Proceedings of the 1. Conference on
Applied Mathematics and Computation
Dubrovnik, Croatia, September 13–18, 1999
pp. 175–184

On a Class of Stochastic Evolution Equations — Pathwise Approach*

Zoran Pasarić†

Abstract. A random evolution equation in the space of random (Colombeau) generalized Banach space valued functions is solved by a construction of appropriate generalized evolution family. For generators described via the Wiener white noise the existence of expectation of evolution family is proved. Moreover, its associated distribution is shown to be a classical evolution family that solves an equation analogous to the diffusion one.

AMS subject classification: 34F05, 60H10, 46F30

Key words: random generalized functions, expectation of solution, white noise

1. Introduction

In this paper the following stochastic differential equation is considered:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{d} w_i(\omega, t)\, A_i(t)\, u \qquad \text{in } X,\ \text{for } t \in \mathbb{R}. \tag{1}$$

Here, $\mathbf{w} = \big(w_i(\omega, t)\big)_{i=1}^{d}$ is d-dimensional Wiener white noise, and $A_i(\cdot)$ are continuous families of bounded linear operators defined on a Banach (or B-)space X. The equation is called *stochastic* because of the presence of white noise. If, instead of white noise, we have ordinary stochastic process $\mathbf{g}(\omega, t)$, the equation would be called *random*. Random equations, that is the equations whose coefficients are random processes with good enough (let's say continuous) trajectories, have sense pathwise. Hence, it is natural to solve such an equation for each ω separately, using corresponding deterministic theory. Then, it only remains to prove measurability of solution. For stochastic equations, because of irregular trajectories, there is no such natural interpretation. This is where, classically, Itô or some other stochastic integral enters the story.

Our goal is to extend the pathwise approach to stochastic equation (1). However, we are not only interested in proving the *existence*, *uniqueness* and *measurability* of the solution. We want also to investigate its mean value, i.e., the *mathematical expectation*.

---

*This work is supported by the Ministry of Science and Technology of the Republic of Croatia under the program 00981305.

†Ruđer Bošković Institute, Center for Marine and Environmental Research, Bijenička 54, 10000 Zagreb, Croatia, e–mail: pasaric@rudjer.irb.hr

As for the physical motivation, let us note that if the operators $A_i(t)$ are replaced by $a_i\partial_{x_i}$, then (1) becomes the equation of conservation of mass, describing the transport of substance in a random velocity field. The function u represents concentration of substance and only the *mean concentration* is of physical interest.

As it is well known, the Wiener white noise may be realized as a probability measure in the space of distributions, but not as a probability measure in the much smaller space of ordinary functions. Moreover, almost any trajectory of white noise is distributional derivative of a continuous function. Thus, pathwise approach leads us to consider nonlinear operations (multiplication) on distributions. This is done within the framework of differential algebra of generalized functions discovered by J. F. Colombeau in the early 1980's.

After recalling basic facts about algebras of deterministic and random X-valued generalized functions, the generalized (random) evolution family of operators is defined as a $L(X)$-valued generalized function having properties similar to the properties of classical evolution families. Then it is used to solve an evolution equation in the Colombeau algebra. The equation (1) is put into the above framework by a suitable embedding, and existence of expectation of solution (as a deterministic generalized function) is proved. Finally, its associated distribution is explicitly calculated and it is shown to be an ordinary evolution family of operators.

Results presented here comprise a part of author's doctoral dissertation [6].

2. Random Colombeau generalized functions

First we recall some basic facts about Colombeau algebra of (deterministic) generalized functions with values in B-algebra X. Details may be found in [5]. In the sequel, all integrals are X–Bochner integrals and all derivatives are X–strong derivatives. When $X = \mathbb{R}$, it is omitted from the notation. For $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we let

$$\mathcal{A}_q(\mathbb{R}) = \left\{ \phi \in \mathcal{D}(\mathbb{R}) \;\middle|\; \int dt\, \phi(t) = 1, \;\; \int dt\, t^k \phi(t) = 0, \;\; 1 \le k \le q \right\}$$

and

$$\mathcal{A}_q(\mathbb{R}^d) = \left\{ \phi^{\otimes d}(t_1, \dots, t_d) = \prod_{j=1}^{d} \phi(t_j) \;\middle|\; \phi \in \mathcal{A}_q(\mathbb{R}) \right\}.$$

Next, let $\mathcal{E}(\mathbb{R}^d, X)$ be the set of all functions on $\mathcal{A}(\mathbb{R}^d)$ with values in $C^\infty(\mathbb{R}^d, X)$. For $u \in \mathcal{E}(\mathbb{R}^d)$ we write $u(\phi, \mathbf{t})$ instead of $u(\phi)(\mathbf{t})$. The set $\mathcal{E}(\mathbb{R}^d, X)$ is a differential algebra. Differentiation is defined by $(\partial_{t_i} u)(\phi, \mathbf{t}) = \partial_{t_i} u(\phi, \mathbf{t})$. The space of distributions $\mathcal{D}'(\mathbb{R}^d, X)$ is embedded in $\mathcal{E}(\mathbb{R}^d, X)$ by the inclusion

$$w \mapsto \big(\phi \mapsto w * \phi\big), \quad w \in \mathcal{D}'(\mathbb{R}^d, X), \quad \phi \in \mathcal{D}(\mathbb{R}^d, X), \tag{2}$$

as a *vector space*. Differentiation is preserved, but even $C^\infty(\mathbb{R}^d, X)$ is *not a subalgebra*. For $\varepsilon > 0$ and $\phi \in \mathcal{A}_0(\mathbb{R}^d)$ we define ϕ_ε by

$$\phi_\varepsilon(\mathbf{t}) = \varepsilon^{-d} \phi(\mathbf{t}/\varepsilon) \in \mathcal{A}_0(\mathbb{R}^d).$$

Now, an element $u \in \mathcal{E}(\mathbb{R}^d, X)$ is called *moderate*, if for every compact set $K \subset \mathbb{R}^d$ and every differential operator $D = \partial_{t_1}^{k_1} \cdots \partial_{t_d}^{k_d}$, $k_j \in \mathbb{N}_0$, there is $N \in \mathbb{N}$ such that:

- for every $\phi \in \mathcal{A}_N(\mathbb{R}^d)$ there exist $C > 0$, $\eta > 0$, such that $\sup_{\mathbf{t} \in K} \|Du(\phi, \mathbf{t})\|_X \leq C\varepsilon^{-N}$, for $0 < \varepsilon < \eta$.

The algebra of all moderate elements is denoted by $\mathcal{E}_M(\mathbb{R}^d, X)$. An ideal $\mathcal{N}(\mathbb{R}^d, X)$ of $\mathcal{E}_M(\mathbb{R}^d, X)$ is defined by: $u \in \mathcal{N}(\mathbb{R}^d, X)$, if for every compact set $K \subset \mathbb{R}^d$ and every differential operator $D = \partial_{t_1}^{k_1} \cdots \partial_{t_d}^{k_d}$, $k_j \in \mathbb{N}_0$, there is $N \in \mathbb{N}$ such that:

- for every $q \geq N$ and every $\phi \in \mathcal{A}_q(\mathbb{R}^d)$ there exist $C > 0$, $\eta > 0$, such that $\sup_{\mathbf{t} \in K} \|Du(\phi, \mathbf{t})\|_X \leq C\varepsilon^{q-N}$, for $0 < \varepsilon < \eta$.

Finally, the algebra $\mathcal{G}(\mathbb{R}^d, X)$ is given by the quotient:

$$\mathcal{G}(\mathbb{R}^d, X) = \mathcal{E}_M(\mathbb{R}^d, X)/\mathcal{N}(\mathbb{R}^d, X).$$

By applying the embedding (2) to the representatives, we have $\mathcal{D}'(\mathbb{R}^d, X) \subset \mathcal{G}(\mathbb{R}^d, X)$. For $f \in C^\infty(\mathbb{R}^d, X)$ the constant embedding $f \mapsto (\phi \mapsto f)$ defines the same element of $\mathcal{G}(\mathbb{R}^d, X)$ as the embedding (2). Hence, we have $C^\infty(\mathbb{R}^d, X) \subset \mathcal{G}(\mathbb{R}^d, X)$ as a *subalgebra*. This is the key idea of the whole construction, and because of "impossibility" result of L. Schwartz [8], the best one can achieve.

It is straightforward to define generalized functions on an open subset of $\mathbb{R}^d$, and then the notion of support. Value at a point, and integral over a compact set are also easily defined as X-valued generalized constants, the notion that may be introduced independently of the space dimension. Namely, by dropping the dependence on $\mathbf{t}$ in definitions of $\mathcal{E}_M(\mathbb{R}, X)$ and $\mathcal{N}(\mathbb{R}, X)$, the algebra $\mathcal{C}_M(X)$ and ideal $\mathcal{I}(X)$ are obtained, and the *algebra of X-valued generalized constants* is defined by $\overline{X} = \mathcal{C}_M(X)/\mathcal{I}(X)$.

The consistency between various operations (e.g., multiplication) in $\mathcal{G}(\mathbb{R}^d, X)$ with corresponding operations on ordinary functions is established through the notion of association. Namely, member $u \in \mathcal{G}(\mathbb{R}^d, X)$ is said to admit an *associated distribution* $w \in \mathcal{D}'(\mathbb{R}^d, X)$, if it has a representative R_u such that:

- for every $\psi \in \mathcal{D}'(\mathbb{R}^d, X)$ there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N(\mathbb{R}^d)$ it holds: $\int d\mathbf{t}\, R_u(\phi_\varepsilon, \mathbf{t})\psi(\mathbf{t}) \to w(\psi)$ as $\varepsilon \to 0$.

Now, the above concepts are shifted into the probabilistic setting, as was sketched in [7]. Let $\{\Omega, \mathcal{F}, P\}$ denote a probability space. A mapping $u : \Omega \to \mathcal{G}(\mathbb{R}^d, X)$ is called *random* (Colombeau) *generalized function* if there is $R_u : \Omega \to \mathcal{E}_M(\mathbb{R}^d, X)$ such that:

(*a*) $R_u(\omega)$ represents $u(\omega)$ for a.e. ω,

(*b*) for every $\phi \in \mathcal{A}_0(\mathbb{R}^d)$ and $\mathbf{t} \in \mathbb{R}^d$, $R_u(\cdot, \phi, \mathbf{t})$ is X-random element.

The algebra of random generalized functions is denoted by $\mathcal{G}_\Omega(\mathbb{R}^d, X)$. In the same way, the algebra $\overline{X}_\Omega$ of random generalized X-valued constants is defined.

Any generalized (X-valued) stochastic process, i.e., a weakly measurable mapping $w : \Omega \to \mathcal{D}'(\mathbb{R}^d, X)$ is identified with a random generalized function, the measurable representative being obtained by applying (2) for a.e. ω.

The algebra $\mathcal{G}_\Omega(\mathbb{R}^d, X)$ is convenient for (pathwise) calculations. However, it is not possible to define mathematical expectation in it, since any class contains non-integrable representatives. Therefore, we consider the vector space $\mathcal{G}\big(\mathbb{R}^d, L_p(\Omega, X)\big)$ of generalized functions with values in B-spaces of p-integrable X-random elements, $p \in \mathbb{N}$, and also the algebra $\mathcal{G}\big(\mathbb{R}^d, L_{(\infty)}(\Omega, X)\big)$, where $L_{(\infty)}(\Omega, X) = \cap_{p\in\mathbb{N}} L_p(\Omega, X)$ is a Frechet algebra. The latter case requires a slight modification of definitions (see [6]).

Proposition 1.

(a) *For any* $u \in \mathcal{G}\big(\mathbb{R}^d, L_1(\Omega, X)\big)$*, the expectation* $Eu \in \mathcal{G}(\mathbb{R}^d, X)$ *is well defined by the representative* $E(R_u)$*.*

(b) *For every* $u \in \mathcal{G}\big(\mathbb{R}^d, L_p(\Omega, X)\big)$*,* $p \in \mathbb{N}$*, it holds* $u^p \in \mathcal{G}\big(\mathbb{R}^d, L_1(\Omega, X)\big)$*.*

Proof. A straightforward calculation based on Hölder inequality, Leibnitz rule, and the inequality $\|Ef\|_X \le \|f\|_{L_1(\Omega,X)}$ that is valid for $f \in \mathrm{L}_1(\Omega, X)$. ■

We say that a function $\widetilde{R} \in \mathcal{E}\big(\mathbb{R}^d, L_p(\Omega, X)\big)$ has $\mathcal{E}(\mathbb{R}^d, X)$-*modification* $R \in \mathcal{E}(\mathbb{R}^d, X)$ if

(a) $R(\omega)$ belongs to $\mathcal{E}(\mathbb{R}^d, X)$ for a.e. ω,

(b) for every $\mathbf{t}$ and every ϕ, the mapping $R(\cdot, \phi, \mathbf{t})$ is measurable,

(c) for every $\mathbf{t}$ and every ϕ, it holds that $R(\cdot, \phi, \mathbf{t}) = \widetilde{R}(\phi, \mathbf{t})$ a.e.

A function $\tilde{u} \in \mathcal{G}\big(\mathbb{R}^d, L_p(\Omega, X)\big)$ is said to have a *version* $u \in \mathcal{G}_\Omega(\mathbb{R}^d, X)$, if there are representatives $\widetilde{R}_u$ and R_u such that R_u is a modification of $R_{\tilde{u}}$.

Lastly, we recall basic facts about the Wiener white noise. It is a generalized stochastic process $w : \Omega \to \mathcal{D}'(\mathbb{R})$ uniquely defined by its characteristic functional $C_w : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$, $C_w(\phi) := \exp\{-1/2\, \|\phi\|_0^2\}$, with $\|\cdot\|_0 := \|\cdot\|_{L_2(\mathbb{R})}$. The derivative in the sense of distributions of w is (the version of) an ordinary Brownian motion (which we denote by b), with a.s. continuous paths. (For details see [1, Theorem 2.4.1. and Examples 1., 3. on pages 36–37]). Via the embedding (2) applied for a.e. ω with $X = \mathbb{R}$, both processes w and b are members of $\mathcal{G}_\Omega(\mathbb{R})$. Furthermore, for every $\phi \in \mathcal{D}(\mathbb{R})$, random variables $b(\phi)$ and $w(\phi)$ belong to $L_{(\infty)}(\Omega)$, since they are Gaussian. Let $\gamma(\omega) := \max_{t\in K} |b(\omega, t)|$ for a compact set $K \subset \mathbb{R}$. Lemma 1 from Section 4 shows that the random variable $\gamma(\omega)$ has moments of all orders, from which easily follows that $b \in C\big(\mathbb{R}, L_{(\infty)}(\Omega)\big)$. Thus, b and $w = b'$ belong to $\mathcal{D}'\big(\mathbb{R}, L_{(\infty)}(\Omega)\big)$. By using the embedding (2) with $X = L_{(\infty)}(\Omega)$, we obtain generalized functions $\tilde{b}$ and $\widetilde{w}$ for which b and w are versions with paths in $\mathcal{G}(\mathbb{R})$.

3. Random generalized evolution family

Let X be a separable B-space and $L(X)$ the B-algebra of bounded linear operators on X. We are going to solve a random evolution equation in $\mathcal{G}(\mathbb{R}, X)$. This will be done by construction of random generalized family of evolution operators.

Definition 1. *Random generalized evolution family is a function $U \in \mathcal{G}_\Omega\big(\mathbb{R}^2, L(X)\big)$ such that for a.e. $\omega \in \Omega$ it holds:*

(*a*) $U(\omega,t,r)\,U(\omega,r,s) = U(\omega,t,s)$ *in* $\mathcal{G}\big(\mathbb{R}^3, L(X)\big)$, *for* $t \le r \le s$,

(*b*) $U(\omega,t,t) = I \in L(X) \subset \overline{L(X)}$.

Note that the usual condition of strong continuity with respect to t, s, is replaced by a mild request that U is a generalized function. With regard to the moderation property of U, we need the following definition [4].

Definition 2. *Let $A \in \mathcal{G}\big(\mathbb{R}, L(X)\big)$. Then A is called locally of logarithmic growth, if there is a representative $R_A \in \mathcal{E}_M\big(\mathbb{R}, L(X)\big)$ with property:*

- *For every compact $K \subset \mathbb{R}$ there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N(\mathbb{R})$ there exist $C > 0$, $\eta > 0$, with $\sup_{t\in K} \|R_A(\phi,t)\|_X \le N \log(C/\varepsilon)$.*

Theorem 1. *Let $A \in \mathcal{G}_\Omega\big(\mathbb{R}, L(X)\big)$ be of locally logarithmic growth. Then there exists evolution family $U \in \mathcal{G}_\Omega\big(\mathbb{R}^2, L(X)\big)$ with properties:*

$$(\partial/\partial t)\,U(\omega,t,s) = A(\omega,t)\,U(\omega,t,s), \quad (\partial/\partial s)\,U(\omega,t,s) = -U(\omega,t,s)\,A(\omega,s), \tag{3}$$

in $\mathcal{G}\big(\mathbb{R}^2, L(X)\big)$ for a.e. $\omega \in \Omega$.

Proof. Let $R_A(\omega,\phi,t) \in \mathcal{E}_M\big(\mathbb{R}, L(X)\big)$ be a measurable representative of A, and let it be a.s. locally of logarithmic growth. Then, the function R_U is defined as follows:

$$\begin{aligned} R_U(\omega,\phi\otimes\phi,t,s) &= I \\ &+ \sum_{k=1}^{\infty} \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{k-1}} dt_k\, R_A(\omega,\phi,t_1)\,R_A(\omega,\phi,t_2)\cdots R_A(\omega,\phi,t_k). \end{aligned} \tag{4}$$

By using the classical Picard's iteration method for a.e. (fixed) ω and ϕ, the following properties of R_U are obtained:

(*i*) $R_U(\omega,\phi\otimes\phi,t,t) = I, \quad R_U(\omega,\phi\otimes\phi,t,r)\,R_U(\omega,\phi\otimes\phi,r,s) = R_U(\omega,\phi\otimes\phi,t,s)$,

(*ii*) $\|R_U(\omega,\phi\otimes\phi,t,s)\|_{L(X)} \le \exp\big\{|t-s| \max_{r\in[s,t]} \|R_A(\omega,\phi,r)\|_{L(X)}\big\}$,

(*iii*) $(t,s) \mapsto R_U(\omega,\phi\otimes\phi,t,s)$ is continuously differentiable,

(*iv*) $(\partial/\partial t)\,R_U(\omega,\phi\otimes\phi,t,s) = R_A(\omega,\phi,t)\,R_U(\omega,\phi\otimes\phi,t,s)$,

(*v*) $(\partial/\partial s)\,R_U(\omega,\phi\otimes\phi,t,s) = -R_U(\omega,\phi\otimes\phi,t,s)\,R_A(\omega,\phi,s)$.

Hence, if we show the moderation property for a.e. ω and measurability for fixed ϕ, t and s, then (*i*) implies (*a*) and (*b*) from Definition 1, while (*iv*) and (*v*) imply (3).

Since $R_A(\omega,\phi,\cdot) \in C^\infty\big(\mathbb{R}, L(X)\big)$, the Leibnitz rule gives the same for $R_U(\omega,\phi\otimes\phi,\cdot,\cdot)$. Now, for a compact set $K \subset \mathbb{R}^2$, let $C_K := \max_{(t,s)\in K} |t-s|$. From (*ii*) there is a compact set $K_1 \subset \mathbb{R}$ such that

$$\sup_{(t,s)\in K} \|R_U(\phi\otimes\phi,t,s)\|_{L(X)} \le \exp\big\{C_K \sup_{r\in K_1} \|R_A(\phi,r)\|_{L(X)}\big\}.$$

If we apply Definition 2 to K_1 and A, it follows

$$\sup_{(t,s)\in K} \|R_U(\phi_\varepsilon \otimes \phi_\varepsilon, t, s)\|_{L(X)} \leq \exp\big[C_K N \ln(C/\varepsilon)\big] = C^{C_K N} \cdot \varepsilon^{-C_K N},$$

which is the desired estimate for the function itself. The estimates for derivatives are obtained by induction using (iv) and (v).

Measurability is proved in the same way as in [3, theorem 1.6]. ∎

Corollary 1. *Let* $f \in \mathcal{G}_\Omega(\mathbb{R}, X)$ *and* $u_0 \in \overline{X}_\Omega$. *The unique solution of initial value problem*

$$\frac{du}{dt} = A(\omega,t)u + f(\omega,t), \quad u(\omega,0) = u_0(\omega)$$

in $\mathcal{G}_\Omega(\mathbb{R}, X)$ *is given by*

$$u(\omega,t) = U(\omega,t,0)\, u_0(\omega) + \int_0^t dr\, U(\omega,t,r) f(\omega,r).$$

Proof. On the level of representatives, the unique solution reads

$$R_u(\omega,\phi,t) = R_U(\omega,\phi\otimes\phi,t,0)\, R_{u_0}(\omega,\phi) + \int_0^t dr\, R_U(\omega,\phi\otimes\phi,t,r)\, R_f(\omega,\phi,r),$$

which gives the existence. If $R_{u_0} \in \mathcal{I}(X)$ and $R_f \in \mathcal{N}(\mathbb{R}, X)$, then $R_u \in \mathcal{N}(\mathbb{R}, X)$ also, since by Theorem 1 R_U is moderated. This proves uniqueness. ∎

4. Expectation and its associated distribution

Now, the obtained results are applied to the family of generators

$$A(\omega,t) = \sum_{i=1}^{d} w_i(\omega,t) A_i(t), \tag{5}$$

where $\mathbf{w} = (w_i)_{i=1}^d$ is a d-dimensional Wiener white noise, and $A_i(\cdot) \in C\big(\mathbb{R}, L(X)\big)$. As (5) is meaningless in the classical sense, since it involves multiplication of distributions and continuous functions, it is interpreted in $\mathcal{G}_\Omega\big(\mathbb{R}, L(X)\big)$ as a product of corresponding embeddings. However, since the embedding (2) applied to w_i does not result in a generalized function locally of logarithmic growth, it is replaced by another generalized function associated with w_i, that enjoys this property. So, the family (5) is replaced by a generalized function (still denoted by A) with representative

$$R_A(\omega,\phi,t) := \sum_{i=1}^{d} \big(w_i(\omega) * \theta(\phi)\big)(t) \cdot (A_i * \phi)(t), \tag{6}$$

where $\theta : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is defined by $\phi \mapsto \theta(\phi) = \psi_{\nu(l(\phi))}$. Here $\psi \in \mathcal{D}(\mathbb{R})$ is a fixed function, $l(\phi) = \sup\{|t| \mid \phi(t) \neq 0\}$ (obviously $l(\phi_\varepsilon) = \varepsilon l(\phi)$), and $\nu : (0,\infty) \to \mathbb{R}$ is the function $\nu(t) := (\ln 1/t)^{-1/2}$ (see [4, Proposition 1.5.]).

The following lemma is the main ingredient in the proof of existence of expectation for the evolution family generated by (5).

Lemma 1. *Let* $\mathbf{g} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ *be a vector-valued Gaussian process with a.s. continuous paths and let*

$$\tilde{g}(\omega) := \max_{1 \le i \le d} \max_{r \in [s,t]} |g(\omega, r)|$$

for fixed $s < t$*. Then there are constants* η*,* $C > 0$ *such that:*

(*a*) $E(e^{a\tilde{g}}) \le e^{a^2 d^2/\eta} + C$*, for every* $a > 0$*,*

(*b*) $E(\tilde{g}^j e^{a\tilde{g}}) \le e^{(j+1)^2 a^2 d^2/\eta} + C$*, for every* $a > 1$*, and* $j \in \mathbb{N}$*.*

Proof. For fixed i let $\tilde{g}_i(\omega) := \max_{r \in [s,t]} |g_i(\omega, r)| < \infty$ a.s. (because of continuity of paths). According to [2] there is a constant η such that random variable $\exp(\eta \tilde{g}_i^2)$ is integrable. Putting $C := \max_{1 \le i \le d} E(e^{\eta \tilde{g}_i^2})$, we have:

$$E(e^{a\tilde{g}_i}) = \int_{|\tilde{g}_i| < a/\eta} dP(\omega)\, e^{a\tilde{g}_i(\omega)} + \int_{|\tilde{g}_i| > a/\eta} dP(\omega)\, e^{a\tilde{g}_i(\omega)} \le e^{a^2/\eta} + C.$$

Hence, for $a > 0$ we have

$$E(e^{a\tilde{g}}) \le \frac{1}{d} \sum_{i=1}^{d} E(e^{ad\tilde{g}_i}) \le e^{a^2 d^2/\eta} + C,$$

which proves (a). Similarly, for $a > 1$,

$$E(\tilde{g}^j e^{a\tilde{g}}) \le E(e^{(a+j)\tilde{g}}) \le e^{(a+j)^2 d^2/\eta} + C \le e^{(j+1)^2 a^2 d^2/\eta} + C,$$

which gives (b). ∎

Proposition 2. *Let* $U \in \mathcal{G}_\Omega\big(\mathbb{R}^2, L(X)\big)$ *be a random generalized evolution family generated by* $A \in \mathcal{G}_\Omega\big(\mathbb{R}, L(X)\big)$ *with representative* (6)*. Then, there is a generalized family* $\widetilde{U} \in \mathcal{G}\big(\mathbb{R}^2, L_{(\infty)}(\Omega, L(X))\big)$ *such that* U *is a version of* $\widetilde{U}$*.*

Proof. The representative R_U of U is given by (4) with R_A taken from (6). It is enough to show that $R_U \in \mathcal{G}\big(\mathbb{R}^2, L_{(\infty)}(\Omega, L(X))\big)$. Take $l(\phi) = 1$ (without loss of generality) and put $\varepsilon_1 = \nu\big(\varepsilon l(\phi)\big) = \nu(\varepsilon)$. In the rest of the proof, C_1, C_2, …, are constants that do not depend on ε. After a short calculation we obtain

$$\|R_A(\omega, \phi_\varepsilon, r)\|_{L(X)} \le \sum_{i=1}^{d} \max_{s_1 \in [r-\varepsilon_1, r+\varepsilon_1]} |b_i(\omega, s_1)| \cdot \max_{s_2 \in [r-\varepsilon, r+\varepsilon]} \|A_i(s_2)\|_{L(X)} \cdot \frac{C_1}{\nu(\varepsilon)}.$$

Now, for any $p \in \mathbb{N}$ and compact set $K \subset \mathbb{R}^2$, there is a compact set $K_1 \subset \mathbb{R}$ and $C_2 = \max_{1 \le i \le d} \max_{s_2 \in K_1} \|A(s_2)\|_{L(X)}$, so that, utilizing ($ii$) from the proof of Theorem 1, we have

$$\|R_U(\cdot, \phi \otimes \phi, t, s)\|^p_{L_p(\Omega, L(X))} \le \int dP(\omega) \left\{ \exp\left[\frac{C_2 |t-s|}{\nu(\varepsilon)} \sum_i \max_{r \in K_1} |b_i(\omega, r)| \right] \right\}^p.$$

According to Lemma 1 (a), the integral on the right-hand side is bounded by $C_3 \exp\{-C_4\, p\, |t-s|^2\}$. After taking supremum over K, the moderation estimate for the function R_U itself follows. Estimates of derivatives are obtained by induction, using Lemma 1 (b). ■

Theorem 2. *Let $\widetilde{U} \in \mathcal{G}\big(\mathbb{R}^2, L_{(\infty)}(\Omega, L(X))\big)$ be a random generalized evolution family from Proposition 2. Let $B(t) = \sum_{i=1}^{d} A_i^2(t)$. Then the expectation $E\widetilde{U} \in \mathcal{G}\big(\mathbb{R}^2, L(X)\big)$ admits an associated distribution $V \in C\big(\mathbb{R}^2, L(X)\big)$ given by*

$$V(t,s) = I + \sum_{k=1}^{\infty} \frac{1}{2^k} \int_s^t dt_1\, B(t_1) \int_s^{t_1} dt_2\, B(t_2) \cdots \int_s^{t_{k-1}} dt_k\, B(t_k) \tag{7}$$

for $s \le t$, and $V(t,s) = V(s,t)$ for $t < s$.

Sketch of the proof. We have to show that $ER_U(\omega, \phi_\varepsilon \otimes \phi_\varepsilon, t, s) \to V(t,s)$ in $\mathcal{D}'\big(\mathbb{R}^2, L(X)\big)$ as $\varepsilon \to 0$, where R_U is given by (4). As this is done by an elementary, but long calculation, we shall not give all the details.

It is easily seen that proof does not depend on the property of logarithmic growth. For this reason, as well as for simplicity of notation, we work with $\phi_\varepsilon \mapsto w_{i,\varepsilon} = w_i * \phi_\varepsilon$. The process $\{w_{i,\varepsilon}\}_{i=1}^d$ is Gaussian with covariance $E\big(w_{i,\varepsilon}(t_1)\, w_{j,\varepsilon}(t_2)\big) = \delta_{ij} q_\varepsilon(t_1,t_2)$, where $q_\varepsilon(t_1,t_2) = \int dr\, \phi_\varepsilon(t_1 - r)\, \phi_\varepsilon(t_2 - r)$ and δ_{ij} is the Kronecker symbol. Note that $q_\varepsilon(t_1,t_2) \to \delta(t_1 - t_2)$ in $\mathcal{D}'(\mathbb{R}^2)$. Furthermore, since $A_{i,\varepsilon} = A_i * \phi_\varepsilon$ is bounded uniformly with respect to ε, there is no essential difference if we work with A_i instead of $A_{i,\varepsilon}$.

Let us denote the expectation of the $2k$-th term in the series (4) by $I_k(t,s)$ (suppressing ϕ_ε in notation). Thus,

$$I_k(t,s) = \sum_{i_1,\dots,i_{2k}=1}^{d} \int_s^t dt_1 \cdots \int_s^{t_{2k-1}} dt_{2k}\, \mu_{i_1,\dots,i_{2k}}(t_1,\dots,t_{2k}) \cdot A_{i_1}(t_1) \cdots A_{i_{2k}}(t_{2k}),$$

where $\mu_{i_1,\dots,i_{2k}}(t_1,\dots,t_{2k}) = E\big(w_{i_1,\varepsilon}(t_1) \cdots w_{i_{2k},\varepsilon}(t_{2k})\big)$ (the moments of odd order are zero). Now, we denote $A_i^{(0)}(t) = A_i(t)$ and

$$A_i^{(j)}(t) := \frac{A_i(t)}{2^j} \int_s^t ds_{k+1-j}\, B(s_{k+1-j}) \cdot \int_s^{s_{k+1-j}} ds_{k+2-j}\, B(s_{k+2-j}) \cdots \int_s^{s_{k-1}} ds_k\, B(s_k),$$

for $j = 1, \dots, k-1$. Then let $Q_\varepsilon(t_1,t_2,s) := \int_s^{t_2} dr\, q_\varepsilon(t_1, r)$, and

$$\mathcal{J}_1^{(j)} = \sum_{i=1}^{d} \int_s^{s+2\varepsilon} dt \left(Q_\varepsilon(t,t,s) - \frac{1}{2} \right) A_i(t)\, A_i^{(j)}(t),$$

$$\mathcal{J}_2^{(j)} = \sum_{i=1}^{d} \int_s^{s+2\varepsilon} dt_1\, A_i(t_1) \int_s^{t_1} dt_2\, q_\varepsilon(t_1,t_2) \big(A_i^{(j)}(t_2) - A_i^{(j)}(t_1) \big),$$

$$\mathcal{J}_3^{(j)}(t) = \sum_{i=1}^{d} \int_{s+2\varepsilon}^{t} dt_1\, A_i(t) \int_{t_1-2\varepsilon}^{t_1} dt_2\, q_\varepsilon(t_1,t_2) \big(A_i^{(j)}(t_2) - A_i^{(j)}(t_1) \big),$$

for $j = 0, 1, \dots, k-1$. Let $I_k^{(0)}(t,s) = I_k(t,s)$ and

$$I_k^{(j)}(t,s) := \sum_{i_1,\dots,i_{2(k-j)}=1}^{d} \int_s^t dt_1 \cdots \int_s^{t_{2(k-j)-1}} dt_{2(k-j)}\, \mu_{i_1,\dots,i_{2(k-j)}}(t_1,\dots,t_{2(k-j)}) \cdot A_{i_1}(t_1)\cdots A_{i_{2(k-j)-1}}(t_{2(k-j)-1})\, A^{(j)}_{i_{2(k-j)}}(t_{2(k-j)}),$$

for $j = 1, \dots, k-1$, and let $I_k^{(k)}$ be the k-th term in (7). Then, for $k \geq 1$, we have

$$\begin{aligned}\|I_k^{(0)}(t,s) - I_k^{(k)}(t,s)\|_{L(X)} &\leq \sum_{j=0}^{k-1} \|I_k^{(j)}(t,s) - I_k^{(j+1)}(t,s)\|_{L(X)} \\ &\leq \sum_{j=0}^{k-1} \|I_{k-j-1}(t,s)\|_{L(X)} \Big[\|\mathcal{J}_1^{(j)}\|_{L(X)} + \|\mathcal{J}_2^{(j)}\|_{L(X)} \\ &\qquad + \max_{r\in(s,t)} \|\mathcal{J}_3^{(j)}(r)\|_{L(X)}\Big] + \|\mathcal{O}_k^{(j)}\|_{L(X)},\end{aligned}$$

where the second inequality is obtained by applying the recursive formula for the moments of Gaussian process

$$\begin{aligned}\mu_{i_1,\dots,i_{2k}}(t_1,\dots,t_{2k}) &= \mu_{i_1,\dots,i_{2k-2}}(t_1,\dots,t_{2k-2})\, \mu_{i_{2k-1},i_{2k}}(t_{2k-1},t_{2k}) \\ &+ \sum_{l=1}^{2k-2} \mu_{i_1,\dots,i_{l-1},i_{l+1},\dots,i_{2k-1}}(t_1,\dots,t_{l-1},t_{l+1},\dots,t_{2k-1})\, \mu_{i_l,i_{2k}}(t_l,t_{2k}),\end{aligned} \tag{8}$$

with $k-j$ instead of k. The term $\mathcal{O}_k^{(j)}$ arises from the sum (8). Now, denoting

$$\begin{aligned}C &= \int dr\, |\phi(r)|, \\ D &= \max_{1\leq i\leq d} \max_{r\in(s,t)} \|A_i(r)\|_{L(X)}, \\ \eta(\varepsilon) &= \max_{1\leq i\leq d} \max_{\substack{t_1,t_2\in(s,t)\\ |t_1-t_2|\leq 2\varepsilon}} \|A_i(t_1) - A_i(t_2)\|_{L(X)},\end{aligned}$$

the following estimates are valid:

$$\begin{aligned}\max_{r\in(s,t)} \|A_i^{(j)}(r)\|_{L(X)} &\leq D\, \frac{(t-s)^j}{2^j\, j!}\, (dD^2)^j, \\ \|\mathcal{J}_1^{(j)}\|_{L(X)} &\leq 2(dD^2)^{j+1} \left(C^2 + \frac{1}{2}\right) \frac{\varepsilon^{j+1}}{j!}, \\ \|\mathcal{J}_2^{(j)}\|_{L(X)} &\leq 4(dD^2)^{j+1} C^2\, \frac{\varepsilon^{j+1}}{j!}, \\ \max_{r\in(s,t)} \|\mathcal{J}_3^{(j)}(r)\|_{L(X)} &\leq dDC^2\, \frac{(dD^2)^j\, (t-s)^j}{2^j (j-1)!} \left(\frac{t-s}{j}\, \eta(\varepsilon) + \delta_{j0}\, 2D\varepsilon\right), \\ \|\mathcal{O}_k^{(j)}(t,s)\|_{L(X)} &\leq D^{2(n-j)}\, \frac{(t-s)^j}{2^j\, j!}\, (dD^2)^j \cdot C^{(k)}.\end{aligned}$$

Constants $C^{(k)}$ for $k \geq 4$ are bounded by $const \cdot (t-s+1)^{k-4}\, C^{2(k-4)}/(k-4)!$. Hence, by using the above estimates and the binomial theorem, the constants C_1, C_2 (that do not depend on ε!) are found such that for $k \geq 2$ we have

$$\|I_k^{(0)}(t,s) - I_k^{(k)}(t,s)\|_{L(X)} \leq C_1 \frac{C_2^{k-2}}{(k-2)!} \big(\varepsilon + \eta(\varepsilon)\big),$$

and similarly for $k = 1$. Finally, by summing over k, we arrive at

$$\|E\big(R_U(\omega, \phi_\varepsilon \otimes \phi_\varepsilon, t, s\big) - V(t,s)\big)\|_{L(X)} \leq C_3\big(\varepsilon + \eta(\varepsilon)\big),$$

for some constant C_3. For $t < s$ the proof is analogous. ∎

When the operators $A_i(t)$ commute with each other the proof is much easier, i.e., it is a straightforward generalization of the calculation of characteristic function for Gaussian process.

From (7) we see that $V(t,s)$ satisfies the equation $(\partial/\partial t)V = (1/2)B(t)V$. If A_i are (formally) replaced by $\partial/\partial x_i$, the classical diffusion equation is obtained, explaining partially the heuristically well-known relationship between white noise and classical diffusion. (See [6].)

References

[1] K. Itô, *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces*, SIAM, Philadelphia, 1984.

[2] H. J. Landau and L. A. Shepp, *On the supremum of a Gaussian process*, Sankhyā, 32 (1970), pp. 369–378.

[3] N. Limić and Z. Pasarić, *The continuity equation with random velocity field and the expectation of its solutions*, J. Math. Anal. Appl., 212 (1997), pp. 209–224.

[4] M. Oberguggenberger, *Hyperbolic systems with discontinuous coefficients: Generalized solutions and a transmission problem in acoustics*, J. Math. Anal. Appl., 142 (1989), pp. 452–467.

[5] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Research Notes Math. 259, Longman Scientific & Technical, Harlow, 1992.

[6] Z. Pasarić, *Random and Stochastic Evolution Equations — Pathwise Approach*, Ph.D. thesis, University of Zagreb, Zagreb, 1999. (in Croatian).

[7] F. Russo, *Colombeau generalized functions and stochastic analysis*, Tech. Rep. 645/7/94, Universität Bielefeld, 1994.

[8] L. Schwartz, *Sur l'impossibilité de la multiplication des distributions*, C. R. Acad. Sci. Paris Sér. I Math., 239 (1954), pp. 847–848.