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A Numerical Approach to the Estimation of Search Effort in a Search for a Moving Object*

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Abstract. A model of diffusion in a bounded domain, randomly killed at a point x at the rate $c(x, \kappa, z)$, is considered as a model of search for a moving target. The search effort κ is a parameter that should be estimated from data and the searcher's path z is a control variable. It is assumed that samples of data are obtained by measuring the minimum detection time and the first exit time up to some prescribed maximum observation time T, for various values of z. The search effort is estimated by the minimum χ^2-method from obtained data and an appropriate statistical model is calculated numerically. Sensitivity of the minimum χ^2-estimation method with respect to the applied numerical methods is illustrated numerically by a realistic example.

AMS subject classification: 60H10, 62F10, 65C05, 65C20

Key words: diffusion processes, stochastic differential equation, minimum χ^2-estimation method

1. Introduction

Search for a moving object is a problem which can be formulated in terms of mathematical models and analyzed efficiently by mathematical methods. Mangel's approach [7] is general enough and therefore chosen for our purposes. This approach is based on splitting the problem into three independent parts and modelling each part separately: target (moving object) is modelled by a diffusion process on a two-dimensional domain, searcher is modelled by a deterministic path, moving velocity along this path and the maximum observation time, while detection model describes random detection process. The diffusion is defined by diffusion tensor and drift velocity (sea current, etc.), the searcher's path and moving velocity are generalized variables, and the detection depends on distance between target and searcher, and certain parameters characterizing detector and its handling. Search effort is the most important

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parameter and the object of our study. Thus the detection model unifies the three independent models into a single one, a model of search for a moving target. The resulting search model is defined by a diffusion killed randomly depending on searcher's variables and detection. This stochastic search model has evidently a deterministic version in which the target is defined by the initial value problem for diffusion equation, and detection by the detection probability depending on searcher's variables and solution of diffusion equation.

The purpose of our analysis is to evaluate numerically a statistical model for estimation of detection parameters from data. A statistical model is formulated for a finite number of detection parameters. Based on numerical stability, the minimum χ^2-estimation method is chosen and analyzed numerically. Efficiency of numerical evaluation of the chosen method is demonstrated by a nontrivial example of a search for a moving target.

Two standard estimation methods for the proposed statistical model are considered in Section 2. Moreover, basic results regarding their asymptotic consistency are also discussed. A review of numerical methods for data simulation and model evaluation is given in Section 3. A nontrivial example of search in a bounded two-dimensional domain is defined at the beginning of Section 4. Then data on detection are generated by Monte Carlo methods and the search effort is estimated numerically by applying the minimum χ^2-estimation. The efficiency of the method is demonstrated by comparison of estimated search effort and its true value.

2. Statistical model

Let D be an open set in the Euclidean space $\mathbb{R}^d$ ($d \geq 2$) and let $X = (X_t, t \geq 0)$ be a diffusion in $\mathbb{R}^d$ represented by stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \quad t \geq 0, \tag{1}$$

of Itô type (see [9]). The set D represents the search region and the process X represents the target. The initial position X_0 of X is a random variable of known distribution with a support in D. The diffusion X is randomly killed with the infinitesimal rate function $(t, x) \mapsto c(t, x, \kappa, z)$ (see [9]), which depends on an unknown parameter κ — search effort, and a control variable z, representing a deterministic search path. The rate function c defines detection by the random time ζ with the conditional distribution

$$\mathbb{P}(\zeta > t|X) = \exp\left(-\int_0^t c(s, X_s, \kappa, z)\, ds\right), \quad t \geq 0.$$

It has to be pointed out that the conditional distribution of this expression tends generally to a number less than 1, for large values of t.

The target is lost after leaving the set D. Therefore, the first exit time of X from D, $\tau_D = \inf\{t \geq 0 \mid X_t \in \mathbb{R}^d \setminus D\}$, is one of variables describing the process. Let η be the minimum of ζ and τ_D. Then η is the detection time. Obviously, the distribution of η depends on the parameter κ and variable z.

We assume that value of κ belongs to an open set $\mathcal{K}$ of an Euclidean space, called the parametric space. The problem is to estimate the unknown parameter κ, based on a sequence of independent observations of random variable η for several (generally different) values of the control variable z, up to some prescribed deterministic time $T > 0$, called the maximum observation time. A realization of η is recorded only if it has a value less than T. Otherwise, the value T is recorded. Let

$$F^{(z)}(t \,|\, \kappa) = \mathbb{P}_\kappa^{(z)}(\eta \le t), \quad t \in \mathbb{R},$$

be the distribution function (CDF) of η. Then CDF of the observation $\min\{\eta, T\}$ is

$$F_T^{(z)}(t \,|\, \kappa) = \mathbb{1}_{\{t<T\}}\, F^{(z)}(t \,|\, \kappa) + \mathbb{1}_{\{t \ge T\}}, \quad t \in \mathbb{R}.$$

Let the observations be taken for l different values $z_1, z_2, \ldots, z_l$, of the control variable z. Then the statistical model is defined by the following family of probability distribution functions:

$$\big\{F_T^{(z_i)}(\,\cdot\,|\,\kappa) \,\big|\, i = 1, 2, \ldots, l;\, \kappa \in \mathcal{K}\big\}. \tag{2}$$

If CDF $F^{(z)}(\,\cdot\,|\,\kappa)$ has a density function (PDF) $f^{(z)}(\,\cdot\,|\,\kappa)$ for all $\kappa \in \mathcal{K}$ and $z = z_i$, $i = 1, 2, \ldots, l$, then the maximum likelihood estimation method provides a consistent and asymptotically efficient estimator of the parameter κ under some general conditions (see [5]). Except in a few cases, it is not possible to obtain PDF in a closed form. So one has to apply numerical methods, including numerical differentiation which is numerically unstable. To avoid unstable numerical procedures we rather choose the minimum χ^2-estimation method as described in the following.

For any of l values z_i, $i = 1, \ldots, l$, of the control variable, we consider a random sample of length n, $n \in \mathbb{N}$, from CDF $F_T^{(z_i)}(\,\cdot\,|\,\kappa)$. For any of these samples, let $X_n^{(z_i)}$ be the sample mean of data strictly less than T, and let $Y_n^{(z_i)}$ be the relative frequency of data equal to T. For any $z = z_i$, let $\mathbb{E}_\kappa^{(z)}[\eta \,|\, \eta < T]$ be the conditional mean of η, given $\{\eta < T\}$, and let $\mathbb{P}_\kappa^{(z)}(\eta \ge T)$ be the probability that η is at least equal to T. The χ^2-function is defined by

$$Q_n(\kappa) := n \sum_{i=1}^{l} \left(\mathbb{P}_\kappa^{(z_i)}(\eta < T)\, \frac{\big(X_n^{(z_i)} - \mathbb{E}_\kappa^{(z_i)}[\eta \,|\, \eta < T]\big)^2}{\mathbb{V}\mathrm{ar}_\kappa^{(z_i)}[\eta \,|\, \eta < T]} + \frac{\big(Y_n^{(z_i)} - \mathbb{P}_\kappa^{(z_i)}(\eta \ge T)\big)^2}{\mathbb{V}\mathrm{ar}_\kappa^{(z_i)}[\mathbb{1}_{\{\eta \ge T\}}]} \right) \tag{3}$$

for any $\kappa \in \mathcal{K}$. The minimum χ^2-estimator of κ is any value $\hat{\kappa}_n \in \mathcal{K}$ which minimizes the function $\kappa \mapsto Q_n(\kappa)$ over $\mathcal{K}$. If $\mathcal{K}$ is a relatively compact set and some identifiable and regularity assumptions are satisfied, then $\hat{\kappa}_n$ exists for sufficiently large n, and it is consistent and asymptotically normally distributed estimator of κ (see [3, 2]).

3. Numerical methods

The estimation procedure of parameter κ in the statistical model (2) consists of three basic steps:

(i) calculation of the model functionals which are used in the definition of χ^2-function (3),

(ii) minimization of (3), and

(iii) simulation of random samples from (2) to obtain an approximate distribution of the estimator.

A diffusion process considered on a bounded domain D up to its first exit time τ_D from D, has a family of one-dimensional PDFs $p(t,\cdot)$, $t \geq 0$. For calculations of the functionals $\mathbb{E}_\kappa^{(z_i)}[\eta \,|\, \eta < T]$, $\mathbb{V}\mathrm{ar}_\kappa^{(z_i)}[\eta \,|\, \eta < T]$ and $\mathbb{P}_\kappa^{(z_i)}(\eta \geq T)$, $\kappa \in \mathcal{K}$, $i = 1, 2, \ldots, l$, from (3), it is sufficient to determine this family.

The function $(t, x) \mapsto p(t, x)$ is a solution of an initial value problem for the second-order parabolic system. In our case the elliptic differential operator of the parabolic system has the form $A = \frac{1}{2}\sum_{i,j} \partial_i a_{ij} \partial_j + \sum_i \partial_i (a_i - b_i)$, where $a_{ij} = \sum_k \sigma_{ik}\sigma_{kj}$, and the functions σ_{ij} and b_i are defined in (1), while $a_i = \frac{1}{2}\sum_k \partial_k a_{ik}$. The corresponding initial value problem for PDFs is defined by the following parabolic system ($p(t) \equiv p(t,\cdot)$):

$$\frac{\partial}{\partial t} p(t) - A(t)p(t) = 0, \quad p(t)\big|_{\partial D} = 0, \quad t \geq 0, \quad p(0) \text{ is given.} \tag{4}$$

In the case when diffusion is killed with the infinitesimal rate function c, the differential operator has an additional term, so that its form is given by

$$A(t) = \frac{1}{2}\sum_{i,j} \partial_i a_{ij} \partial_j + \sum_i \partial_i (a_i - b_i) + c(t). \tag{5}$$

The initial value problem (4) with the elliptic operator (5) has solutions $t \mapsto p(t)$, representing one-dimensional PDFs of the diffusion (1), which is killed with the infinitesimal rate c.

A numerical procedure to be used to approximate numerically the solution $p(t)$ of (4) consists of two main steps.

3.1. Space discretization

The orthogonal coordinate system in $\mathbb{R}^2$ is determined by unit vectors $\mathbf{e}_i$. For each $n \in \mathbb{N}$, points $\mathbf{x} = h \sum_{l=1}^{2} k_l \mathbf{e}_l$, $h = 2^{-n}$, $k_l \in \mathbb{Z}$, define a numerical grid G_n on $\mathbb{R}^2$. Elements of G_n are called grid knots. The operator (5) is approximated on $G_n(D) = G_n \cap D$ by finite-dimensional matrices $A_n(t)$. In this way, the initial value problem (4) is replaced by a sequence of ODE:

$$\frac{d}{dt} p_n(t) - A_n(t)p_n(t) = 0, \quad p_n(0) \text{ is given,} \tag{6}$$

where $p_n(t)$ is a column with $\mathrm{card}(G_n(D))$ elements, representing a numerical approximation of the PDF $p(t)$. The matrix valued functions $A_n(t)$ must have compartmental structure [6]. This ensures that the columns $p_n(t)$ are nonnegative, and $\|p_n(t)\|_1 = 1$ when $c = 0$.

3.2. Time discretization

To solve ODE (6), one has to define points $t_k = \tau k$, $k = 0, 1, \ldots$, and calculate numerically solutions $p_n(t)$ at the discrete set of times t_k. Let p_n^k be a numerical approximation of $p_n(t_k)$ obtained by a method for ODE. The chosen numerical method must fulfill certain conditions:

(i) the columns p_n^k must have nonnegative elements,

(ii) $\|p_n^{k+1}\|_1 = \|p_n^k\|_1$ when $c = 0$,

(iii) there exist $L \in \mathbb{N}$ and a positive function $t \mapsto \rho(t)$, such that for each $T > 0$, $K \in \mathbb{N}$ and $\tau = T/K$:

$$\sup_{k \le K} \|p_n^k - p_n(t_k)\|_1 \le \rho(T)\tau^L.$$

In this case one says that the method converges with the order equal to L.

Among linear one-step methods only the implicit Euler's method has all the needed properties (i)–(iii), unconditionally with respect to τ. We rather use a class of newly developed methods [6]. Methods are explicit, of any order of convergence, and unconditionally stable. When applied to the example described in the next section, our third-order method happens to be 700 times faster than the implicit Euler's method.

Simulation of random samples from (2) consists of obtaining realizations of the random variable

$$\eta_T = \zeta \wedge \tau_D \wedge T$$

by using Monte Carlo methods. Realizations of both random variables, ζ and τ_D, depend on sample paths of the diffusion X until the observation time T. Moreover, for ζ, we use the fact that the random variable

$$\xi = \int_0^\zeta c(s, X_s, \kappa, z)\, ds$$

is exponentially distributed with expectation 1 and independent of X (see, e.g., [1]).

3.3. Numerical approximation of diffusion

Let the interval $[0, T]$ be partitioned by points $t_k = k\Delta$, $k = 0, 1, \ldots, K$, where $K \in \mathbb{N}$ and $\Delta = T/K$. The stochastic differential equation (1) on $[0, T]$ is approximated with a system of finite differences, obtained by applying the Euler's scheme to (1) with the time increment Δ. The corresponding sequence of solutions is denoted by Y_k^Δ, $k = 0, 1, \ldots, K$. Random variables Y_k^Δ define an approximation of the diffusion process $X(\cdot)$ at times t_k.

We say that a discrete time process $(Y_k^\Delta,\ k = 0, 1, \ldots, K)$ is a pathwise approximation of $(X_t,\ t \in [0, T])$ at time T, if (see [4])

$$\lim_{\Delta \to 0} \mathbb{E}[\,|X_T - Y_K^\Delta|\,] = 0.$$

We say that a discrete time approximation $(Y_k^\Delta,\ k = 0, 1, \ldots, K)$ converges strongly to X at time T, with the order $\gamma > 0$ at T, if there exists a positive constant C, independent of Δ, such that

$$\mathbb{E}[\,|X_T - Y_K^\Delta|\,] \le C\Delta^\gamma,$$

for each $\Delta > 0$. In our case the Euler's scheme is used. Therefore, the order of convergence is equal to 0.5 (see [4, 10]). Numerical approximations of $(X_t, t \in [0, T])$ are obtained by simulating the discrete time process $(Y_k^\Delta,\ k = 0, 1, \ldots, K)$.

3.4. Simulation of sample from (2)

Having a discrete time approximation $(Y_k^\Delta,\ k = 0, 1, \ldots, K)$ of X, we simulate an exponentially distributed value ξ with expectation 1 independently from $(Y_k^\Delta,\ k = 0, 1, \ldots, K)$. The approximate value of ζ is the first time point t_l of the interval $[0, T]$, which satisfies

$$\Delta \sum_{i=0}^{l} c(t_i, Y_i^\Delta, \kappa, z) \ge \xi,$$

and the approximate value of τ_D is the first time point t_l of the interval $[0, T]$, which satisfies $Y_l^\Delta \notin D$.

4. An example

For the sake of model simplicity, we assume that the target moves as a two-dimensional scaled Brownian motion

$$X_t = X_0 + \sigma W_t, \quad t \ge 0, \quad (\sigma^2 = 0.5),$$

with initial position $X_0 = (0.5, 0.5)$, the center of the square $D = \langle 0, 1\rangle \times \langle 0, 1\rangle$ where the search is taking place. The searcher tries to detect the target by using one of ten different paths z_i, $i = 1, 2, \ldots, 10$, over D, defined by

$$z_i(t) = \begin{cases} x_{i0} + t v_i^{(1)}, & 0 \le t \le \dfrac{13}{24}T, \\ z_i\left(\dfrac{13}{24}T\right) + \left(t - \dfrac{13}{24}T\right) v_i^{(2)}, & \dfrac{13}{24}T < t \le T, \end{cases} \qquad T \ge 0.$$

The initial positions x_{i0} and the velocities $v_i^{(1)}$, $v_i^{(2)}$, for $i = 1, 2, \ldots, 10$, are presented in Figure 1. The rate function of detection, c, is

$$c(t, x, \kappa, z) = 10\kappa\, \mathbb{1}_{\{|x - z(t)| < 1/\sqrt{40}\}},$$

where the search effort κ is one-dimensional parameter. The maximum observation (search) time is $T = 1.2$.

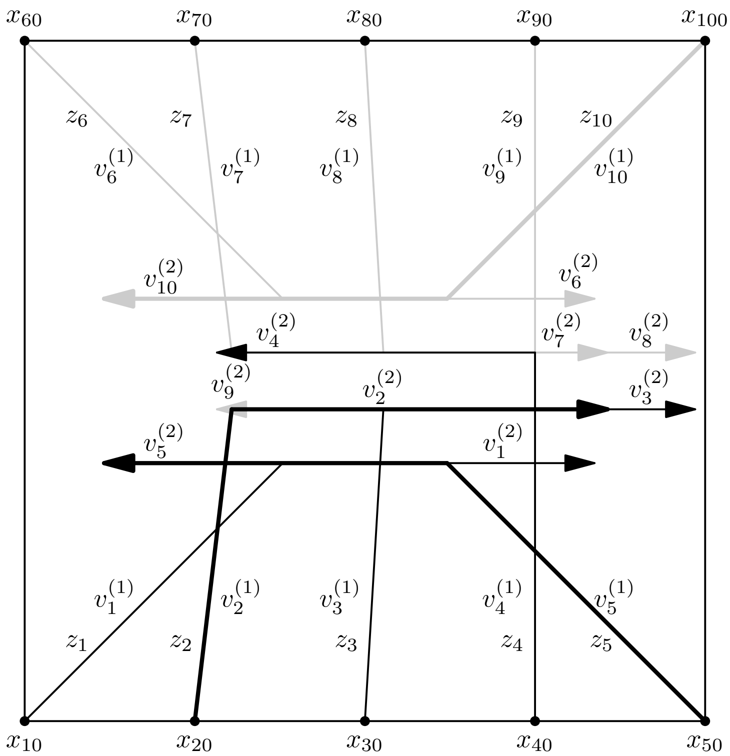

Figure 1. The initial positions and velocity directions of z_i, $i = 1, 2, \ldots, 10$.

To test sensitivity of the minimum χ^2-estimation method of Section 2 with respect to the applied numerical methods of Section 3, we simulated $M = 1001$ random samples of length $10n = 10000$ from the statistical model (2) ($n = 1000$ data per each path z_i, $i = 1, 2, \ldots, 10$). We assumed that the true value of the search effort κ is 1. For each sample we calculated the minimum χ^2-estimate $\hat{\kappa}_n$ over presumed parameter space $\mathcal{K} = \langle 0.75, 1.25 \rangle$. In this way, we obtained the sample of M realizations of $\hat{\kappa}_n$. From the so simulated data, we estimated the mean $\mathbb{E}[\hat{\kappa}_n]$, and the standard deviation $\mathrm{std}[\hat{\kappa}_n]$ of the estimator $\hat{\kappa}_n$.

We repeated the whole procedure three times, each time using a different combination of discretization parameters in numerical calculations of the model functionals in (3). We chose combinations of the space discretization parameter h and the time discretization parameter τ (see Table 1 below), yielding relative errors not larger than 3%. In all cases we simulated random samples by using the Euler's scheme for pathwise approximations of X with time discretization parameter $\Delta = 1/200$ (see Section 3). The summary data are shown in Table 1.

(h, τ)	$(1/40, 1/400)$	$(1/60, 1/1000)$	$(1/80, 1/2400)$
$\widehat{\mathbb{E}}[\hat{\kappa}_n]$	1.04025	1.02078	1.004845
$\widehat{\mathrm{std}}[\hat{\kappa}_n]$	2.7539e−02	2.6978e−02	2.6515e−02

Table 1.

We notice from the table that the component of bias due to the numerical methods, decreases when both discretization parameters h and τ decrease. On the other hand, it seems that the standard deviation is not significantly sensitive to the same parameters.

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