there exists $\varepsilon_0 > 0$ such that the function $\boldsymbol{P}$ is injective; especially one has

$$g(y) = (\det \nabla \boldsymbol{P}(y))^2 = \left(1 - R_{12}(y^1)y^2 - R_{13}\, y^3\right)^2 > 0, \quad y \in \Omega^\varepsilon.$$

In the sequel we restrict ourselves to $\varepsilon \in (0, \varepsilon_0)$. The domain, see Figure 1,

$$\widetilde{\Omega}^\varepsilon = \boldsymbol{P}(\Omega^\varepsilon)$$

is called the curved rod; its lateral boundary and its cross-sections at $\boldsymbol{\Phi}(y^1)$ are, respectively,

$$\widetilde{\Gamma}^\varepsilon = \boldsymbol{P}(\Gamma^\varepsilon), \quad \widetilde{B}^\varepsilon_{y^1} = \boldsymbol{P}(B^\varepsilon_{y^1}).$$

$\widetilde{B}^\varepsilon_0$ and $\widetilde{B}^\varepsilon_\ell$ are the bases of $\widetilde{\Omega}^\varepsilon$. Because of (1), the curve $\mathcal{C}$ passes through the centers of mass of $\widetilde{B}^\varepsilon_{y^1}$, $y^1 \in [0, \ell]$, thus we say that $\mathcal{C}$ is the middle curve of the curved rod $\widetilde{\Omega}^\varepsilon$.

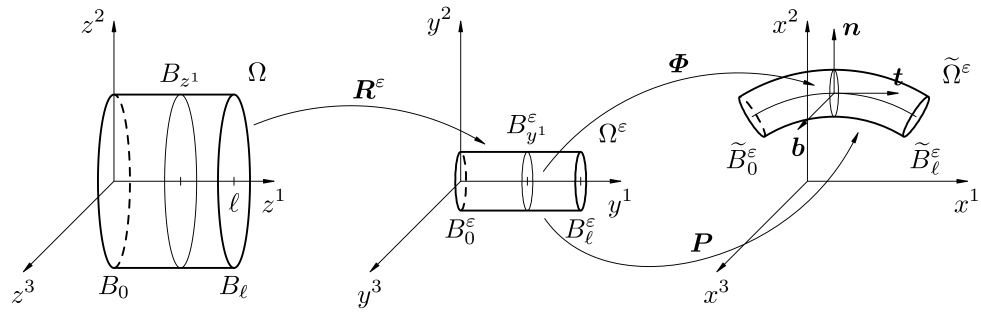

Figure 1. Canonical domain, thin cylinder and curved rod.

3. The three-dimensional evolution equations

We assume that $\overline{\widetilde{\Omega}}^\varepsilon$ is the natural state of a linearized isotropic elastic body with the Lamé constants λ and μ. Let $\widetilde{\Omega}^\varepsilon$ be clamped on its bases $\widetilde{B}^\varepsilon_0$ and $\widetilde{B}^\varepsilon_\ell$ and force-free on its lateral boundary $\widetilde{\Gamma}^\varepsilon$, and let $\widetilde{\boldsymbol{U}}^\varepsilon_0$ and $\widetilde{\boldsymbol{U}}^\varepsilon_1$ be the initial displacement and velocity of $\widetilde{\Omega}^\varepsilon$, respectively. Let $T > 0$. Then the oscillations of $\widetilde{\Omega}^\varepsilon$ are described by a function $\widetilde{\boldsymbol{U}}^\varepsilon$ which is formally a solution of

$$\begin{aligned}
&\tilde{\varrho}^\varepsilon \partial_{tt}\widetilde{\boldsymbol{U}}^\varepsilon - \operatorname{div}\big(\lambda \operatorname{tr}\mathbf{e}(\widetilde{\boldsymbol{U}}^\varepsilon)\mathbf{I} + 2\mu\,\mathbf{e}(\widetilde{\boldsymbol{U}}^\varepsilon)\big) = \widetilde{\boldsymbol{F}}^\varepsilon \quad \text{in } (0,T)\times\widetilde{\Omega}^\varepsilon,\\
&\widetilde{\boldsymbol{U}}^\varepsilon = 0 \quad \text{on } (0,T)\times\big(\widetilde{B}^\varepsilon_0 \cup \widetilde{B}^\varepsilon_\ell\big),\\
&\lambda \operatorname{tr}\big(\mathbf{e}(\widetilde{\boldsymbol{U}}^\varepsilon)\big)\widetilde{\boldsymbol{\nu}} + 2\mu\,\mathbf{e}(\widetilde{\boldsymbol{U}}^\varepsilon)\widetilde{\boldsymbol{\nu}} = 0 \quad \text{on } (0,T)\times\widetilde{\Gamma}^\varepsilon,\\
&\widetilde{\boldsymbol{U}}^\varepsilon|_{t=0} = \widetilde{\boldsymbol{U}}^\varepsilon_0, \quad \partial_t\widetilde{\boldsymbol{U}}^\varepsilon|_{t=0} = \widetilde{\boldsymbol{U}}^\varepsilon_1 \quad \text{in } \widetilde{\Omega}^\varepsilon.
\end{aligned}$$

Here $\widetilde{\boldsymbol{F}}^\varepsilon$ is the volume force density acting on the curved rod, $\tilde{\varrho}^\varepsilon$ is the mass density of the rod, $\widetilde{\boldsymbol{\nu}}$ is the unit outer normal at the boundary of the curved rod, and $\mathbf{e}(\widetilde{\boldsymbol{V}})$ denotes the symmetrized gradient of the function $\widetilde{\boldsymbol{V}}$, i.e.,

$$\mathbf{e}(\widetilde{\boldsymbol{V}}) = \frac{1}{2}\big(\nabla\widetilde{\boldsymbol{V}} + (\nabla\widetilde{\boldsymbol{V}})^T\big).$$

More precisely, let us introduce the function space

$$\mathcal{V}(\widetilde{\Omega}^\varepsilon) = \{\widetilde{\boldsymbol{V}} \in H^1(\widetilde{\Omega}^\varepsilon)^3 \mid \widetilde{\boldsymbol{V}}|_{\widetilde{B}_0^\varepsilon} = \widetilde{\boldsymbol{V}}|_{\widetilde{B}_\ell^\varepsilon} = 0\},$$

which is a Hilbert space for the scalar product of $H^1(\Omega)^3$. Differential equations are then formally equivalent to:

$$\int_{\widetilde{\Omega}^\varepsilon} \tilde{\varrho}^\varepsilon \partial_{tt}\widetilde{\boldsymbol{U}}^\varepsilon \cdot \widetilde{\boldsymbol{V}}\, dx + \int_{\widetilde{\Omega}^\varepsilon} \lambda \operatorname{tr} \mathbf{e}(\widetilde{\boldsymbol{U}}^\varepsilon) \operatorname{tr} \mathbf{e}(\widetilde{\boldsymbol{V}}) + 2\mu\, \mathbf{e}(\widetilde{\boldsymbol{U}}^\varepsilon)\cdot \mathbf{e}(\widetilde{\boldsymbol{V}})\, dx$$

$$= \int_{\widetilde{\Omega}^\varepsilon} \widetilde{\boldsymbol{F}}^\varepsilon \cdot \widetilde{\boldsymbol{V}}\, dx, \quad \widetilde{\boldsymbol{V}} \in \mathcal{V}(\widetilde{\Omega}^\varepsilon), \quad 0 < t < T, \tag{2}$$

$$\widetilde{\boldsymbol{U}}^\varepsilon|_{t=0} = \widetilde{\boldsymbol{U}}_0^\varepsilon, \quad \partial_t \widetilde{\boldsymbol{U}}^\varepsilon|_{t=0} = \widetilde{\boldsymbol{U}}_1^\varepsilon. \tag{3}$$

Lemma 1. *Let $\widetilde{\boldsymbol{F}}^\varepsilon \in L^2(0,T;L^2(\widetilde{\Omega}^\varepsilon)^3)$, $\widetilde{\boldsymbol{U}}_0^\varepsilon \in \mathcal{V}(\widetilde{\Omega}^\varepsilon)$ and $\widetilde{\boldsymbol{U}}_1^\varepsilon \in L^2(\widetilde{\Omega}^\varepsilon)^3$. Then there exists a unique solution $\widetilde{\boldsymbol{U}}^\varepsilon$ of the problem* (2), (3)*, such that*

$$\widetilde{\boldsymbol{U}}^\varepsilon \in C([0,T];\mathcal{V}(\widetilde{\Omega}^\varepsilon)), \quad \partial_t \widetilde{\boldsymbol{U}}^\varepsilon \in C([0,T];L^2(\widetilde{\Omega}^\varepsilon)), \quad \partial_{tt}\widetilde{\boldsymbol{U}}^\varepsilon \in L^2(0,T;\mathcal{V}(\widetilde{\Omega}^\varepsilon)').$$

At the end of this section we rewrite (2), (3) in curvilinear coordinates defined by $\boldsymbol{P}$. Covariant basis of the curved rod is defined by

$$\boldsymbol{g}_i = \partial_i \boldsymbol{P} : \Omega^\varepsilon \to \mathbb{R}^3, \quad i = 1,2,3.$$

Vectors $(\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3)$ satisfying

$$\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j \quad \text{on } \widetilde{\Omega}^\varepsilon, \quad i,j = 1,2,3,$$

where δ_i^j is the Kronecker symbol, form the contravariant basis on $\widetilde{\Omega}^\varepsilon$. The contravariant metric tensor $\mathbf{G} = (g^{ij})$ and the Christoffel symbols Γ_{jk}^i of the curved rod $\widetilde{\Omega}^\varepsilon$ are defined by

$$g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \quad \Gamma_{jk}^i = \boldsymbol{g}^i \cdot \partial_j \boldsymbol{g}_k \quad \text{on } \widetilde{\Omega}^\varepsilon, \quad i,j,k = 1,2,3.$$

The corresponding function space to $\mathcal{V}(\widetilde{\Omega}^\varepsilon)$ is the space

$$\mathcal{V}(\Omega^\varepsilon) = \{\boldsymbol{V} \in H^1(\Omega^\varepsilon)^3 \mid \boldsymbol{V}|_{B_0^\varepsilon} = \boldsymbol{V}|_{B_\ell^\varepsilon} = 0\}.$$

The displacements and velocities are rewritten in contravariant basis, while the force density in covariant basis

$$\widetilde{\boldsymbol{U}}^\varepsilon \circ \boldsymbol{P} = U_i^\varepsilon \boldsymbol{g}^i, \quad \widetilde{\boldsymbol{U}}_0^\varepsilon \circ \boldsymbol{P} = (U_0^\varepsilon)_i \boldsymbol{g}^i, \quad \widetilde{\boldsymbol{U}}_1^\varepsilon \circ \boldsymbol{P} = (U_1^\varepsilon)_i \boldsymbol{g}^i,$$
$$\widetilde{\boldsymbol{V}} \circ \boldsymbol{P} = V_i \boldsymbol{g}^i, \quad \widetilde{\boldsymbol{F}}^\varepsilon \circ \boldsymbol{P} = F^{\varepsilon i} \boldsymbol{g}_i.$$

We define vector functions

$$\boldsymbol{U}^\varepsilon = U_i^\varepsilon \boldsymbol{e}_i, \quad \boldsymbol{U}_0^\varepsilon = (U_0^\varepsilon)_i \boldsymbol{e}_i, \quad \boldsymbol{U}_1^\varepsilon = (U_1^\varepsilon)_i \boldsymbol{e}_i, \quad \boldsymbol{V} = V_i \boldsymbol{e}_i, \quad \boldsymbol{F}^\varepsilon = F^{\varepsilon i} \boldsymbol{e}_i.$$

Let $\varrho^\varepsilon = \tilde{\varrho}^\varepsilon \circ \boldsymbol{P}$ and

$$\boldsymbol{\gamma}(\boldsymbol{V}) = \mathbf{e}(\boldsymbol{V}) - V_i \boldsymbol{\Gamma}^i,$$
$$\mathbf{A}\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 = \lambda \operatorname{tr} \boldsymbol{\sigma}^1 \operatorname{tr} \boldsymbol{\sigma}^2 + 2\mu\, \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2, \quad \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \mathrm{Sym}(\mathbb{R}^3),$$

where $\mathrm{Sym}(\mathbb{R}^3)$ denotes the space of all real symmetric matrices of order 3. The system (2), (3) is then equivalent to the following equation of motion in curvilinear coordinates:

$$\int_{\Omega^\varepsilon} \varrho^\varepsilon \mathbf{G}\, \partial_{tt} \boldsymbol{U}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, dy + \int_{\Omega^\varepsilon} \mathbf{A}\boldsymbol{\gamma}(\boldsymbol{U}^\varepsilon) \cdot \boldsymbol{\gamma}(\boldsymbol{V}) \sqrt{g}\, dy$$
$$= \int_{\Omega^\varepsilon} \boldsymbol{F}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, dy, \quad \boldsymbol{V} \in \mathcal{V}(\Omega^\varepsilon), \quad 0 < t < T, \tag{4}$$
$$\boldsymbol{U}^\varepsilon|_{t=0} = \boldsymbol{U}_0^\varepsilon, \quad \partial_t \boldsymbol{U}^\varepsilon|_{t=0} = \boldsymbol{U}_1^\varepsilon. \tag{5}$$

4. A priori estimates on fixed domain

Our main goal is to find the limit of $(\widetilde{\boldsymbol{U}}^\varepsilon, \varepsilon > 0)$ as ε tends to zero, as well as the equations satisfied by the limit. Problems for both $\widetilde{\boldsymbol{U}}^\varepsilon$ and $\boldsymbol{U}^\varepsilon$ are posed on ε-dependent domains. Now we transform the problem (4), (5) to ε-independent domain, see Figure 1. As a consequence, the coefficients of the resulting weak formulation will depend on ε explicitly, and calculation of the limit will be enabled.

Let $\Omega = (0, \ell) \times S$, and let $\boldsymbol{R}^\varepsilon : \overline{\Omega} \to \mathbb{R}^3$ be defined by

$$\boldsymbol{R}^\varepsilon(z) = (z^1, \varepsilon z^2, \varepsilon z^3), \quad z \in \Omega, \quad \varepsilon \in (0, \varepsilon_0).$$

By Γ and S_{z^1} we denote the lateral surface of Ω and its cross-section at $z^1 \in [0, \ell]$, respectively. To the functions $\boldsymbol{U}^\varepsilon$, $\boldsymbol{U}_0^\varepsilon$, $\boldsymbol{U}_1^\varepsilon$, $\boldsymbol{F}^\varepsilon$, g, $\boldsymbol{g}_i$, $\boldsymbol{g}^i$, ϱ^ε, $\mathbf{G}$, $\mathbf{R}$, Γ^i_{jk}, $i, j, k = 1, 2, 3$, defined on Ω^ε, we associate the functions $\boldsymbol{u}(\varepsilon)$, $\boldsymbol{u}_0(\varepsilon)$, $\boldsymbol{u}_1(\varepsilon)$, $\boldsymbol{f}(\varepsilon)$, $g(\varepsilon)$, $\boldsymbol{g}_i(\varepsilon)$, $\boldsymbol{g}^i(\varepsilon)$, $\varrho(\varepsilon)$, $\mathbf{G}(\varepsilon)$, $\mathbf{R}(\varepsilon)$, $\Gamma^i_{jk}(\varepsilon)$, $i, j, k = 1, 2, 3$, defined on Ω by composition with $\boldsymbol{R}^\varepsilon$. Let

$$\mathcal{V}(\Omega) = \big\{ \boldsymbol{v} = (v_1, v_2, v_3) \in H^1(\Omega)^3 \;\big|\; \boldsymbol{v}|_{B_0} = \boldsymbol{v}|_{B_\ell} = 0 \big\},$$
$$a(\varepsilon) : \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to \mathbb{R}, \quad a(\varepsilon)(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \mathbf{A}(\varepsilon) \frac{1}{\varepsilon} \boldsymbol{\gamma}^\varepsilon(\boldsymbol{v}) \cdot \frac{1}{\varepsilon} \boldsymbol{\gamma}^\varepsilon(\boldsymbol{w}) \sqrt{g(\varepsilon)}\, dz,$$
$$b(\varepsilon) : L^2(\Omega)^3 \times L^2(\Omega)^3 \to \mathbb{R}, \quad b(\varepsilon)(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \frac{1}{\varepsilon^2}\, \varrho(\varepsilon)\, \mathbf{G}(\varepsilon) \boldsymbol{v} \cdot \boldsymbol{w} \sqrt{g(\varepsilon)}\, dz,$$
$$\boldsymbol{\gamma}^\varepsilon(\boldsymbol{v}) = \frac{1}{\varepsilon} \boldsymbol{\gamma}_z(\boldsymbol{v}) + \boldsymbol{\gamma}_y(\boldsymbol{v}) - v_i \boldsymbol{\Gamma}^i(\varepsilon),$$
$$\boldsymbol{\gamma}_y(\boldsymbol{v}) = \begin{pmatrix} \partial_1 v_1 & \frac{1}{2}\partial_1 v_2 & \frac{1}{2}\partial_1 v_3 \\ \frac{1}{2}\partial_1 v_2 & 0 & 0 \\ \frac{1}{2}\partial_1 v_3 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\gamma}_z(\boldsymbol{v}) = \begin{pmatrix} 0 & \frac{1}{2}\partial_2 v_1 & \frac{1}{2}\partial_3 v_1 \\ \frac{1}{2}\partial_2 v_1 & \partial_2 v_2 & \frac{1}{2}(\partial_2 v_3 + \partial_3 v_2) \\ \frac{1}{2}\partial_3 v_1 & \frac{1}{2}(\partial_2 v_3 + \partial_3 v_2) & \partial_3 v_3 \end{pmatrix},$$

$$\mathbf{A}(\varepsilon)\boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 = \lambda \operatorname{tr} \mathbf{G}(\varepsilon)\boldsymbol{\sigma}^1 \operatorname{tr} \mathbf{G}(\varepsilon)\boldsymbol{\sigma}^2 + 2\mu\, \mathbf{G}(\varepsilon)\boldsymbol{\sigma}^1 \mathbf{G}(\varepsilon) \cdot \boldsymbol{\sigma}^2, \quad \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \operatorname{Sym}(\mathbb{R}^3).$$

The problem (4), (5) is then equivalent to the following problem:

Find $\boldsymbol{u}(\varepsilon)$ such that

$$\frac{d}{dt} b(\varepsilon)\big(\partial_t \boldsymbol{u}(\varepsilon), \boldsymbol{v}\big) + a(\varepsilon)\big(\boldsymbol{u}(\varepsilon), \boldsymbol{v}\big) = \left(\frac{1}{\varepsilon^2} \boldsymbol{f}(\varepsilon)\sqrt{g(\varepsilon)} \,\middle|\, \boldsymbol{v}\right)_{L^2(\Omega)^3},$$
$$\boldsymbol{v} \in \mathcal{V}(\Omega), \quad 0 < t < T, \tag{6}$$

$$\boldsymbol{u}(\varepsilon)|_{t=0} = \boldsymbol{u}_0(\varepsilon), \quad \partial_t \boldsymbol{u}(\varepsilon)|_{t=0} = \boldsymbol{u}_1(\varepsilon). \tag{7}$$

The family $(\boldsymbol{u}(\varepsilon), 0 < \varepsilon < \varepsilon_0)$ satisfies

$$\boldsymbol{u}(\varepsilon) \in C([0,T]; \mathcal{V}(\Omega)), \quad \partial_t \boldsymbol{u}(\varepsilon) \in C([0,T]; L^2(\Omega)^3), \quad \partial_{tt} \boldsymbol{u}(\varepsilon) \in L^2(0,T; \mathcal{V}(\Omega)').$$

In general, this family is not bounded with respect to ε. The conditions ensuring certain uniform boundedness are stated in the following theorem.

Theorem 1 (*a priori* estimates). *Let us assume that*

$$\boldsymbol{f}(\varepsilon) = \varepsilon^2 \boldsymbol{f}, \quad \varrho(\varepsilon) = \varepsilon^2 \varrho, \tag{8}$$

$$\left\| \frac{1}{\varepsilon} \boldsymbol{\gamma}^\varepsilon(\boldsymbol{u}_0(\varepsilon)) \right\|_{L^2(\Omega)^9} \le C_u, \quad \boldsymbol{u}_1(\varepsilon) \rightharpoonup \boldsymbol{u}_1 \quad \textit{weakly in } L^2(\Omega)^3, \tag{9}$$

where ϱ and C_u are constants independent of ε. Then there exists $C > 0$ such that

$$\|\boldsymbol{u}(\varepsilon)\|_{\mathcal{V}(\Omega)} \le C, \quad \|\partial_t \boldsymbol{u}(\varepsilon)\|_{L^2(\Omega)^3} \le C, \quad \left\| \frac{1}{\varepsilon} \boldsymbol{\gamma}^\varepsilon(\boldsymbol{u}(\varepsilon)) \right\|_{L^2(\Omega)^9} \le C, \quad \varepsilon \in (0, \varepsilon_0). \tag{10}$$

Proof of Theorem 1 is long and we omit it here; details can be found in [8]. A simple consequence of *a priori* estimate (10) and smoothness of $\boldsymbol{u}(\varepsilon)$ is the following convergence result.

Corollary 1. *There is a subsequence of $(\boldsymbol{u}(\varepsilon), \varepsilon > 0)$ (still denoted by ε) and functions $\boldsymbol{u}$ and $\boldsymbol{\gamma}$, $\boldsymbol{u} \in L^\infty(0,T; \mathcal{V}(\Omega))$, $\partial_t \boldsymbol{u} \in L^\infty(0,T; L^2(\Omega)^3)$, $\boldsymbol{\gamma} \in L^\infty(0,T; L^2(\Omega)^9)$, such that*

$$\boldsymbol{u}(\varepsilon) \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \textit{weak} * \textit{in } L^\infty(0,T; \mathcal{V}(\Omega)),$$
$$\partial_t \boldsymbol{u}(\varepsilon) \overset{*}{\rightharpoonup} \partial_t \boldsymbol{u} \quad \textit{weak} * \textit{in } L^\infty(0,T; L^2(\Omega)^3),$$
$$\frac{1}{\varepsilon} \boldsymbol{\gamma}^\varepsilon(\boldsymbol{u}(\varepsilon)) \overset{*}{\rightharpoonup} \boldsymbol{\gamma} \quad \textit{weak} * \textit{in } L^\infty(0,T; L^2(\Omega)^9).$$

5. The limit problem

The relevant function spaces for description of the limit problem are the following Hilbert spaces

$$V_0 = H_0^1(0,\ell)\times H_0^2(0,\ell)\times H_0^2(0,\ell),\quad \mathcal{V}_0 = \{\boldsymbol{v}\in V_0 \mid v_1' - R_{12}v_2 - R_{13}v_3 = 0\},$$
$$H_0 = L^2(0,\ell)^3,\quad \mathcal{H}_0 = \{\boldsymbol{v}\in H_0^1(0,\ell)\times L^2(0,\ell)\times L^2(0,\ell) \mid v_1' - R_{12}v_2 - R_{13}v_3 = 0\},$$
$$W_0 = V_0\times H_0^1(0,\ell),\quad \mathcal{W}_0 = \mathcal{V}_0\times H_0^1(0,\ell),$$

with norms

$$\|\boldsymbol{v}\|_{V_0}^2 = \|v_1'\|_{L^2(0,\ell)}^2 + \|v_2''\|_{L^2(0,\ell)}^2 + \|v_3''\|_{L^2(0,\ell)}^2,$$
$$\|\boldsymbol{v}\|_{H_0}^2 = \|v_1\|_{L^2(0,\ell)}^2 + \|v_2\|_{L^2(0,\ell)}^2 + \|v_3\|_{L^2(0,\ell)}^2,$$
$$\|\mathbf{v}\|_{W_0}^2 = \|\boldsymbol{v}\|_{V_0}^2 + \|\psi'\|_{L^2(0,\ell)}^2,\quad \mathbf{v} = (\boldsymbol{v},\psi)\in W_0.$$

V_0 is the usual function space for rods, but the limit displacement from Corollary 1 describes the inextensible oscillations of the rod, i.e., it belongs to the subspace $\mathcal{V}_0$ of V_0. Also note that $\mathcal{H}_0$ is the closure of $\mathcal{V}_0$ in the norm of H_0.

Let us introduce bilinear forms $a_0 : W_0\times W_0 \to \mathbb{R}$ and $b_0 : H_0\times H_0\to\mathbb{R}$ by

$$a_0(\mathbf{v},\mathbf{w}) := \int_0^\ell \mathbf{H}\boldsymbol{a}(\mathbf{v})\cdot\boldsymbol{a}(\mathbf{w})\,dz^1,\quad b_0(\boldsymbol{v},\boldsymbol{w}) := (\varrho A\boldsymbol{v}\mid\boldsymbol{w})_{H_0},$$

where $\boldsymbol{a}(\mathbf{v})$ is defined for $\mathbf{v} = (\boldsymbol{v},\psi)\in W_0$ by

$$a_1(\mathbf{v}) = \psi' + R_{12}(v_3' + R_{13}v_1 + R_{23}v_2) - R_{13}(v_2' + R_{12}v_1 - R_{23}v_3),$$
$$a_2(\mathbf{v}) = -(v_3' + R_{13}v_1 + R_{23}v_2)' + R_{12}\psi - R_{23}(v_2' + R_{12}v_1 - R_{23}v_3),$$
$$a_3(\mathbf{v}) = -(v_2' + R_{12}v_1 - R_{23}v_3)' - R_{13}\psi + R_{23}(v_3' + R_{13}v_1 + R_{23}v_2).$$

The positive definite symmetric matrix $\mathbf{H}$ depends on geometry of the rod and on its elastic properties. Precisely, let the moments of inertia of the cross-section S and the area of S be denoted by

$$I_{23} = -\int_S z^2z^3\,dz^2\,dz^3,\quad I_\alpha = \int_S (z^\alpha)^2\,dz^2\,dz^3,\quad \alpha = 2,3,\quad A = \int_S dz^2\,dz^3,$$

and let $p\in H^1(S)$ be the warping function, i.e., a unique solution of the problem

$$\Delta p = 0\quad\text{in } S,\quad \frac{\partial p}{\partial\boldsymbol{\nu}} = \begin{pmatrix} z^3\\ -z^2\end{pmatrix}\cdot\boldsymbol{\nu}\quad\text{on }\partial S,\quad \int_S p\,dz^2\,dz^3 = 0,$$

where $\boldsymbol{\nu}$ denotes the unit outer normal on S. It can be shown that the number

$$K = \int_S \left((\partial_2 p - z^3)^2 + (\partial_3 p + z^2)^2\right)dz^2\,dz^3$$

is positive; μK is called the torsion rigidity. Now,

$$\mathbf{H} = \begin{pmatrix} \mu K & 0 & 0 \\ 0 & EI_3 & -EI_{23} \\ 0 & -EI_{23} & EI_2 \end{pmatrix},$$

where E is the Young modulus.

Theorem 2. *Let us assume that*

$$\boldsymbol{u}_0(\varepsilon) \rightharpoonup \boldsymbol{u}_0 \quad \text{weakly in } \mathcal{V}(\Omega), \quad \boldsymbol{u}_1 \in \mathcal{H}_0. \tag{11}$$

Let $\boldsymbol{u}$ and $\boldsymbol{\gamma}$ be the limits from Corollary 1. Then there exists $\phi \in L^\infty(0,T;H_0^1(0,\ell))$ such that $\mathbf{u} = (\boldsymbol{u}, \phi)$ satisfies

$$\boldsymbol{u} \in L^\infty(0,T;\mathcal{V}_0), \quad \partial_t \boldsymbol{u} \in L^\infty(0,T;\mathcal{H}_0), \quad \partial_{tt}\boldsymbol{u} \in L^2(0,T;\mathcal{V}_0'),$$

$$\frac{d}{dt} b_0(\partial_t \boldsymbol{u}, \boldsymbol{v}) + a_0(\mathbf{u}, \mathbf{v}) = (\overline{\boldsymbol{f}} \,|\, \boldsymbol{v})_{H_0}, \quad \mathbf{v} = (\boldsymbol{v}, \psi) \in \mathcal{W}_0, \tag{12}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t \boldsymbol{u}(0) = \boldsymbol{u}_1, \tag{13}$$

where

$$\overline{\boldsymbol{f}} = \int_S \boldsymbol{f}\, dz^2\, dz^3 \in L^2(0,T;\mathcal{H}_0).$$

Moreover, the limit function $\boldsymbol{\gamma}$ is of the form

$$\boldsymbol{\gamma} = \begin{pmatrix} a_3(\mathbf{u})z^2 + a_2(\mathbf{u})z^3 & \cdot & \cdot \\ \frac{1}{2}a_1(\mathbf{u})(\partial_2 p - z^3) & -\nu\big(a_3(\mathbf{u})z^2 + a_2(\mathbf{u})z^3\big) & \cdot \\ \frac{1}{2}a_1(\mathbf{u})(\partial_3 p + z^2) & 0 & -\nu\big(a_3(\mathbf{u})z^2 + a_2(\mathbf{u})z^3\big) \end{pmatrix},$$

where ν is the Poisson ratio.

The problem (12), (13) is not a classical one, because there is no time derivative of ϕ. The equivalent form of (12) is

$$a_0((\boldsymbol{u},\phi),(0,\psi)) = 0, \quad \psi \in H_0^1(0,\ell),$$

$$\frac{d}{dt} b_0(\partial_t \boldsymbol{u}, \boldsymbol{v}) + a_0((\boldsymbol{u},\phi),(\boldsymbol{v},0)) = (\overline{\boldsymbol{f}} \,|\, \boldsymbol{v})_{H_0}, \quad \boldsymbol{v} \in \mathcal{V}_0. \tag{14}$$

It follows that there exists a linear continuous operator $D : \mathcal{V}_0 \to H_0^1(0,\ell)$ such that

$$\phi = D\boldsymbol{u}. \tag{15}$$

From (14) it follows that $\boldsymbol{u}$ is a solution of the following standard problem

$$\frac{d}{dt} b_0(\partial_t \boldsymbol{u}, \boldsymbol{v}) + d_0(\boldsymbol{u}, \boldsymbol{v}) = (\overline{\boldsymbol{f}} \,|\, \boldsymbol{v})_{H_0}, \quad \boldsymbol{v} \in \mathcal{V}_0, \tag{16}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t \boldsymbol{u}(0) = \boldsymbol{u}_1, \tag{17}$$

where

$$d_0 : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathbb{R}, \quad d_0(\boldsymbol{v}, \boldsymbol{w}) = a_0((\boldsymbol{u}, 0), (\boldsymbol{v}, 0)) - a_0((0, D\boldsymbol{u}), (0, D\boldsymbol{v})).$$

Since bilinear forms b_0 and d_0 are elliptic, the classical theory of evolution equations implies that the problem (16), (17) has a unique solution $\boldsymbol{u}$ such that

$$\boldsymbol{u} \in \mathcal{V}_0^c(0,T) = \{\boldsymbol{v} \in C([0,T]; \mathcal{V}_0) \mid \partial_t \boldsymbol{v} \in C([0,T]; \mathcal{H}_0),\ \partial_{tt} \boldsymbol{v} \in L^2(0,T; \mathcal{V}_0')\}.$$

The function ϕ is uniquely determined by (15) and $\phi \in C([0,T]; H_0^1(0,\ell))$. It follows that the limit functions $\boldsymbol{u}$ and $\boldsymbol{\gamma}$ are unique, hence whole families $(\boldsymbol{u}(\varepsilon), 0 < \varepsilon < \varepsilon_0)$ and $((1/\varepsilon)\boldsymbol{\gamma}^\varepsilon(\boldsymbol{u}(\varepsilon)), 0 < \varepsilon < \varepsilon_0)$ are convergent. Thus we proved

Theorem 3. *Let the function $\boldsymbol{u}(\varepsilon)$, for $0 < \varepsilon < \varepsilon_0$, be the solution of* (6), (7), *and let the assumptions* (8), (9) *and* (11) *be fulfilled. Then*

$$\boldsymbol{u}(\varepsilon) \in C([0,T];\, \mathcal{V}(\Omega)), \quad \partial_t \boldsymbol{u}(\varepsilon) \in C([0,T];\, L^2(\Omega)^3), \quad \partial_{tt} \boldsymbol{u}(\varepsilon) \in L^2(0,T;\, \mathcal{V}(\Omega)'),$$

and

$$\boldsymbol{u}(\varepsilon) \stackrel{*}{\rightharpoonup} \boldsymbol{u} \quad \textit{weak} * \textit{in } L^\infty(0,T; \mathcal{V}(\Omega)),$$

$$\partial_t \boldsymbol{u}(\varepsilon) \stackrel{*}{\rightharpoonup} \partial_t \boldsymbol{u} \quad \textit{weak} * \textit{in } L^\infty(0,T; L^2(\Omega)^3),$$

$$\partial_{tt} \boldsymbol{u}(\varepsilon) \rightharpoonup \partial_{tt} \boldsymbol{u} \quad \textit{weakly in } L^2(0,T; \mathcal{V}(\Omega)'),$$

$$\frac{1}{\varepsilon} \boldsymbol{\gamma}^\varepsilon(\boldsymbol{u}(\varepsilon)) \stackrel{*}{\rightharpoonup} \boldsymbol{\gamma} \quad \textit{weak} * \textit{in } L^\infty(0,T; L^2(\Omega)^9),$$

when $\varepsilon \to 0$, where $\mathbf{u} = (\boldsymbol{u}, \phi) \in \mathcal{V}_0^c(0,T) \times C([0,T]; H_0^1(0,\ell))$ is a unique solution of (12), (13) *and $\boldsymbol{\gamma}$ as given in Theorem* 2.

6. Dynamic curved rod model

The system (12), (13) is posed on the canonical domain Ω. The evolution equations of a curved rod in curvilinear coordinates follow from (12), (13), by the change of variables $\boldsymbol{R}^\varepsilon : \Omega \to \Omega^\varepsilon$ (see Figure 1). It is easy to show that

$$I_{23}^\varepsilon = \varepsilon^4 I_{23}, \quad I_\alpha^\varepsilon = \varepsilon^4 I_\alpha, \quad \alpha = 2, 3, \quad A^\varepsilon = \varepsilon^2 A,$$

are moments of inertia and area of εS, respectively. Also, the warping function p^ε of εS is given by $p^\varepsilon \circ \boldsymbol{R}^\varepsilon = \varepsilon^2 p$, so $K^\varepsilon = \varepsilon^4 K$. Thus the matrix

$$\mathbf{H}^\varepsilon = \begin{pmatrix} \mu K^\varepsilon & 0 & 0 \\ 0 & EI_3^\varepsilon & -EI_{23}^\varepsilon \\ 0 & -EI_{23}^\varepsilon & EI_2^\varepsilon \end{pmatrix} = \varepsilon^4 \mathbf{H}$$

is symmetric and positive definite. The forms $a_0^\varepsilon : W_0 \times W_0 \to \mathbb{R}$ and $b_0^\varepsilon : H_0 \times H_0 \to \mathbb{R}$ defined by

$$a_0^\varepsilon(\mathbf{v}, \mathbf{w}) := \int_0^\ell \mathbf{H}^\varepsilon \boldsymbol{a}(\mathbf{v}) \cdot \boldsymbol{a}(\mathbf{w})\, dz^1, \quad b_0^\varepsilon(\boldsymbol{v}, \boldsymbol{w}) := (\varrho^\varepsilon A^\varepsilon \boldsymbol{v} \mid \boldsymbol{w})_{H_0},$$

correspond to the forms a_0 and b_0. It can be shown by simple calculation that the problem (12), (13) is then equivalent to

$$\frac{d}{dt} b_0^\varepsilon(\partial_t \boldsymbol{u}, \boldsymbol{v}) + a_0^\varepsilon(\mathbf{u}, \mathbf{v}) = (\overline{\boldsymbol{F}}^\varepsilon \mid \boldsymbol{v})_{H_0}, \quad \mathbf{v} = (\boldsymbol{v}, \psi) \in \mathcal{W}_0,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \partial_t \boldsymbol{u}(0) = \boldsymbol{u}_1,$$

where

$$\overline{\boldsymbol{F}}^\varepsilon = \int_{\varepsilon S} \boldsymbol{F}^\varepsilon \, dy^2 \, dy^3.$$

References

[1] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Volume 5, Evolution Problems I*, Springer–Verlag, Berlin, 1992.

[2] R. Jamal and É. Sanchez–Palencia, *Théorie asymptotique des tiges courbes anisotropes*, C. R. Acad. Sci. Paris Sér. I Math., 322 (1996), pp. 1099–1106.

[3] M. Jurak and J. Tambača, *Derivation and justification of a curved rod model*, Math. Models Methods Appl. Sci., vol. 9, no. 7 (1999), pp. 991–1014.

[4] M. Jurak and J. Tambača, *Linear curved rod model. General curve*, in preparation.

[5] Xiao Li-ming, *Asymptotic analysis of dynamic problems for linearly elastic shells – Justification of equations for dynamic membrane shells*, Asymptotic Anal., 17 (1998), pp. 121–134.

[6] A. Raoult, *Construction d'un modèle d'évolution de plaques avec terme d'inerte de rotation*, Ann. Mat. Pura Appl. (4), 139 (1985), pp. 361–400.

[7] J. Tambača, *One-dimensional models in theory of elasticity*, M.Sc. thesis, Department of Mathematics, University of Zagreb, 1999. (In Croatian).

[8] J. Tambača, *Evolution model of curved rods*, Ph.D. thesis, Department of Mathematics, University of Zagreb, 2000.