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Upscaling of Two–Phase Flow

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Abstract. We consider the upscaling of two-phase flow in the case of high Peclet number. We treat periodic media with one rock-type by the method of asymptotic expansion and we propose a new upscaling method that is easy to generalize to nonperiodic media.

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1. Introduction

When building petroleum reservoir model one usually faces the problem of small scale heterogeneities that influence large scale behaviour of the reservoir. More precisely, the characteristics of the reservoir like porosity, permeability, relative permeabilities, capillary pressure, etc., are variable at very small length scale compared to the field scale. These variations are known at best in the terms of the probability model. On the other hand, these microscopic variations influence macroscopic (field scale) flow behaviour. Since the flow simulation is usually not feasible at the microscopic scale, we have to *upscale* these oscillating quantities to the field scale or to some intermediate scale determined by feasibility of numerical flow simulation. In other words, we have to calculate *effective reservoir properties*.

Small scale heterogeneities will influence large scale flow behaviour differently in different flow regimes. Quantity that characterizes the flow regime is the ratio of capillary and viscous forces at the microscopic level, or in mathematical terms, the ratio of diffusion and convection in governing equations; that quantity can be measured by Peclet number. For low Peclet number flows (strong capillary forces, slow flow) the upscaling problem is well studied (see [5, 4, 2, 7]). The situation is different for high Peclet number flows, the problem that we address in this article.

We will make a number of simplifications to describe the upscaling problem. First, we will assume that there are only two well separated scales: microscopic or *local* one, determined by the length of the small scale heterogeneities, and macroscopic or *global* scale, given by dimensions of the whole reservoir. In real situations, the media would be variable at all length scales. Secondly, we assume that the structure of the medium is *periodic*. Although unrealistic, this assumption permits us to make effective calculations and obtain the results that we can try to generalize to a more realistic

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model of *random medium* (see [9]). To make such a generalization possible, we should never neglect capillary forces (diffusion), no matter how small they may be. On the other hand, we know that the periodic media can be regarded as a special case of more general random media that is statistically homogeneous and ergodic (see [9]). Therefore, we do not consider the periodicity hypothesis as a purely academic case. Our third assumption is that the functions of the saturation — relative permeabilities and capillary pressure — do not oscillate at the microscale. That is, we consider porous medium with just one *rock type*, and the upscaling problem is then reduced to the upscaling of the absolute permeability[1]. Naturally, we are more interested in porous media with multiple rock types, since there the influence of high Peclet number on the upscaling procedure is much stronger than in the one rock-type case. But, for simplicity, we treat here only the one rock-type problem. Our method could be equally applied to the multiple rock-type upscaling problem and we will treat it in a subsequent paper.

The plan of the paper is the following one. In the second section we write two-phase flow equations in periodic setting. In the third section we introduce the Peclet number and we scale the flow equations. Then, in the fourth section we describe the well-known upscaling procedure for slow flows (low Peclet number), and in the fifth section we give a new upscaling procedure, corresponding to the fast flow (high Peclet number). Finally, in the sixth section we specialize this new method to the one-dimensional flow problem. There we can make explicit calculations of the upscaled absolute permeability.

2. Two-phase flow model

Let $\Omega \subset \mathbb{R}^3$ be a bounded set with regular boundary that represents heterogeneous porous medium. The porous medium is characterized by its porosity ϕ and absolute permeability $\mathbf{K}$. The porosity function ϕ satisfies $0 < \phi(x) < 1$ in Ω, while the permeability tensor $\mathbf{K}$ is symmetric, uniformly bounded and uniformly positive definite in Ω.

In our formal approach we will assume that the porous medium is *periodic*. That means that there exist a cell $Y = (0, L_1) \times (0, L_2) \times (0, L_3)$, a small positive parameter ε, and Y–periodic functions ϕ and $\mathbf{K}$, such that the porosity ϕ^ε and the absolute permeability $\mathbf{K}^\varepsilon$ take the form

$$\phi^\varepsilon(x) = \phi\Big(\frac{x}{\varepsilon}\Big), \quad \mathbf{K}^\varepsilon(x) = \mathbf{K}\Big(\frac{x}{\varepsilon}\Big), \quad x \in \Omega.$$

The parameter ε is proportional to the characteristic length of the heterogeneities in the reservoir. For small values of ε the rock properties of the porous medium are highly oscillating. We explicitly denote the dependence of the porosity and permeability on ε, since we are interested in the effective behaviour of the porous medium for small values of ε.

[1] Upscaling of the porosity is always straightforward, since the pore volume is an additive quantity.

We consider two-phase flow through periodic porous medium Ω. Fluids are supposed to be incompressible and immiscible, with constant viscosities μ_w, μ_n, and constant mass densities ρ_w, ρ_n (indices w and n stand for *wetting* and *non-wetting* phase, respectively). Through the capillarity law we can eliminate the pressure of the wetting phase and write down the flow equations in the terms of wetting phase saturation S^ε and non-wetting phase pressure p^ε as follows (see, e.g., [8]). For simplicity, we will neglect the effects of gravity.

$$\begin{cases} \phi^\varepsilon \dfrac{\partial S^\varepsilon}{\partial t} + \operatorname{div}\big(b(S^\varepsilon)\vec{q}_t^{\,\varepsilon}\big) = \operatorname{div}\big(\mathbf{K}^\varepsilon a(S^\varepsilon)\,\nabla S^\varepsilon\big) \\ \vec{q}_t^{\,\varepsilon} = -d(S^\varepsilon)\,\mathbf{K}^\varepsilon\big(\nabla p^\varepsilon + a_1(S^\varepsilon)\,\nabla S^\varepsilon\big) \\ \operatorname{div}(\vec{q}_t^{\,\varepsilon}) = 0. \end{cases} \tag{1}$$

Here $\vec{q}_t^{\,\varepsilon}$ denotes total velocity, and the functions of saturation are defined as follows:

$$\lambda_w(S) = \frac{kr_w(S)}{\mu_w}, \quad \lambda_n(S) = \frac{kr_n(S)}{\mu_n}, \quad d(S) = \lambda_w(S) + \lambda_n(S),$$
$$b(S) = \frac{\lambda_w(S)}{d(S)}, \quad a(S) = -\frac{\lambda_w(S)\,\lambda_n(S)}{d(S)}\frac{dp_c}{dS}(S), \quad a_1(S) = -\frac{\lambda_w(S)}{d(S)}\frac{dp_c}{dS}(S),$$

where $kr_w(S)$ and $kr_n(S)$ are relative permeabilities of wetting and non-wetting phase, respectively, and $p_c(S)$ is the capillary pressure. For a general discussion on the physical principles behind this model we refer, e.g., to Marle [10].

Equations (1) should be completed by certain boundary and initial conditions for the pressure and the saturation. It can be proved (see [8] or [1]) that the initial boundary value problem for (1), when properly posed, has at least one solution. Since for further development, the boundary conditions are not essential, we will not write them down. We refer to [8] for a discussion on boundary conditions.

In the case considered here, relative permeability functions and capillary pressure are not oscillating quantities. In other words, we consider the case of *one rock-type*. Upscaling of two-phase flow with multiple rock-types is a more difficult problem. For voluminous engineering literature on that problem we refer to Barker and Thibeau [3] and references therein. From mathematical work we mention the work of Bourgeat and Hidani [6] that rigorously justifies formal development in Saez et al. [15]. See also Quintard and Whitaker [13], where somewhat different approach is taken.

3. Flow regime

As mentioned in the introduction, the effective behaviour of the porous medium is dependent on the flow regime. In order to quantify different flow regimes we write the system (1) in dimensionless form. We introduce different dimensionless quantities denoted by a prime:

$$x = lx', \quad \vec{q}_t^{\,\varepsilon} = q^0\vec{q}_t^{\,\prime\varepsilon}, \quad \mathbf{K}^\varepsilon = k^0\mathbf{K}'_\varepsilon, \quad \mu_n = \mu^0\mu'_n,$$
$$\mu_w = \mu^0\mu'_w, \quad \phi^\varepsilon = \phi^0\phi'_\varepsilon, \quad p_c(S) = Pc^0p'_c(S), \quad p = P^0p',$$

where characteristic values are denoted by the superscript 0. The parameter l is the characteristic length of the heterogeneities in the porous medium, and equals εL, where L is the characteristic length of the whole reservoir. Let us note that taking a different value for P^0 is equivalent to a rescaling of dimensionless pressure p' and, therefore, one can usually take $P^0 = Pc^0$, but it is not necessary.

In dimensionless equations (but still with physical time), we can see that there exist two characteristic times. They are

$$\tau_d = \frac{l^2 \mu^0 \phi^0}{k^0 Pc^0}, \quad \tau_c = \frac{l\phi^0}{q^0},$$

local diffusion time τ_d and local convection time τ_c. If we replace l by L in the above definitions, we get global diffusion time T_d and global convection time T_c. Since $l/L = \varepsilon$, we have $\tau_d = \varepsilon^2 T_d$ and $\tau_c = \varepsilon T_c$.

From these two times we can build one dimensionless parameter, that is local *Peclet number*

$$Pe_l = \frac{\tau_d}{\tau_c} = \frac{q^0 l \mu^0}{k^0 Pc^0}.$$

By using global times instead of local ones, we obtain global Peclet number Pe_G. Obviously, we have

$$Pe_G = \frac{1}{\varepsilon} Pe_l.$$

Let us note here that the Peclet number can be regarded as ratio of viscous and capillary forces or convective and diffusive terms in the convection–diffusion equation for the saturation. Large Peclet number corresponds to convection dominated flow.

In order to write the system (1) in dimensionless form we have to choose dimensionless time. The choice depends on Peclet number. If global Peclet number is of order one ($Pe_G = O(1)$), then global diffusion and convection times (T_d and T_c) are of the same order in ε and we have just one global time scale. After making the time dimensionless, we get the following equations (we do not write primes for simplicity):

$$\begin{cases} \phi_\varepsilon \dfrac{\partial S^\varepsilon}{\partial t} + Pe_G \operatorname{div}\big(b(S^\varepsilon)\vec{q}_t^{\,\varepsilon}\big) = \operatorname{div}\big(\mathbf{K}_\varepsilon a(S^\varepsilon)\,\nabla S^\varepsilon\big) \\ \vec{q}_t^{\,\varepsilon} = -\dfrac{1}{Pe_G}\, d(S^\varepsilon)\,\mathbf{K}_\varepsilon\big(\nabla p^\varepsilon + a_1(S^\varepsilon)\,\nabla S^\varepsilon\big) \\ \operatorname{div}(\vec{q}_t^{\,\varepsilon}) = 0, \end{cases} \tag{2}$$

where we took $P^0 = Pc^0$. We see that the equations (2) have the same form as the equations (1) written in physical variables. So if we apply the method of asymptotic expansion (see Section 4) to the system (2) (or (1)), we introduce the assumption $Pe_G = O(1)$. Corresponding effective equations can give a good approximation to the microscale equations only under that flow regime.

Let us now consider the case $Pe_G = O(1/\varepsilon)$. In that case the viscous forces dominate over capillary forces at the global level. Then there are two different global characteristic times and thus two possible global time scales: global time scale of diffusion and of convection. Since we are mainly interested in convective effects, we write dimensionless equations in global time scale of convection (without primes for simplicity) as:

$$\begin{cases} \phi_\varepsilon \dfrac{\partial S^\varepsilon}{\partial t} + Pe_l \operatorname{div}\big(b(S^\varepsilon)\vec{q}_t^{\,\varepsilon}\big) = \varepsilon \operatorname{div}\big(\mathbf{K}_\varepsilon a(S^\varepsilon)\,\nabla S^\varepsilon\big) \\ \vec{q}_t^{\,\varepsilon} = -\dfrac{1}{Pe_l}\, d(S^\varepsilon)\,\mathbf{K}_\varepsilon\big(\nabla p^\varepsilon + \varepsilon\, a_1(S^\varepsilon)\,\nabla S^\varepsilon\big) \\ \operatorname{div}(\vec{q}_t^{\,\varepsilon}) = 0, \end{cases} \tag{3}$$

where we took $P^0 = Pc^0/\varepsilon$. This is proper scaling of the system (1) in the case $Pe_G = O(1/\varepsilon)$. The effective equations that correspond to nonhomogeneous media in this flow regime can be obtained by asymptotic expansion method starting from the system (3).

4. Slow flow upscaling method

Let us now apply the technique of asymptotic expansion to the system (2) (or, equivalently, (1)). The details of the technique can be found in [9]. We assume that the solution of the system (2) can be, at least formally, expanded in Taylor series with respect to the small parameter ε.

$$S^\varepsilon(x,t) = S^0(x,t) + \varepsilon S^1(x,y,t) + \varepsilon^2 S^2(x,y,t) + \cdots, \quad y = \frac{x}{\varepsilon}, \tag{4}$$

$$p^\varepsilon(x,t) = p^0(x,t) + \varepsilon p^1(x,y,t) + \varepsilon^2 p^2(x,y,t) + \cdots, \quad y = \frac{x}{\varepsilon}, \tag{5}$$

where the functions $S^k(x,y,t)$ and $p^k(x,y,t)$ are defined on $\Omega \times \mathbb{R}^3 \times [0,T]$ and are Y–periodic with respect to the variable y. Then we substitute the above expansions into the system (2) and identify successive problems that couples (S^k, p^k) have to satisfy. Zero-order term (S^0, p^0) does not contain oscillations and the equations that it satisfies are *effective* (or *upscaled*) equations of the two-phase flow through periodic nonhomogeneous media. By standard calculation we find:

$$\begin{cases} \langle\phi\rangle\, \dfrac{\partial S^0}{\partial t} + \operatorname{div}\big(b(S^0)\vec{q}_t^{\,0}\big) = \operatorname{div}\big(\mathbf{K}^h a(S^0)\,\nabla S^0\big) \\ \vec{q}_t^{\,0} = -d(S^0)\,\mathbf{K}^h\big(\nabla p^0 + a_1(S^0)\,\nabla S^0\big) \\ \operatorname{div}(\vec{q}_t^{\,0}) = 0, \end{cases} \tag{6}$$

where generally we set

$$\langle f\rangle = \frac{1}{|Y|}\int_Y f(y)\,dy,$$

and where *effective absolute permeability* tensor is given by:

$$\mathbf{K}^h\vec{\xi} = \langle \mathbf{K}(\nabla\chi + \vec{\xi})\rangle, \quad \vec{\xi} \in \mathbb{R}^3. \tag{7}$$

Here, for a given $\vec{\xi} \in \mathbb{R}^3$, $\chi \in H^1(Y)$ is a solution of the following *local problem*:

$$\begin{cases} \operatorname{div}\big(\mathbf{K}(\nabla\chi + \vec{\xi})\big) = 0 & \text{in } \mathbb{R}^3, \\ \chi \text{ is } Y\text{–periodic} \quad \text{and} & \langle\chi\rangle = 0. \end{cases}$$

We see that the governing laws in effective homogeneous medium are the same as in nonhomogeneous medium. Only new effective coefficients (porosity and permeability) have to be calculated. That is a consequence of strong capillary force at the local level, which actually enables us to apply monophase upscaling method (see, e.g., [9]) to multiphase flow. That makes this upscaling method simple and efficient, but the results will deteriorate for large Peclet number flows.

The solution (S^0, p^0) of the system (6) can be obtained as a weak limit of the solution $(S^\varepsilon, p^\varepsilon)$ of the system (1), as ε tends to zero. That provides a justification of the effective equations. In the periodic setting, the effective equations are justified in A. Bourgeat [5, 4]; in a more general *stochastic setting* that describes random media, justification is done in Bourgeat et al. [7].

5. Fast flow upscaling method

In this section we make an asymptotic expansion of the form (4), (5) in the system (3). Since the diffusive term is of order ε, the problem for (S^0, p^0) does not contain the diffusion term. In order to retain capillary forces in our effective flow model, it is necessary to take into consideration the term (S^1, p^1). In other words, we cannot find effective flow equations by passing to a weak limit as ε tends to zero in the system (3). We have to build certain "first-order approximation".

Our approach is based on formal asymptotic expansion. We observe that the function

$$S^*(x,t) = S^0(x,t) + \varepsilon\,\frac{\langle \phi\, S^1(x,\cdot,t)\rangle}{\langle\phi\rangle}$$

satisfies approximately[2], after neglecting certain terms of second order in ε, one equation of convection–diffusion type with the diffusion of order ε. In that way we obtain the correct diffusion term in our effective equations. Similar approach is taken in linear case in Panfilov [12] (see also [11]). In the pressure equation we work with two different pressures: $p^0(x,t)$ and $\overline{p}^1(x,t) = \langle p^1(x,\cdot,t)\rangle$, since the problem for the first-order approximation to the pressure, $p^0 + \varepsilon\overline{p}^1$, is not well posed.

[2]Here we use mean value weighted by porosity for technical reasons.

With the procedure described above, we arrive at the following effective equations:

$$\begin{cases} \langle\phi\rangle \dfrac{\partial S^*}{\partial t} + \operatorname{div}\big(b(S^*)\big[\vec{q}_t^{\,(0)} + \varepsilon \vec{q}_t^{\,(1)}\big]\big) = \varepsilon \operatorname{div}\big(\mathbf{D}^h(S^*, \nabla p^0)\, \nabla S^*\big) \\ \operatorname{div} \vec{q}_t^{\,(0)} = 0, \quad \operatorname{div} \vec{q}_t^{\,(1)} = 0 \\ \vec{q}_t^{\,(1)} = -d(S^*)\,\mathbf{K}^h \nabla \overline{p}^1 - d(S^*)\,\mathbf{N}^h \cdot \nabla^2 p^0 - \mathbf{M}^h(S^*, \nabla_x p^0)\, \nabla S^* \\ \vec{q}_t^{\,(0)} = -d(S^*)\,\mathbf{K}^h \nabla p^0. \end{cases} \tag{8}$$

Effective pressure is $p^* = p^0 + \varepsilon p^1$ and effective total velocity is $\vec{q}_t^{\,(*)} = \vec{q}_t^{\,(0)} + \varepsilon \vec{q}_t^{\,(1)}$. They are calculated in two steps by solving two elliptic equations, as it is usual when constructing asymptotic expansion for an elliptic equation. In order to form the effective equations it is necessary to calculate the following effective quantities: effective absolute permeability $\mathbf{K}^h$, "second-order effective permeability" $\mathbf{N}^h$, tensor $\mathbf{M}^h(S^*, \nabla_x p^0)$ that is "effective value" of $a_1(S)\,\mathbf{K}^\varepsilon$ in the pressure equation, and tensor $\mathbf{D}^h(S^*, \nabla p^0)$ that is "effective value" of $a(S)\,\mathbf{K}^\varepsilon$ in the saturation equation. We note that the last two effective tensors depend on S^* and ∇p^0, while oscillating quantities depend only on S^ε. That is a well-known effect of *dispersion* at the macroscopic scale where diffusion depends on (macroscopic) velocity, here through ∇p^0.

In order to calculate all these effective quantities we have to solve four different *local problems*. Two of them (for $\mathbf{K}^h$, $\mathbf{N}^h$) are classical ones from the elliptic homogenization theory (see, e.g., [9]), and the other two contain coupling between global and local scales. They have to be solved for each value of the saturation S^* and each pressure gradient ∇p^0. The local problems are:

First local problem. For $k = 1, 2, 3$:

$$\begin{cases} \operatorname{div}_y\big(\vec{F}_k^0\big) = 0 \quad \text{in } \mathbb{R}^3, \\ \vec{F}_k^0(y) = -\mathbf{K}(y)(\nabla_y \chi_k + \vec{e}_k), \\ \chi_k \text{ is } Y\text{–periodic} \quad \text{and} \quad \langle \chi_k \rangle = 0. \end{cases}$$

Then we define

$$\vec{Q}_t^{(0)}(x, y, t) = \sum_{k=1}^{3} \vec{F}_k^0(y)\, \frac{\partial p^0}{\partial x_k}(x, t),$$

and we have

$$\vec{q}_t^{\,(0)} = d(S^*) \sum_{k=1}^{3} \langle \vec{F}_k^0 \rangle\, \frac{\partial p^0}{\partial x_k} = -d(S^*)\,\mathbf{K}^h \nabla_x p^0,$$

where $\mathbf{K}^h$ is the same tensor as in (7).

Second local problem. For $k, l = 1, 2, 3$:

$$\begin{cases} \operatorname{div}_y\big(\mathbf{K}(\nabla_y \chi_{k,l} + \vec{e}_l \chi_k)\big) = \big(\vec{F}_k^0 - \langle \vec{F}_k^0 \rangle\big) \cdot \vec{e}_l, \\ \chi_{k,l} \text{ is } Y\text{–periodic} \quad \text{and} \quad \langle \chi_{k,l} \rangle = 0. \end{cases}$$

Third local problem. For $k = 1, 2, 3$ (and $c(S) = a(S)/(d(S)\, b'(S)) > 0$):

$$\begin{cases} -c(S^*)\,\mathrm{div}_y\big(\mathbf{K}(\nabla_y \psi_k + \vec{e}_k)\big) + \mathrm{div}_y\big(\psi_k \vec{Q}_t^{(0)}\big) = \left(\dfrac{\phi}{\langle\phi\rangle}\,\langle \vec{Q}_t^{(0)}\rangle - \vec{Q}_t^{(0)}\right)\cdot \vec{e}_k, \\ \psi_k \text{ is } Y\text{–periodic} \quad \text{and} \quad \langle \phi\, \psi_k\rangle = 0. \end{cases}$$

Fourth local problem. For $k = 1, 2, 3$:

$$\begin{cases} -\,\mathrm{div}_y\big(\mathbf{K}\nabla_y w_k\big) - a_1(S^*)\,\mathrm{div}_y\big(\mathbf{K}(\nabla_y \psi_k + \vec{e}_k)\big) + \dfrac{d'(S^*)}{d(S^*)}\,\mathrm{div}_y\big(\psi_k \vec{Q}_t^{(0)}\big) = 0, \\ w_k \text{ is } Y\text{–periodic} \quad \text{and} \quad \langle w_k\rangle = 0. \end{cases}$$

Effective tensors are given by:

$$\mathbf{N}^h = (\mathbf{N}^h_{k,l}), \quad \mathbf{N}^h_{l,k} = \big\langle \mathbf{K}(\nabla_y \chi_{k,l} + \vec{e}_l \chi_k)\big\rangle,$$
$$\mathbf{M}^h(S^*, \nabla_x p^0)\vec{e}_l = d(S^*)\big\langle \mathbf{K}[\nabla_y w_l + a_1(S^*)(\nabla_y \psi_l + \vec{e}_l)]\big\rangle - d'(S^*)\big\langle \psi_l \vec{Q}_t^{(0)}\big\rangle,$$
$$\mathbf{D}^h(S^*, \nabla_x p^0)\vec{e}_l = a(S^*)\big\langle \mathbf{K}(\nabla_y \psi_l + \vec{e}_l)\big\rangle - b'(S^*)d(S^*)\big\langle \psi_l \vec{Q}_t^{(0)}\big\rangle.$$

The tensors $\mathbf{M}^h$ and $\mathbf{D}^h$ differ significantly, since a part of $\mathbf{D}^h$ comes from convective term in the saturation equation, while there does not exist such an effect in the pressure equation. Furthermore, one can show that $\mathbf{D}^h$ is of the form $a(S^*)\,\mathbf{D}^h_1(S^*, \nabla p^0)$, where $\mathbf{D}^h_1$ is a positive definite tensor, although nonsymmetric. That will ensure that the initial boundary value problem for (8) is well posed. We should note also that all local problems are obviously well posed.

To conclude, we have found very complicated system of effective equations that is not easy either to analyse theoretically or to verify numerically. The main characteristic of the system is a coupling between local and global variables through local problems that depend on global variables. On the other hand, this coupling cannot be avoided in the case of the fast flow. In engineering literature there is a number of upscaling methods (for multiple rock-type porous media), the so-called *dynamic methods*, that realize this coupling through certain “restricted” microscale simulation — in other words, numerically (see [3] for details). The advantage of our approach is that the coupling is given in an explicit way. We presume that the method could be extended to the stochastic setting by appropriate change of boundary conditions (see [7] and [2]).

6. One-dimensional case

In one space dimension we can simplify the obtained effective equations, since the pressure and the saturation equation are completely separated by constantness of the total velocity. We will consider only the saturation equation.

Let us suppose that $\Omega = (0,1)$, and that constant positive total velocity q^0 is prescribed at $x = 0$. Then the total velocity is equal to q^0 in the whole domain and, consequently, there are no oscillations in the convective term of the saturation equation. In other words, we have

$$q_t^{(0)} = q^0 \quad \text{and} \quad q_t^{(1)} = 0.$$

By direct calculation we can see that the effective saturation equation has the following form:

$$\langle \phi \rangle \frac{\partial S^*}{\partial t} + q^0 \frac{d}{dx} b(S^*) = \varepsilon \frac{d}{dx}\left(a(S^*) k^{hom}(S^*) \frac{dS^*}{dx} \right),$$

where the only effective quantity is effective absolute permeability k^{hom}, given by

$$k^{hom}(S^*) = k^*(\beta), \quad \text{where} \quad \beta = \frac{b'(S^*)}{a(S^*)} q^0,$$

and

$$k^*(\beta) = \frac{\beta}{1 - e^{-\beta/k^h}} \frac{\langle \phi e^{\beta r} \rangle \langle \phi e^{-\beta r} \rangle}{\langle \phi \rangle^2} - \frac{\beta}{\langle \phi \rangle^2} \int_0^1 \phi(y) e^{\beta r(y)} \int_0^y \phi(t) e^{-\beta r(t)}\, dt\, dy,$$

$$r(y) = \int_0^y \frac{dt}{k(t)} \quad \text{and} \quad k^h = \left(\int_0^1 \frac{dt}{k(t)} \right)^{-1} \quad \text{(harmonic mean).}$$

One can prove that k^* is bounded, continuous and increasing function with the limits

$$\lim_{\beta \to 0} k^*(\beta) = k^h, \quad \lim_{\beta \to +\infty} k^*(\beta) = \frac{1}{\langle \phi \rangle^2} \int_0^1 \phi^2(y) k(y)\, dy.$$

We see that effective permeability is a function of the saturation and it varies between harmonic mean (that is the effective permeability in the case of slow flow) and weighted arithmetic mean. Preliminary numerical calculations indicate that the fast flow upscaling method adds bigger amount of diffusion in the effective equation compared to the slow flow method, producing better approximation to the solution of the heterogeneous problem. We remark finally that these formulas are easy to generalize to nonperiodic case.

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