Some estimates of the number of Diophantine quadruples

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Abstract

A Diophantine m-tuple with the property $D(n)$, where n is an integer, is defined as a set of m positive integers such that the product of its any two distinct elements increased by n is a perfect square. In the present paper we show that if $|n|$ is sufficiently large and $n \equiv 1 \pmod 8$, or $n \equiv 4 \pmod{32}$, or $n \equiv 0 \pmod{16}$, then there exist at least six, and if $n \equiv 8 \pmod{16}$, or $n \equiv 13,\ 21 \pmod{24}$, or $n \equiv 3,\ 7 \pmod{12}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.

1 Introduction

The Greek mathematician Diophantus of Alexandria noted that the numbers x, $x+2$, $4x+4$ and $9x+6$, where $x = \frac{1}{16}$, have the following property: the product of any two of them increased by 1 is a square of a rational number (see [3, pp. 103–104, 232]). The first set of four positive integers with the above property was found by Fermat, and it was $\{1, 3, 8, 120\}$. In 1969, Baker and Davenport [1] showed that if positive integers 1, 3, 8 and d have this property then d must be 120.

In [2] and [4], the more general problem was considered. Let n be an integer. A set of positive integers $\{a_1, a_2, \ldots, a_m\}$ is said to have *the property of Diophantus of order* n, symbolically $D(n)$, if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$. Such a set is called *a Diophantine m-tuple*. It was proved in [2] that if n is an integer of the form $4k+2$, $k \in \mathbf{Z}$, then there does not exist Diophantine quadruple with the property $D(n)$ (see also [8, p. 802] and [9, Theorem 10]). In [4, Theorems 5 and 6], it was proved that if an integer n is not of the form $4k+2$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one, and if

[0] *Mathematics subject classification* (1991): Primary 11D09. Secondary 11A41, 11C08, 11Y50.

Key words and phrases: Diophantine quadruple, property of Diophantus, Schinzel-Sierpiński conjecture

$n \notin S \cup T$, where $T = \{-15, -12, -7, 7, 13, 15, 21, 24, 28, 32, 48, 60, 84\}$, then there exist at least two distinct Diophantine quadruples with the property $D(n)$ (see also [5, p. 315]).

In the present paper we give some improvements of these results. Namely, we show that if $|n|$ is sufficiently large and $n \equiv 1 \pmod 8$, or $n \equiv 4 \pmod{32}$, or $n \equiv 0 \pmod{16}$, then there exist at least six, and if $n \equiv 8 \pmod{16}$, or $n \equiv 13, 21 \pmod{24}$, or $n \equiv 3, 7 \pmod{12}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.

2 Some polynomial formulas for Diophantine quadruples

The proof of [4, Theorems 5 and 6] is based on the fact that the sets

$$\{x, x(3y+1)^2 + 2y, x(3y+2)^2 + 2y + 2, 9x(2y+1)^2 + 8y + 4\}, \tag{1}$$

$$\{x, xy^2 - 2y - 2, x(y+1)^2 - 2y, x(2y+1)^2 - 8y - 4\} \tag{2}$$

have the property $D(2x(2y+1)+1)$. The formulas of this type were systematically derived in [6]. It was shown in [6, Theorems 1 and 2] that the set

$$\{x, xy^2 + 2y - 2, x(y+1)^2 + 2y + 4, x(2y+1)^2 + 8y + 4\} \tag{3}$$

has the property $D(2x(2y+1)+9)$, the set

$$\begin{gathered}\{x, xy^2 + 2(y^2+y+1), x(y-1)^2 + 2y(y-1),\\ x(y+1)^2 + 2(y+1)(y+2)\}\end{gathered} \tag{4}$$

has the property $D(2x(y^2-1) + (2y+1)^2)$, and the set

$$\begin{gathered}\{x, x(3y+1)^2 + 2(3y^2+3y+1), x(3y+2)^2 + 2(y+1)(3y+2),\\ 9xy^2 + 2y(3y+1)\}\end{gathered} \tag{5}$$

has the property $D(2xy(3y+2) + (2y+1)^2)$.

3 Some estimates of the number of Diophantine quadruples

THEOREM **1** *If n is an integer such that $n \equiv 1 \pmod 8$ and $n \notin V_1 = \{-15, -7, 17, 33\}$, then there exist at least six distinct Diophantine quadruples with the property $D(n)$.*

PROOF. The proof is based on the facts that the sets

$$\{4, 9k^2 - 5k, 9k^2 + 7k + 2, 36k^2 + 4k\}, \tag{6}$$

$$\{4, k^2 - 3k, k^2 + k + 2, 4k^2 - 4k\}, \tag{7}$$

$$\{8, \frac{1}{2}k(k + 3) + 3, \frac{1}{2}k(k - 5) + 1, 2k^2 - 2k\}, \tag{8}$$

$$\{8, \frac{1}{2}k(9k - 11) + 1, \frac{1}{2}k(9k + 13) + 3, 18k^2 + 2k\} \tag{9}$$

have the property $D(8k + 1)$, the sets

$$\{m - 3, 4m, 9m - 1, 16m - 8\}, \tag{10}$$

$$\{4m, 25m + 1, 49m + 3, 144m + 8\} \tag{11}$$

have the property $D(16m + 1)$, and the sets

$$\{m, 16m + 8, 25m + 14, 36m + 20\}, \tag{12}$$

$$\{m - 1, 4m, 9m + 5, 16m + 8\} \tag{13}$$

have the property $D(16m + 9)$.

The sets (6) and (7) are exactly the sets [4, (8) and (9)]. The set (8) is obtained from (3), for $x = 8$ and $y = \frac{k-3}{4}$. From (1), for $x = 8$ and $y = \frac{k-2}{4}$ we get the set (9), and for $x = 4m$ and $y = \frac{1}{2}$ we get the set (11). From (4), for $x = m - 3$ and $y = 3$ we get the set (10), and for $x = m - 1$ and $y = -3$ we get the set (13). Finally, the set (12) is obtained from (5), for $x = m$ and $y = -2$.

We are left with the task of determining the values of k and m for which the above sets have at least two equal elements or elements with different signs, and the values of k and m for which the corresponding sets coincide. An easy computation shows that the above cases appear in the sets (6)$-$(9) iff $k \in \{-5, -2, -1, 0, 1, 2, 3, 4, 7\}$, in the sets (10) and (11) iff $m \in \{-1, 0, 1, 2, 3\}$, and in the sets (12) and (13) iff $m \in \{-1, 0, 1\}$.

Comparing the sets (6)$-$(9) with the sets (10) and (11) we conclude that for all integers n of the form $16m + 1$, where $m \notin \{-2, -1, 0, 1, 2, 3\}$, there exist at least six distinct Diophantine quadruples with the property $D(n)$.

The same conclusion can be drawn for all integers n of the form $16m+9$, where $m \notin \{-3,-1,0,1,3\}$.

Thus we have proved that for every integer n such that $n \equiv 1 \pmod 8$ and $n \notin \{-39,-31,-15,-7,1,9,17,25,33,49,57\}$ there exist at least six distinct Diophantine quadruples with the property $D(n)$. But for the numbers 1, 9, 25 and 49 the assertion of Theorem is valid since they are perfect squares (see [4]). From (6)$-$(13) for $n=-39$ and $n=57$ we get five, and for $n=-31$ we get four distinct Diophantine quadruples with the property $D(n)$. A trivial verification shows that the sets $\{1,40,47,56\}$ and $\{1,40,287,320\}$ have the property $D(-31)$, and the sets $\{1,43,48,3520\}$ and $\{1,7,24,232\}$ have the properties $D(-39)$ and $D(57)$ respectively, which completes the proof. ∎

Corollary **1** *If n is an integer such that $n \equiv 4 \pmod{32}$ and $n \notin V_2=\{-28,68\}$, then there exist at least six distinct Diophantine quadruples with the property $D(n)$.*

Proof. Since multiplying all elements of the set with the property $D(8k+1)$ by 2 we get the set with the property $D(32k+4)$, by Theorem 1, it is sufficient to prove the Corollary for $n=-60$ and $n=132$. But the sets $\{1,60,736,1216\}$, $\{1,64,96,316\}$, $\{1,124,256,736\}$, $\{4,15,19,64\}$, $\{4,19,31,96\}$ and $\{8,48,92,272\}$ have the property $D(-60)$, and the sets $\{1,12,37,64\}$, $\{1,12,64,1312\}$, $\{2,6,32,272\}$, $\{3,64,103,148\}$, $\{8,248,348,1184\}$ and $\{16,102,202,596\}$ have the property $D(132)$. ∎

Remark **1** For the elements of the sets V_1 and V_2, the following holds: the set $\{4,24,46,136\}$ has the property $D(-15)$, the set $\{1,8,11,16\}$ has the property $D(-7)$, the sets $\{1,8,19,208\}$ and $\{4,26,52,152\}$ have the property $D(17)$, the sets $\{1,3,16,136\}$, $\{4,124,174,592\}$ and $\{8,51,101,296\}$ have the property $D(33)$, the sets $\{1,32,37,352\}$, $\{1,32,172,352\}$, $\{2,16,22,32\}$, $\{4,7,11,32\}$ and $\{4,23,43,128\}$ have the property $D(-28)$, and the sets $\{1,13,32,1376\}$, $\{1,32,53,76\}$, $\{2,16,38,416\}$, $\{4,127,179,608\}$ and $\{8,52,104,304\}$ have the property $D(68)$.

Theorem **2** *If n is an integer such that $n \equiv 8 \pmod{16}$ and $n \notin V_3=\{-8,8,24,40\}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.*

Proof. The proof is based on the fact that the sets

$$\{1,4k^2-8k-4,4k^2-4k+1,16k^2-24k-7\}, \tag{14}$$

$$\{1, 36k^2 + 20k + 1, 36k^2 + 32k + 8, 144k^2 + 104k + 17\}, \tag{15}$$

$$\{1, k^2 - 10k + 1, k^2 - 8k + 8, 4k^2 - 36k + 17\}, \tag{16}$$

$$\{1, 9k^2 + 2k + 1, 9k^2 - 4k - 4, 36k^2 - 4k - 7\} \tag{17}$$

have the property $D(16k + 8)$.

The sets (14) and (15) are obtained directly from [4, (20) and (10)]. Multiplying all elements of the sets (2) and (1) by 4, for $x = \frac{1}{4}$ and $y = k-1$, we get the sets (16) and (17) respectively.

Analysis similar to that in the proof of Theorem 1 shows that for all integers n of the form $16k+8$, where $k \notin \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, there exist at least four distinct Diophantine quadruples with the property $D(n)$.

Therefore, the proof is completed by showing that the assertion of Theorem is valid for $n \in Y = \{-24, 56, 72, 88, 104, 120, 136, 152, 168\}$. For every $n \in Y$ the sets (14), (15) and (17) give three distinct Diophantine quadruples with the property $D(n)$. A trivial verification shows that the sets $\{3, 8, 11, 35\}$, $\{1, 25, 44, 65\}$, $\{7, 72, 127, 391\}$, $\{3, 11, 36, 91\}$, $\{1, 17, 185, 220\}$, $\{1, 49, 76, 4641\}$, $\{1, 33, 305, 540\}$, $\{11, 232, 347, 1147\}$ and $\{1, 57, 793, 1276\}$ have the properties $D(-24)$, $D(56)$, $D(72)$, $D(88)$, $D(104)$, $D(120)$, $D(136)$, $D(152)$ and $D(168)$ respectively, which completes the proof. ∎

Remark **2** For the elements of the set V_3, the following holds: the sets $\{1, 8, 9, 33\}$ and $\{1, 12, 17, 57\}$ have the property $D(-8)$, the set $\{1, 57, 76, 265\}$ has the property $D(24)$, and the sets $\{1, 24, 41, 129\}$, $\{1, 185, 216, 801\}$ and $\{3, 52, 83, 267\}$ have the property $D(40)$. No Diophantine quadruple with the property $D(8)$ is known.

Theorem **3** *If n is an integer such that $n \equiv 0 \pmod{16}$ and $n \notin V_4 = \{-16, 32, 48, 80\}$, then there exist at least six Diophantine quadruples with the property $D(n)$.*

Proof. If $n \equiv 0 \pmod{16}$, then necessarily n can be represented in one of the forms

$$32k + 16, \quad 64k + 32, \quad 128k + 64, \quad 128k,$$

and the proof will be divided into four cases.

Let us first observe that the sets

$$\{1, k^2 - 6k + 1, k^2 - 4k + 4, 4k^2 - 20k + 9\}, \tag{18}$$

$$\{1, 9k^2 - 8k, 9k^2 - 2k + 1, 36k^2 - 20k + 1\} \tag{19}$$

have the property $D(8k)$, and the sets

$$\{1, k^2 - 20k + 20, k^2 - 18k + 33, 4k^2 - 76k + 105\}, \tag{20}$$

$$\{1, 9k^2 - 14k - 7, 9k^2 - 8k, 36k^2 - 44k - 15\}, \tag{21}$$

$$\{1, k^2 - 6k - 3, k^2 - 2k + 5, 4k^2 - 16k\}, \tag{22}$$

$$\{1, 9k^2 - 2k - 3, 9k^2 + 10k + 5, 36k^2 + 16k\} \tag{23}$$

have the property $D(32k + 16)$.

The sets (18) and (19) are exactly the sets (20) and (1) from [4]. Multiplying all elements of the sets (2) and (1) by 8, for $x = \frac{1}{8}$ and $y = k - 2$, we get the sets (20) and (21) respectively, and multiplying the same elements by 4, for $x = 1$ and $y = \frac{k-1}{2}$, we get the sets (22) and (23).

Analyzing the sets (18)−(23), as in the proof of Theorem 1, we conclude that for all integers n of the form $32k+16$, where $k \notin \{-2, -1, 0, \ldots, 18, 19\}$, there exist at least six distinct Diophantine quadruples with the property $D(n)$. It is easy to check on a computer that for all of the remaining cases, except for $n \in \{-16, 48, 80\}$, there exist al least six Diophantine quadruples with the property $D(n)$. This proves the theorem in case $n \equiv 16 \pmod{32}$.

Let now $n = 32k$. For $k \notin \{0, 1\}$ the sets (18) and (19) give two distinct Diophantine quadruples with the property $D(n)$ (see [4, Theorem 6]). Each of these two quadruples contain the number 1. Multiplying all elements of the sets (18) and (19) by 2 we get the sets with the property $D(32k)$. By the proof of [4, Theorem 6], for $k \notin \{0, 1, 2, 3, 4, 5, 6\}$ these sets are two distinct Diophantine quadruples which do not contain the number 1, and therefore they are different from two quadruples obtained before.

Let $n = 64k + 32$. By Theorem 2, for $k \notin \{-1, 0, 1, 2\}$ there exist at least four distinct Diophantine quadruples with the property $D(16k + 8)$. Multiplying all elements of these sets by 2 we get four Diophantine quadruples with even elements with the property $D(64k + 32)$. Therefore, for $k \notin \{-1, 0, 1, 2\}$ there exist at least six Diophantine quadruples with the property $D(64k + 32)$.

Consider now the case $n = 128k + 64$. As we have proved before, for $k \notin \{-1, 1, 2\}$ there exist at least six distinct Diophantine quadruples with the property $D(32k + 16)$. Multiplying all elements of these quadruples by 2 we get the quadruples with the property $D(128k + 64)$. All elements of those quadruples are even and, accordingly, they do not contain the number 1. Thus we proved that for $k \notin \{-1, 1, 2\}$ there exist at least eight distinct Diophantine quadruples with the property $D(128k + 64)$.

It remains to consider the case $n = 128k$. But we have already proved that for $k \notin \{0, 1, 2, 3, 4, 5, 6\}$ there exist at least four distinct Diophantine quadruples with the property $D(32k)$. Multiplying all elements of those quadruples by 2 we get four Diophantine quadruples with the property $D(128k)$ which do not contain the number 1. Therefore, for $k \notin \{0, 1, 2, 3, 4, 5, 6\}$ there exist at least six Diophantine quadruples with the property $D(128k)$.

An easy verification on a computer shows that for every $n \in \{-32, 96, 160, -64, 192, 320, 0, 128, 256, 384, 512, 768\}$ there exist six distinct Diophantine quadruples with the property $D(n)$, which completes the proof. ∎

Remark **3** For the elements of the set V_4, the following holds: the sets $\{1, 16, 17, 65\}$ and $\{1, 41, 52, 185\}$ have the property $D(-16)$, the set $\{1, 112, 137, 497\}$ has the property $D(32)$, the set $\{1, 276, 313, 1177\}$ has the property $D(48)$, and the sets $\{1, 41, 64, 209\}$, $\{1, 820, 881, 3401\}$ and $\{4, 29, 61, 176\}$ have the property $D(80)$.

Theorem **4** *If n is an integer such that $n \equiv 13 \pmod{24}$ and $n \notin V_5 = \{-11, 13\}$, or $n \equiv 21 \pmod{24}$ and $n \notin V_6 = \{-27, -3, 21, 45, 117\}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.*

Proof. The proof in the case $n = 24k + 13$ is based on the fact that the sets

$$\{6, 54k^2 + 38k + 6, 54k^2 + 74k + 26, 216k^2 + 224k + 58\}, \tag{24}$$

$$\{6, 6k^2 - 2k - 2, 6k^2 + 20k + 6, 24k^2 + 16k + 2\} \tag{25}$$

have the property $D(24k + 13)$.

These sets are obtained from (1) and (2), for $x = 6$ and $y = k$. Analyzing the sets (24), (25) and the sets (9) and (19) from [4] we conclude that for $k \notin \{-1, 0\}$ there exist at least four distinct Diophantine quadruples with the property $D(24k + 13)$, which is the desired conclusion.

Let us now consider the case $n = 24k+21$. We start with the observation that the sets

$$\{2, 2k^2 - 6k - 6, 2k^2 - 2k + 2, 8k^2 - 16k - 10\}, \tag{26}$$

$$\{6, 6k^2+2k-2, 6k^2+14k+10, 24k^2+32k+10\} \tag{27}$$

have the property $D(24k+21)$.

The set (26) is obtained by multiplication of all elements of the set (2) by 3, for $x=\frac{2}{3}$ and $y=k$, and the set (27) is obtained from (3), for $x=6$ and $y=k$.

From (26), (27) and [4, (9) and (19)] it follows that for $k \notin \{-2,-1,0,1,2,3,4\}$ there exist at least four distinct Diophantine quadruples with the property $D(24k+21)$. But the sets $\{6,62,110,170\}$ and $\{22,154,294,874\}$ have the properties $D(69)$ and $D(93)$ respectively, which completes the proof. ∎

Remark **4** For the exceptions from the sets V_5 and V_6, the following holds: the sets $\{2,6,10,30\}$, $\{2,10,18,30\}$ and $\{2,30,46,150\}$ have the property $D(-11)$, the set $\{2,34,54,174\}$ has the property $D(13)$, the sets $\{2,26,38,126\}$ and $\{2,194,234,854\}$ have the property $D(-27)$, the set $\{2,102,134,470\}$ has the property $D(21)$, the sets $\{2,38,62,198\}$ and $\{2,522,590,2222\}$ have the property $D(45)$, and the sets $\{2,362,422,1566\}$, $\{2,3726,3902,15254\}$ and $\{6,102,162,522\}$ have the property $D(117)$. No Diophantine quadruple with the property $D(-3)$ is known.

Corollary **2** *If n is an integer such that $n \equiv 52 \pmod{96}$ and $n \notin V_7=\{52\}$, or $n \equiv 84 \pmod{96}$ and $n \notin V_8=\{-108,-12,84,180\}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.*

Proof. The corollary is direct consequence of Theorem 4, Remark 4 and the fact that the sets $\{3,15,20,276\}$ and $\{1,1132,2668,7276\}$ have the properties $D(-44)$ and $D(468)$ respectively. ∎

Remark **5** Note that the sets $\{3,36,84,228\}$ and $\{4,531,9559,14596\}$ have the properties $D(-108)$ and $D(180)$ respectively. Thus, from Remark 4 it follows that there exist at least three Diophantine quadruples with the properties $D(-108)$ and $D(180)$.

Theorem **5** *If n is an integer such that $n \equiv 3 \pmod{12}$ and $n \notin V_9=\{-9,3,15,27,63\}$, or $n \equiv 7 \pmod{12}$ and $n \notin V_{10}=\{-5,7\}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.*

Proof. Let $n=12k+3$. The sets

$$\{1, k^2-8k+1, k^2-6k+6, 4k^2-28k+13\}, \tag{28}$$

$$\{3, 3k^2 - 4k - 1, 3k^2 + 2k + 2, 12k^2 - 4k - 1\} \tag{29}$$

have the property $D(12k + 3)$.

The set (28) is obtained by multiplication of all elements of the set (2) by 3, for $x = \frac{1}{3}$ and $y = k - 1$, and the set (29) is obtained from (3), for $x = 3$ and $y = k - 1$.

From (28), (29) and [4, (7) and (17)] it follows that for $k \notin \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$ there exist at least four distinct Diophantine quadruples with the property $D(12k + 3)$. The fact that the sets $\{3, 35, 62, 95\}$, $\{1, 13, 70, 145\}$, $\{1, 69, 94, 325\}$, $\{1, 2413, 12013, 25194\}$ and $\{1, 70, 801, 1345\}$ have the properties $D(39)$, $D(51)$, $D(75)$, $D(87)$ and $D(99)$ respectively, establishes the first part of the theorem.

Let us now consider the case $n = 12k + 7$. The sets

$$\{3, 27k^2 + 20k + 3, 27k^2 + 38k + 14, 108k^2 + 116k + 31\}, \tag{30}$$

$$\{3, 3k^2 - 2k - 2, 3k^2 + 4k + 3, 12k^2 + 4k - 1\}\,. \tag{31}$$

have the property $D(12k + 7)$.

These sets are obtained from (1) and (2), for $x = 3$ and $y = k$. The formulas (30), (31) and [4, (7) and (17)] imply that for $k \notin \{-1, 0, 1\}$ there exist at least four distinct Diophantine quadruples with the property $D(12k+7)$. But the set $\{1, 17, 30, 45\}$ has the property $D(19)$, and the proof is complete. ■

Remark **6** For the elements of the sets V_9 and V_{10}, the following holds: the sets $\{1, 10, 13, 45\}$ and $\{1, 45, 58, 205\}$ have the property $D(-9)$, the set $\{1, 106, 129, 469\}$ has the property $D(15)$, the sets $\{1, 22, 37, 117\}$, $\{1, 373, 414, 1573\}$ and $\{11, 18, 59, 143\}$ have the property $D(27)$, the sets $\{1, 193, 226, 837\}$, $\{1, 2146, 2241, 8773\}$ and $\{3, 54, 87, 279\}$ have the property $D(63)$, the sets $\{1, 5, 6, 21\}$ and $\{1, 14, 21, 69\}$ have the property $D(-5)$, and the set $\{1, 18, 29, 93\}$ has the property $D(7)$. No Diophantine quadruple with the property $D(3)$ is known.

Note that by [4, Remark 3], the number of Diophantine quadruples with the property $D(16k+12)$ is equal to the number of Diophantine quadruples with the property $D(4k+3)$. Thus we can rephrase Theorem 5 as follows.

Corollary **3** *If n is an integer such that $n \equiv 12 \pmod{48}$ and $n \notin V_{11} = \{-36, 12, 60, 108, 252\}$, or $n \equiv 28 \pmod{48}$ and $n \notin V_{12} = \{-20, 28\}$, then there exist at least four distinct Diophantine quadruples with the property $D(n)$.*

4 Connection with the Schinzel-Sierpiński conjecture

Let U denote the set of all integers n, not of the form $4k+2$, such that there exist at most two distinct Diophantine quadruple with the property $D(n)$. One open question is whether the set U is finite or not. The following corollary is the direct consequence of the results of Section 3.

COROLLARY **4** *If* $n \in U \setminus U_1$, *where* $U_1 = \{-36, -27, -20, -16, -15, -12, -9, -8, -7, -5, -3, 3, 7, 8, 12, 13, 15, 17, 21, 24, 28, 32, 45, 48, 52, 60, 84\}$, *then* n *can be represented in one of the following forms:*

$$12k+11, \quad 24k+5, \quad 48k+44, \quad 96k+20.$$

PROOF. Let $U_2 = \bigcup_{i=1}^{12} V_i$, where V_i, $i = 1, \ldots, 12$, are defined in Section 3. Then $U_1 = U_2 \setminus U_3$, where $U_3 = \{-108, -28, -11, 27, 33, 40, 63, 68, 80, 108, 117, 180, 252\}$. It is clear from Remarks 1 – 6 that $U_3 \cap U = \emptyset$. It implies that $U \setminus U_2 = U \setminus U_1$, which completes the proof. ∎

Note that multiplying all elements of quadruples with the properties $D(12k+11)$ and $D(24k+5)$ by 2, we obtain the quadruple with the properties $D(48k+44)$ and $D(96k+20)$, and by [4, Remark 3], all quadruples with the property $D(48k+44)$ can be obtained on this way.

In [7, Theorems 1 and 2], it was proved that the elements of the set U which have the form $4k+3$ or $8k+5$ must satisfy some primality conditions. The main idea was to analyze the construction of the polynomial formulas for Diophantine quadruples from [6]. It was shown that the additional Diophantine quadruples with the property $D(n)$ can be obtained if factors of the values of some linear polynomials in n are known. These results can be rephrased as follows.

THEOREM **6** *Let* n *be an integer such that* $n \equiv 11 \pmod{12}$, $n \notin \{-1, 11\}$ *and* $n \in U$. *Then the integers* $|n-1|/2$, $|n-9|/2$ *and* $|9n-1|/2$ *are primes. Furthermore, either* $|n|$ *is prime or* n *is the product of twin primes.*

THEOREM **7** *Let* n *be an integer such that* $n \equiv 5 \pmod{24}$, $n \neq 5$ *and* $n \in U$. *Then the integers* $|n|$, $|n-1|/4$, $|n-9|/4$ *and* $|9n-1|/4$ *are primes.*

COROLLARY **5** *Let* n *be an integer such that* $n \in U$ *and* $|n| \leq 10000$. *Then* $n \in W = U_1 \cup W_1$, *where* U_1 *is defined in Corollary 4, and* $W_1 = \{-8563, -7732, -7723, -7492, -6892, -6637, -6427, -6073, -5923, -5413, -5233, -5107, -4603, -4363, -4243, -3508, -3028, -2188, -1933, -1873, -1723, -877, -757, -652, -547, -268, -172, -163, -148, -67, -52, -43,$

$-37, -19, -13, -4, -1, 5, 11, 20, 23, 44, 83, 92, 167, 173, 227, 293, 332, 668, 908, 983, 1172, 1487, 2477, 2903, 3167, 3533, 3932, 4283, 4373, 4703, 5507, 5948, 8573, 9908\}$.

PROOF. If $n \notin U_1$ then, by Corollary 4, n has one of the following forms:

$$12k + 11, \quad 24k + 5, \quad 48k + 44, \quad 96k + 20.$$

Let $n = 12k + 11$ and $n \notin \{-1, 11\}$. Then, by Theorem 6, the integers $|n - 1|/2$, $|n - 9|/2$ and $|9n - 1|/2$ are primes, and either $|n|$ is prime or n is a product of twin primes. There exist exactly 25 integers n, $|n| \leq 10000$, which satisfy these conditions. Note that the sets $\{1, 494, 989, 2881\}$, $\{1, 2, 737, 26197\}$, $\{1, 146, 9073, 11521\}$ and $\{1, 3421, 24158, 45761\}$ have the properties $D(35)$, $D(47)$, $D(143)$ and $D(1763)$ respectively. Hence, we proved that if $n \equiv 11 \pmod{12}$, $|n| \leq 10000$ and $n \notin W_2 = \{ -6637, -6073, -5413, -5233, -1933, -1873, -877, -757, -37, -13, -1, 11, 23, 83, 167, 227, 983, 1487, 2903, 3167, 4283, 4703, 5507\}$, then there exist at least three distinct Diophantine quadruples with the property $D(n)$.

It implies that if $n \equiv 44 \pmod{48}$, $|n| \leq 10000$ and $n \notin W_3 = \{-7732, -7492, -3508, -3028, -148, -52, -4, 44, 92, 332, 668, 908, 3932, 5948\}$, then there exist at least three distinct Diophantine quadruples with the property $D(n)$.

Let $n = 24k + 5$, $n \neq 5$. Then, by Theorem 7, the integers $|n|$, $|n - 1|/4$, $|n - 9|/4$ and $|9n - 1|/4$ are primes. There exist exactly 19 integers n, $|n| \leq 10000$, which satisfy these conditions. Hence, we proved that if $n \equiv 5 \pmod{24}$, $|n| \leq 10000$ and $n \notin W_4 = \{-8563, -7723, -6427, -5923, -5107, -4603, -4363, -1723, -547, -163, -67, -43, -19, 5, 173, 293, 2477, 3533, 4373, 8573\}$, then there exist at least three distinct Diophantine quadruples with the property $D(n)$.

From this and the fact that the sets $\{4, 23, 35, 1540\}$ and $\{1, 92, 7772, 7957\}$ have the properties $D(-76)$ and $D(692)$ respectively, we conclude that if $n \equiv 20 \pmod{96}$, $|n| \leq 10000$ and $n \notin W_5 = \{-6892, -2188, -652, -268, -172, 20, 1172, 9908\}$, then there exist at least three distinct Diophantine quadruples with the property $D(n)$.

This proves the corollary, since it is obvious that

$$W_1 = W_2 \cup W_3 \cup W_4 \cup W_5 \,.$$

■

It is not yet known, whether the set U is finite or not. Note that if U is infinite then at least one of the sets

$$A = \{k \in \mathbf{Z} : |6k + 1|, |6k + 5|, |12k + 11| \text{ and } |54k + 49| \text{ are primes}\},$$

$$B = \{l \in \mathbf{N} : 6l-1, 6l+1, 18l^2-5, 18l^2-1 \text{ and } 162l^2-5 \text{ are primes}\},$$
$$C = \{k \in \mathbf{Z} : |6k-1|, |6k+1|, |24k+5| \text{ and } |54k+11| \text{ are primes}\}$$

is infinite. Let us observe that the polynomials appearing in the sets A, B and C satisfy the conditions of following Schinzel-Sierpiński conjecture ([11], [10, p. 312]):

Let $s \geq 1$, let $f_1(x), \ldots, f_s(x)$ be irreducible polynomials with integral coefficients and positive leading coefficients. Assume that the following condition holds:

There does not exist any integer $n > 1$ dividing all the products $f_1(k)f_2(k)\cdots f_s(k)$ for every integer k.

Then there exist infinitely many natural numbers m such that all numbers $f_1(m), f_2(m), \ldots, f_s(m)$ are primes.

Therefore, the validity of the Schinzel-Sierpiński conjecture would imply that the sets A, B and C are infinite.

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