Heat kernels of non-symmetric jump processes: beyond the stable case

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Abstract

Let J be the Lévy density of a symmetric Lévy process in $\mathbb{R}^d$ with its Lévy exponent satisfying a weak lower scaling condition at infinity. Consider the non-symmetric and non-local operator

$$\mathcal{L}^\kappa f(x) := \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x)) \kappa(x,z) J(z)\, dz\,,$$

where $\kappa(x,z)$ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $0 < \kappa_0 \le \kappa(x,z) \le \kappa_1$, $\kappa(x,z) = \kappa(x,-z)$ and $|\kappa(x,z) - \kappa(y,z)| \le \kappa_2 |x-y|^\beta$ for some $\beta \in (0,1]$. We construct the heat kernel $p^\kappa(t,x,y)$ of $\mathcal{L}^\kappa$, establish its upper bound as well as its fractional derivative and gradient estimates. Under an additional weak upper scaling condition at infinity, we also establish a lower bound for the heat kernel p^κ.

AMS 2010 Mathematics Subject Classification: Primary 60J35; Secondary 60J75.
Keywords and phrases: heat kernel estimates, subordinate Brownian motion, symmetric Lévy process, non-symmetric operator, non-symmetric Markov process

1 Introduction

Suppose that $d \ge 1$, $\alpha \in (0,2)$ and $\kappa(x,z)$ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$0 < \kappa_0 \le \kappa(x,z) \le \kappa_1\,, \quad \kappa(x,z) = \kappa(x,-z)\,, \tag{1.1}$$

and for some $\beta \in (0,1]$,

$$|\kappa(x,z) - \kappa(y,z)| \le \kappa_2 |x-y|^\beta\,. \tag{1.2}$$

The operator

$$\mathcal{L}^\kappa_\alpha f(x) = \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}}\, dz \tag{1.3}$$

[*]This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. 2016R1E1A1A01941893)

[†]Research supported in part by a grant from the Simons Foundation (208236)

[‡]Research supported in part by the Croatian Science Foundation under the project 3526

is a non-symmetric and non-local stable-like operator. In the recent paper [6], Chen and Zhang studied the heat kernel of $\mathcal{L}^\kappa_\alpha$ and its sharp two-sided estimates. As the main result of the paper, they proved the existence and uniqueness of a non-negative jointly continuous function $p^\kappa_\alpha(t,x,y)$ in $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving the equation

$$\partial_t p^\kappa_\alpha(t,x,y) = \mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t,\cdot,y)(x)\,, \quad x \neq y\,,$$

and satisfying four properties - an upper bound, Hölder's estimate, fractional derivative estimate and continuity, cf. [6, Theorem 1.1] for details. They also proved some other properties of the heat kernel $p^\kappa_\alpha(t,x,y)$ such as conservativeness, Chapman-Kolmogorov equation, lower bound, gradient estimate and studied the corresponding semigroup. Their paper is the first one to address these questions for not necessarily symmetric non-local stable-like operators. These operators can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context the Hölder continuity of $\kappa(\cdot,z)$ in (1.2) is a natural assumption.

The goal of this paper is to extend the results of [6] to more general operators than the ones defined in (1.3). These operators will be non-symmetric and not necessarily stable-like. We will replace the kernel $\kappa(x,z)|z|^{-d-\alpha}$ with a kernel $\kappa(x,z)J(z)$ where κ still satisfies (1.1) and (1.2), but $J(z)$ is the Lévy density of a rather general symmetric Lévy process. Here are the precise assumptions that we make.

Let $\phi : (0,\infty) \to (0,\infty)$ be a Bernstein function without drift and killing. Then

$$\phi(\lambda) = \int_{(0,\infty)} \left(1 - e^{-\lambda t}\right) \mu(dt),$$

where μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (t \wedge 1)\mu(dt) < \infty$. Here and throughout this paper, we use the notation $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. Without loss of generality we assume that $\phi(1) = 1$. Define $\Phi : (0,\infty) \to (0,\infty)$ by $\Phi(r) = \phi(r^2)$ and let Φ^{-1} be its inverse. The function $x \mapsto \Phi(|x|) =: \Phi(x)$, $x \in \mathbb{R}^d$, $d \geq 1$, is negative definite and hence it is the characteristic exponent of an isotropic Lévy process on $\mathbb{R}^d$. This process can be obtained by subordinating a d-dimensional Brownian motion by an independent subordinator with Laplace exponent ϕ. The Lévy measure of this process has a density $j(|y|)$ where $j : (0,\infty) \to (0,\infty)$ is the function given by

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(dt)\,.$$

Thus we have

$$\Phi(x) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(x \cdot y))\, j(|y|)\, dy\,.$$

Note that when $\phi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$, we have $\Phi(r) = r^\alpha$, the corresponding subordinate Brownian motion is an isotropic α-stable process and $j(r) = c(d,\alpha)\, r^{-d-\alpha}$.

Our main assumption is the following *weak lower scaling condition at infinity*: There exist $\delta_1 \in (0,2]$ and $a_1 \in (0,1)$ such that

$$a_1 \lambda^{\delta_1} \Phi(r) \le \Phi(\lambda r), \quad \lambda \ge 1, r \ge 1. \tag{1.4}$$

This condition implies that $\lim_{\lambda\to\infty} \Phi(\lambda) = \infty$ and hence $\int_{\mathbb{R}^d\setminus\{0\}} j(|y|)dy = \infty$ (i.e., the subordinate Brownian motion is not a compound Poisson process). The weak lower scaling condition at infinity governs the short-time small-space behavior of the subordinate Brownian motion. We also need a weak condition on the behavior of Φ near zero. We assume that

$$\int_0^1 \frac{\Phi(r)}{r}\, dr = C_* < \infty. \tag{1.5}$$

The following function will play a prominent role in the paper. For $t > 0$ and $x \in \mathbb{R}^d$ we define

$$\rho(t,x) = \rho^{(d)}(t,x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right)\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}. \tag{1.6}$$

In case when $\Phi(r) = r^\alpha$ we see that $\rho(t,x) = (t^{1/\alpha} + |x|)^{-d-\alpha}$. It is well known that $t(t^{1/\alpha} + |x|)^{-d-\alpha}$ is comparable to the heat kernel $p(t,x)$ of the isotropic α-stable process in $\mathbb{R}^d$. We will prove later in this paper (see Proposition 3.2) that $t\rho(t,x)$ is an upper bound of the heat kernel of the subordinate Brownian motion with characteristic exponent Φ.

We assume that $J : \mathbb{R}^d \to (0,\infty)$ is symmetric in the sense that $J(x) = J(-x)$ for all $x \in \mathbb{R}^d$ and there exists $\gamma_0 > 0$ such that

$$\gamma_0^{-1} j(|y|) \le J(y) \le \gamma_0 j(|y|), \quad \text{for all } y \in \mathbb{R}^d. \tag{1.7}$$

Following (1.3), we define a non-symmetric and non-local operator

$$\mathcal{L}^\kappa f(x) = \mathcal{L}^{\kappa,0} f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x))\kappa(x,z)J(z)\, dz := \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x), \tag{1.8}$$

where

$$\mathcal{L}^{\kappa,\varepsilon} f(x) := \int_{|z|>\varepsilon} (f(x+z) - f(x))\kappa(x,z)J(z)\, dz, \quad \varepsilon > 0.$$

The following theorem is the main result of this paper.

Theorem 1.1 *Assume that Φ satisfies* (1.4) *and* (1.5)*, that J satisfies* (1.7)*, and that κ satisfies* (1.1) *and* (1.2)*. Suppose there exists a function $g : \mathbb{R}^d \to (0,\infty)$ such that*

$$\lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \mathcal{L}^\kappa g(x)/g(x) \text{ is bounded from above.} \tag{1.9}$$

Then there exists a unique non-negative jointly continuous function $p^\kappa(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ solving

$$\partial_t p^\kappa(t,x,y) = \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x), \quad x \neq y, \tag{1.10}$$

and satisfying the following properties:
(i) (Upper bound) For every $T \geq 1$, there is a constant $c_1 > 0$ so that for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d$,

$$p^\kappa(t,x,y) \leq c_1 t\rho(t,x-y)\,. \tag{1.11}$$

(ii) (Fractional derivative estimate) For any $x,y \in \mathbb{R}^d$, $x \neq y$, the map $t \mapsto \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ is continuous in $(0,\infty)$, and, for each $T \geq 1$ there is a constant $c_2 > 0$ so that for all $t \in (0,T]$, $\varepsilon \in [0,1]$ and $x,y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t,\cdot,y)(x)| \leq c_2\rho(t,x-y)\,. \tag{1.12}$$

(iii) (Continuity) For any bounded and uniformly continuous function $f:\mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p^\kappa(t,x,y)f(y)\,dy - f(x)\right| = 0\,. \tag{1.13}$$

Moreover, the constants c_1 and c_2 can be chosen so that they depend only on T, $\Phi^{-1}(T^{-1})$, d, a_1, δ_1, C_*, β, γ_0, κ_0, κ_1 and κ_2.

The assumption (1.9) is a quite mild one. For example, if $\int_{|z|>1}|z|^\varepsilon j(|z|)dz < \infty$ for some $\varepsilon > 0$, then (1.9) holds, see Remark 5.2 below.

Some further properties of the heat kernel $p^\kappa(t,x,y)$ are listed in the following result.

Theorem 1.2 *Suppose that the assumptions of Theorem 1.1 are satisfied.*
(1) (Conservativeness) For all $(t,x) \in (0,\infty)\times\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t,x,y)\,dy = 1\,. \tag{1.14}$$

(2) (Chapman-Kolmogorov equation) For all $s,t > 0$ and all $x,y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t,x,z)p^\kappa(s,z,y)\,dz = p^\kappa(t+s,x,y)\,. \tag{1.15}$$

(3) (Joint Hölder continuity) For every $T \geq 1$ and $\gamma \in (0,\delta_1)\cap(0,1]$, there is a constant $c_3 = c_3(T,d,\delta_1,a_1,\beta,C_*,\Phi^{-1}(T^{-1}),\gamma_0,\kappa_0,\kappa_1,\kappa_2) > 0$ such that for all $0 < s \leq t \leq T$ and $x,x',y \in \mathbb{R}^d$,

$$|p^\kappa(s,x,y) - p^\kappa(t,x',y)| \leq c_3\left(|t-s| + |x-x'|^\gamma t\,\Phi^{-1}(t^{-1})\right)\left(\rho(s,x-y)\vee\rho(s,x'-y)\right). \tag{1.16}$$

Furthermore, if the constant δ_1 in (1.4) belongs to $(2/3,2)$ and the constant β in (1.2) satisfies $\beta+\delta_1 > 1$ then (1.16) holds with $\gamma = 1$.
(4) (Gradient estimate) If $\delta_1 \in (2/3,2)$, and $\beta+\delta_1 > 1$, then for every $T \geq 1$, there exists $c_4 = c_4(T,d,\delta_1,a_1,\beta,C_*,\Phi^{-1}(T^{-1}),\gamma_0,\kappa_0,\kappa_1,\kappa_2) > 0$ so that for all $x,y \in \mathbb{R}^d$, $x \neq y$, and $t \in (0,T]$,

$$|\nabla_x p^\kappa(t,x,y)| \leq c_4\Phi^{-1}(t^{-1})t\rho(t,|x-y|)\,. \tag{1.17}$$

Note that the gradient estimate (1.17) is an improvement of the corresponding estimate [6, (4.19)] in the sense that the parameter δ_1 could be smaller than one as long as it is still larger than $2/3$ and $\beta+\delta_1>1$.

For $t>0$, define the operator P^κ_t by

$$P^\kappa_t f(x)=\int_{\mathbb{R}^d} p^\kappa(t,x,y)f(y)\,dy\,,\quad x\in\mathbb{R}^d\,, \tag{1.18}$$

where f is a non-negative (or bounded) Borel function on $\mathbb{R}^d$, and let $P^\kappa_0=\mathrm{Id}$. Then by Theorems 1.1 and 1.2, $(P^\kappa_t)_{t\geq 0}$ is a Feller semigroup with the strong Feller property. Let $C^{2,\varepsilon}_b(\mathbb{R}^d)$ be the space of bounded twice differentiable functions in $\mathbb{R}^d$ whose second derivatives are uniformly Hölder continuous. We further have

Theorem 1.3 *Suppose that the assumptions of Theorem 1.1 are satisfied.*
(1) (Generator) Let $\varepsilon>0$. For any $f\in C^{2,\varepsilon}_b(\mathbb{R}^d)$, we have

$$\lim_{t\downarrow 0}\frac{1}{t}\left(P^\kappa_t f(x)-f(x)\right)=\mathcal{L}^\kappa f(x)\,, \tag{1.19}$$

and the convergence is uniform.
(2) (Analyticity) The semigroup $(P^\kappa_t)_{t\geq 0}$ of $\mathcal{L}^\kappa$ is analytic in $L^p(\mathbb{R}^d)$ for every $p\in[1,\infty)$.

Finally, under an additional assumption, we prove by probabilistic methods a lower bound for the heat kernel $p^\kappa(t,x,y)$. The *weak upper scaling condition* means that there exist $\delta_2\in(0,2)$ and $a_2>0$ such that

$$\Phi(\lambda r)\leq a_2\lambda^{\delta_2}\Phi(r)\,,\quad \lambda\geq 1, r\geq 1\,. \tag{1.20}$$

Theorem 1.4 *Suppose that Φ satisfies* (1.4)*,* (1.20) *and* (1.5)*, that J satisfies* (1.7)*, and that κ satisfies* (1.1) *and* (1.2)*. Suppose also that there exists a function $g:\mathbb{R}^d\to(0,\infty)$ such that* (1.9) *holds. For every $T\geq 1$, there exists $c_5=c_5(T,d,\delta_1,\delta_2,\gamma_0,C_*,\Phi^{-1}(T^{-1}),a_1,a_2,\beta,\kappa_0,\kappa_1,\kappa_2)>0$ such that for all $t\in(0,T]$,*

$$p^\kappa(t,x,y)\geq c_5\begin{cases}\Phi^{-1}(t^{-1})^d & \text{if } |x-y|\leq 3\Phi^{-1}(t^{-1})^{-1},\\ tj\left(|x-y|\right) & \text{if } |x-y|>3\Phi^{-1}(t^{-1})^{-1}.\end{cases} \tag{1.21}$$

In particular, for all $T,M\geq 1$, there exists $c_6=c_6(T,d,\delta_1,\delta_2,\gamma_0,C_*,\Phi^{-1}(T^{-1}),a_1,a_2,\beta,\kappa_0,\kappa_1,\kappa_2)>0$ for all $t\in(0,T]$ and $x,y\in\mathbb{R}^d$ with $|x-y|\leq M$,

$$p^\kappa(t,x,y)\geq c_6 t\rho(t,x-y)\,. \tag{1.22}$$

Theorems 1.1-1.4 generalize [6, Theorem 1.1]. Note that the lower bound (1.22) of $p^\kappa(t,x,y)$ is stated only for $|x-y|\leq M$. This is natural in view of the fact that (1.4) and (1.20) only give information about short-time small-space behavior of the underlying subordinate Brownian motion. We remark in passing that, the upper bound (1.11) may not be sharp under the assumptions (1.4) and (1.5). When Φ satisfies scaling conditions both near infinity and near the origin, see [11, (H1) and (H2)], the upper bound (1.11) is sharp in the sense that the lower bound (1.22) is valid for all $x, y \in \mathbb{R}^d$.

The assumptions (1.4), (1.5), (1.9) and (1.20) are very weak conditions and they are satisfied by many subordinate Brownian motions. For the reader's convenience, we list some examples of ϕ, besides the Laplace exponent of the stable subordinator, such that $\Phi(r) = \phi(r^2)$ satisfies these assumptions.

(1) $\phi(\lambda) = \lambda^{\alpha_1} + \lambda^{\alpha_2}$, $0 < \alpha_1 < \alpha_2 < 1$;

(2) $\phi(\lambda) = (\lambda + \lambda^{\alpha_1})^{\alpha_2}$, $\alpha_1, \alpha_2 \in (0, 1)$;

(3) $\phi(\lambda) = (\lambda + m^{1/\alpha})^{\alpha} - m$, $\alpha \in (0, 1)$, $m > 0$;

(4) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, 1 - \alpha_1]$;

(5) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{-\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, \alpha_1)$;

(6) $\phi(\lambda) = \lambda / \log(1 + \lambda^{\alpha})$, $\alpha \in (0, 1)$.

The functions in (1)–(5) satisfy (1.4), (1.5), (1.20) and (1.9) (see (3.1) and Remark 5.2); while the function in (6) satisfies (1.4), (1.5) and (1.9), but does not satisfy (1.20). The function $\phi(\lambda) = \lambda / \log(1 + \lambda)$ satisfies (1.4), but does not satisfy the other two conditions.

In order to prove our main results, we follow the ideas and the road-map from [6]. At many stages we encounter substantial technical difficulties due to the fact that in the stable-like case one deals with power functions while in the present situation the power functions are replaced with a quite general Φ and its variants. We also strive to simplify the proofs and streamline the presentation. In some places we provide full proofs where in [6] only an indication is given. On the other hand, we skip some proofs which would be almost identical to the corresponding ones in [6]. Below is a detailed outline of the paper with emphasis on the main differences from [6].

In Section 2 we start by introducing the basic setup, state again the assumptions, and derive some of the consequences. In Subsection 2.1 we discuss convolution inequalities, cf. Lemma 2.6. While in [6] these involve power functions, the most challenging task in the present setting was to find appropriate versions of these inequalities. The main new technical result here is Lemma 2.6.

In Section 3 we first study the heat kernel $p(t, x)$ of a symmetric Lévy process Z with Lévy density j_Z comparable to the Lévy density j of the subordinate Brownian motion with characteristic exponent Φ. We prove the joint Lipschitz continuity of $p(t, x)$ and then, based on a result from [10], that $t\rho(t, x)$ is the upper bound of $p(t, x)$ for all $x \in \mathbb{R}^d$ and small t, cf. Proposition 3.2. In Subsection 3.1, we provide some useful estimates on functions of $p(t, x)$. In Subsection 3.2, we specify j_Z by assuming $j_Z(z) = \mathfrak{K}(z)J(z)$, with $\mathfrak{K}$ being symmetric and bounded between two positive constants. Let $\mathcal{L}^{\mathfrak{K}}$ be the infinitesimal generator of the corresponding process and let $p^{\mathfrak{K}}$ be its heat kernel. We look at the continuous dependence of $p^{\mathfrak{K}}$ with respect to $\mathfrak{K}$. This subsection follows the ideas and proofs from [6] with additional technical difficulties.

Given a function κ satisfying (1.1) and (1.2), we define, for a fixed $y \in \mathbb{R}^d$, $\mathfrak{K}_y = \kappa(y, \cdot)$ and denote by $p_y(t,x)$ the heat kernel of the freezing operator $\mathcal{L}^{\mathfrak{K}_y}$. Various estimates and joint continuity of $p_y(t,x)$ are shown in Subsection 4.1. The rest of Section 4 is devoted to constructing the heat kernel $p^\kappa(t,x,y)$ of the operator $\mathcal{L}^\kappa$. The heat kernel should have the form

$$p^\kappa(t,x,y) = p_y(t,x-y) + \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,, \tag{1.23}$$

where according to Levi's method the function $q(t,x,y)$ solves the integral equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x-z)q(s,z,y)\,dz\,ds\,, \tag{1.24}$$

with $q_0(t,x,y) = (\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y})p_y(t,x-y)$. The main result is Theorem 4.5 showing existence and joint continuity of $q(t,x,y)$ satisfying (1.24). We follow [6, Theorem 3.1], and give a full proof. Joint continuity and various estimates of $p^\kappa(t,x,y)$ defined by (1.23) are given in Subsection 4.3.

Section 5 contains proofs of Theorems 1.1–1.4. We start with a version of a non-local maximum principle in Theorem 5.1 which is somewhat different from the one in [6, Theorem 4.1], continue with two results about the semigroup $(P_t^\kappa)_{t\geq 0}$ and then complete the proofs.

In this paper, we use the following notations. We will use ":=" to denote a definition, which is read as "is defined to be". For any two positive functions f and g, $f \asymp g$ means that there is a positive constant $c \geq 1$ so that $c^{-1}\,g \leq f \leq c\,g$ on their common domain of definition. For a set W in $\mathbb{R}^d$, $|W|$ denotes the Lebesgue measure of W in $\mathbb{R}^d$. For a function space $\mathbb{H}(U)$ on an open set U in $\mathbb{R}^d$, we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f \text{ has compact support}\}$, $\mathbb{H}_0(U) := \{f \in \mathbb{H}(U) : f \text{ vanishes at infinity}\}$ and $\mathbb{H}_b(U) := \{f \in \mathbb{H}(U) : f \text{ is bounded}\}$.

Throughout the rest of this paper, the positive constants $\delta_1, \delta_2, \gamma_0, a_1, a_2, \beta, \kappa_0, \kappa_1, \kappa_2, C_i$, $i = 0, 1, 2, \ldots$, can be regarded as fixed. In the statements of results and the proofs, the constants $c_i = c_i(a,b,c,\ldots)$, $i = 0,1,2,\ldots$, denote generic constants depending on $a,b,c,\ldots$, whose exact values are unimportant. They start anew in each statement and each proof. The dependence of the constants on the dimension $d \geq 1$, C_*, $\Phi^{-1}((2T)^{-1})$, $\Phi^{-1}(T^{-1})$ and γ_0 may not be mentioned explicitly.

2 Preliminaries

It is well known that the Laplace exponent ϕ of a subordinator is a Bernstein function and

$$\phi(\lambda t) \leq \lambda\phi(t) \qquad \text{for all } \lambda \geq 1, t > 0\,. \tag{2.1}$$

For notational convenience, in this paper, we denote $\Phi(r) = \phi(r^2)$ and without loss of generality we assume that $\Phi(1) = 1$.

Throughout this paper ϕ is the Laplace exponent of a subordinator and $\Phi(r) = \phi(r^2)$ satisfies the weak lower scaling condition (1.4) at infinity. This can be reformulated as follows: There exist $\delta_1 \in (0, 2]$ and a positive constant $a_1 \in (0, 1]$ such that for any $r_0 \in (0, 1]$,

$$a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(r) \le \Phi(\lambda r), \quad \lambda \ge 1, r \ge r_0 . \tag{2.2}$$

In fact, suppose $r_0 \le r < 1$ and $\lambda \ge 1$. Then, $\Phi(\lambda r) \ge a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(1) \ge a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(r)$ if $\lambda r > 1$, and $\Phi(\lambda r) \ge \Phi(r) \ge a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(r)$ if $\lambda r \le 1$.

Since ϕ is a Bernstein function and we assume (2.2), it follows that Φ is strictly increasing and $\lim_{\lambda \to \infty} \Phi(\lambda) = \infty$. We denote by $\Phi^{-1} : (0, \infty) \to (0, \infty)$ the inverse function of Φ.

From (2.1) we have

$$\Phi^{-1}(\lambda r) \ge \lambda^{1/2} \Phi^{-1}(r), \quad \lambda \ge 1, r > 0 . \tag{2.3}$$

Moreover, by (2.2), Φ^{-1} satisfies the following weak upper scaling condition at infinity: For any $r_0 \in (0, 1]$,

$$\Phi^{-1}(\lambda r) \le a_1^{-1/\delta_1} \Phi^{-1}(r_0)^{-1} \lambda^{1/\delta_1} \Phi^{-1}(r), \quad \lambda \ge 1, r \ge r_0 . \tag{2.4}$$

In fact, from (2.2) we get $\Phi^{-1}(\lambda r) \le a_1^{-1/\delta_1} \lambda^{1/\delta_1} \Phi^{-1}(r)$ for $\lambda \ge 1$ and $r \ge 1$. Suppose $r_0 \le r < 1$. Then, $\Phi^{-1}(\lambda r) \le 1 \le a_1^{-1/\delta_1} \Phi^{-1}(r_0)^{-1} \lambda^{1/\delta_1} \Phi^{-1}(r)$ if $\lambda r \le 1$, and $\Phi^{-1}(\lambda r) \le a_1^{-1/\delta_1} \lambda^{1/\delta_1} r^{1/\delta_1} \le a_1^{-1/\delta_1} \lambda^{1/\delta_1} \Phi^{-1}(r_0)^{-1} \Phi^{-1}(r)$ if $\lambda r > 1$.

For $t > 0$ and $x \in \mathbb{R}^d$, we define functions $r(t, x)$ and $\rho(t, x)$ by

$$r(t, x) = \Phi^{-1}(t^{-1})^d \wedge \frac{t \Phi(|x|^{-1})}{|x|^d}$$

and

$$\rho(t, x) = \rho^{(d)}(t, x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x| \right)^{-1} \right) \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x| \right)^{-d} . \tag{2.5}$$

Note that, by [2, Lemma 17],

$$t\Phi(|x|^{-1})|x|^{-d} \ge \Phi^{-1}(t^{-1})^d \quad \text{if and only if} \quad t\Phi(|x|^{-1}) \ge 1. \tag{2.6}$$

Proposition 2.1 *For all $t > 0$ and $x \in \mathbb{R}^d$, $t\rho(t, x) \le r(t, x) \le 2^{d+2} t \rho(t, x)$.*

Proof. *Case 1:* $t\Phi(|x|^{-1}) \ge 1$. In this case, by (2.6) we have that $r(t, x) = \Phi^{-1}(t^{-1})^d$. Since $|x| \le \frac{1}{\Phi^{-1}(t^{-1})}$, we have

$$\frac{1}{\Phi^{-1}(t^{-1})} \le \frac{1}{\Phi^{-1}(t^{-1})} + |x| \le \frac{2}{\Phi^{-1}(t^{-1})} . \tag{2.7}$$

This and (2.1) imply that

$$t^{-1} = \Phi(\Phi^{-1}(t^{-1})) \ge \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x| \right)^{-1} \right) \ge \Phi(2^{-1}\Phi^{-1}(t^{-1})) \ge \frac{1}{4}\Phi(\Phi^{-1}(t^{-1})) = \frac{1}{4}t^{-1}$$

and

$$2^{-d}\Phi^{-1}(t^{-1})^{-d} \le \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d} \le \Phi^{-1}(t^{-1})^{-d}.$$

The last two displays imply that $2^{-d-2}\Phi^{-1}(t^{-1})^d \le t\rho(t,x) \le \Phi^{-1}(t^{-1})^d$.
Case 2: $t\Phi(|x|^{-1}) \le 1$. In this case, by (2.6) we have that $r(t,x) = \frac{t\Phi(|x|^{-1})}{|x|^d}$. Since $|x| \ge \frac{1}{\Phi^{-1}(t^{-1})}$, we have

$$|x|^{-1} \ge \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1} \ge 2^{-1}|x|^{-1}.$$

This with (2.1) implies that

$$\Phi(|x|^{-1}) \ge \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right) \ge \Phi(2^{-1}|x|^{-1}) \ge \frac{1}{4}\Phi(|x|^{-1}).$$

The last two displays imply the conclusion of the proposition in Case 2. □

Lemma 2.2 *Let* $T \ge 1$ *and* $c = (2(2/a_1)^{1/\delta_1}/\Phi^{-1}((2T)^{-1}))^{d+2}$.
(a) For all $0 < s < t \le T$ *and* $x, z \in \mathbb{R}^d$,

$$\rho(t-s, x-z)\rho(s,z) \le c\big(\rho(t-s,x-z) + \rho(s,z)\big)\rho(t,x). \tag{2.8}$$

(b) For every $x \in \mathbb{R}^d$ *and* $0 < t/2 \le s \le t \le T$, $\rho(t,x) \le \rho(s,x) \le 2c\rho(t,x)$.

Proof. (a) By (2.4) we have that for all $0 < t, s \le T$,

$$\frac{1}{\Phi^{-1}((t+s)^{-1})} \le \frac{1}{\Phi^{-1}(2^{-1}(t \vee s)^{-1})} \le c_1\left(\frac{1}{\Phi^{-1}(t^{-1})} + \frac{1}{\Phi^{-1}(s^{-1})}\right), \tag{2.9}$$

where $c_1 = (2/a_1)^{1/\delta_1}/\Phi^{-1}((2T)^{-1}) \ge 1$.

Define $\varrho : (0,\infty) \to (0,\infty)$ by $\varrho(r) := r^d/\Phi(r^{-1})$, so that $\rho(t,x) = (\varrho(\frac{1}{\Phi^{-1}(t^{-1})} + |x|))^{-1}$. For all $a, b > 0$, $(a+b)^d \le 2^d(a \vee b)^d$ and, by (2.1), $\Phi((a+b)^{-1}) \ge \Phi(2^{-1}(a\vee b)^{-1}) \ge 4^{-1}\Phi((a \vee b)^{-1})$. Therefore, for all $a, b > 0$,

$$\varrho(a+b) \le 2^{d+2}\varrho(a \vee b) \le 2^{d+2}(\varrho(a) + \varrho(b)). \tag{2.10}$$

Moreover, (2.1) implies that for $r > 0$,

$$\varrho(c_1 r) = \frac{(c_1 r)^d}{\Phi(c_1^{-1} r^{-1})} \le c_1^{d+2}\frac{r^d}{\Phi(r^{-1})} = c_1^{d+2}\varrho(r). \tag{2.11}$$

By using (2.9)–(2.11), we have

$$\varrho\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right) \le \varrho\left(c_1\left(\left(\frac{1}{\Phi^{-1}((t-s)^{-1})} + |x-z|\right) + \left(\frac{1}{\Phi^{-1}(s^{-1})} + |z|\right)\right)\right)$$

$$\leq c_1^{d+2}\varrho\left(\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)\right)$$
$$\leq (2c_1)^{d+2}\left(\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)\right). \tag{2.12}$$

Thus we have that for $0<s<t\leq T$ and $x,z\in\mathbb{R}^d$,

$$\begin{aligned}
&(\rho(t-s,x-z)+\rho(s,z))\,\rho(t,x)\\
&= \frac{\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)}{\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)}\frac{1}{\varrho\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x|\right)}\\
&\geq (2c_1)^{-d-2}\frac{1}{\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)}\\
&= (2c_1)^{-d-2}\rho(t-s,x-z)\rho(s,z)\,.
\end{aligned}$$

(b) This follows from (2.12) by taking $s=t/2$, $z=0$ and by using that ϱ is increasing. □

2.1 Convolution inequalities

Let $B(a,b)$ be the beta function, i.e., $B(a,b)=\int_0^1 s^{a-1}(1-s)^{b-1}ds$, $a,b>0$.

Lemma 2.3 *Let $\beta,\gamma,\eta,\theta\in\mathbb{R}$ be such that $\mathbf{1}_{\beta\geq 0}(\beta/2)+\mathbf{1}_{\beta<0}(\beta/\delta_1)+1-\theta>0$ and $\mathbf{1}_{\gamma\geq 0}(\gamma/2)+\mathbf{1}_{\gamma<0}(\gamma/\delta_1)+1-\eta>0$. Then for every $t>0$, we have*

$$\int_0^t u^{-\eta}\Phi^{-1}(u^{-1})^{-\gamma}(t-u)^{-\theta}\Phi^{-1}((t-u)^{-1})^{-\beta}\,du\leq Ct^{1-\eta-\theta}\Phi^{-1}(t^{-1})^{-\gamma-\beta}\,. \tag{2.13}$$

Moreover, if $\beta\geq 0$ and $\gamma\geq 0$ then (2.13) *holds for all $t>0$ with $C=B(\beta/2+1-\theta,\gamma/2+1-\eta)$.*

Proof. Let I denote the integral in (2.13). By the change of variables $s=u/t$ we get that

$$I=t^{1-\eta-\theta}\int_0^1 s^{-\eta}\Phi^{-1}(t^{-1}s^{-1})^{-\gamma}(1-s)^{-\theta}\Phi^{-1}(t^{-1}(1-s)^{-1})^{-\beta}\,ds\,.$$

Since $s^{-1}\geq 1$ and $(1-s)^{-1}\geq 1$, we have by (2.3) that $\Phi^{-1}(t^{-1}s^{-1})\geq s^{-1/2}\Phi^{-1}(t^{-1})$ and $\Phi^{-1}(t^{-1}(1-s)^{-1})\geq(1-s)^{-1/2}\Phi^{-1}(t^{-1})$. Moreover, when $t\in(0,T]$, by (2.4) we have

$$\Phi^{-1}(t^{-1}s^{-1})\leq a_1^{-1/\delta_1}\Phi^{-1}(T^{-1})^{-1}s^{-1/\delta_1}\Phi^{-1}(t^{-1})$$

and

$$\Phi^{-1}(t^{-1}(1-s)^{-1})\leq a_1^{-1/\delta_1}\Phi^{-1}(T^{-1})^{-1}(1-s)^{-1/\delta_1}\Phi^{-1}(t^{-1}).$$

Hence,

$$\begin{aligned} I &\leq c_1 t^{1-\eta-\theta}\Phi^{-1}(t^{-1})^{-\gamma-\beta}\int_0^1 s^{\mathbf{1}_{\gamma\geq 0}(\gamma/2)+\mathbf{1}_{\gamma<0}(\gamma/\delta_1)-\eta}(1-s)^{\mathbf{1}_{\beta\geq 0}(\beta/2)+\mathbf{1}_{\beta<0}(\beta/\delta_1)-\theta}\,ds \\ &= C\Phi^{-1}(t^{-1})^{-\gamma-\beta}. \end{aligned}$$

When $\beta \geq 0$ and $\gamma \geq 0$ then the above inequality holds for all $t > 0$ with $c_1 = 1$ so $C = B(\beta/2+1-\theta, \gamma/2+1-\eta)$. □

Lemma 2.4 *Suppose that* $0 < t_1 \leq t_2 < \infty$. *Under the assumptions of Lemma 2.3, we have*

$$\lim_{h\to 0}\sup_{t\in[t_1,t_2]}\left(\int_0^h + \int_{t-h}^t\right) u^{-\eta}\Phi^{-1}(u^{-1})^{-\gamma}(t-u)^{-\theta}\Phi^{-1}((t-u)^{-1})^{-\beta}\,du = 0.$$

Proof. Under the assumptions of this lemma, by repeating the argument in the proof of Lemma 2.3, we have that for all $t \in [t_1, t_2]$,

$$\begin{aligned} &\left(\int_0^h + \int_{t-h}^t\right) u^{-\eta}\Phi^{-1}(u^{-1})^{-\gamma}(t-u)^{-\theta}\Phi^{-1}((t-u)^{-1})^{-\beta}\,du \\ &\leq \left(t_1^{1-\eta-\theta}\vee t_2^{1-\eta-\theta}\right)\left(\Phi^{-1}(t_1^{-1})^{-\gamma-\beta}\vee\Phi^{-1}(t_2^{-1})^{-\gamma-\beta}\right) \\ &\quad\times\left(\int_0^{h/t_1} + \int_{1-h/t_1}^1\right) s^{\mathbf{1}_{\gamma\geq 0}(\gamma/2)+\mathbf{1}_{\gamma<0}(\gamma/\delta_1)-\eta}(1-s)^{\mathbf{1}_{\beta\geq 0}(\beta/2)+\mathbf{1}_{\beta<0}(\beta/\delta_1)-\theta}\,ds. \end{aligned}$$

Now the conclusion of the lemma follows immediately. □

For $\gamma, \beta \in \mathbb{R}$, we define

$$\rho_\gamma^\beta(t,x) := \Phi^{-1}(t^{-1})^{-\gamma}(|x|^\beta \wedge 1)\rho(t,x), \quad t > 0, x \in \mathbb{R}^d.$$

Note that $\rho_0^0(t,x) = \rho(t,x)$.

Remark 2.5 Recall that Φ is increasing. Thus it is straightforward to see that the following inequalities are true: for $T \geq 1$,

$$\rho_{\gamma_1}^\beta(t,x) \leq \Phi^{-1}(T^{-1})^{\gamma_2-\gamma_1}\rho_{\gamma_2}^\beta(t,x), \qquad (t,x) \in (0,T]\times\mathbb{R}^d, \quad \gamma_2 \leq \gamma_1, \tag{2.14}$$

$$\rho_\gamma^{\beta_1}(t,x) \leq \rho_\gamma^{\beta_2}(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d, \quad 0 \leq \beta_2 \leq \beta_1. \tag{2.15}$$

We record the following inequality: for every $T \geq 1$, $t \in (0,T]$ and $\beta < \delta_1$,

$$\begin{aligned} \int_{\Phi^{-1}(T^{-1})/\Phi^{-1}(t^{-1})}^1 r^{\beta-1}\Phi(r^{-1})dr &\leq \frac{1}{a_1(\delta_1-\beta)}\left(\frac{\Phi^{-1}(T^{-1})}{\Phi^{-1}(t^{-1})}\right)^\beta \Phi\!\left(\frac{\Phi^{-1}(t^{-1})}{\Phi^{-1}(T^{-1})}\right) \\ &\leq \frac{\Phi^{-1}(T^{-1})^{\beta-2}}{a_1(\delta_1-\beta)}t^{-1}\Phi^{-1}(t^{-1})^{-\beta}. \end{aligned} \tag{2.16}$$

The first inequality follows immediately by using the lower scaling to get that for $1 \geq r \geq \lambda^{-1}$, $\Phi(r^{-1}) \leq a_1^{-1}\lambda^{-\delta_1}r^{-\delta_1}\Phi(\lambda)$. The second inequality follows from (2.1).

For the remainder of this paper we always assume that (1.5) holds. The following result is a generalization of [6, Lemma 2.1].

Lemma 2.6 *(a) For every $T \ge 1$, there exists $c_1 = c_1(d, \delta_1, a_1, C_*, T, \Phi^{-1}(T^{-1})) > 0$ such that for $0 < t \le T$, all $\beta \in [0, \delta_1)$ and $\gamma \in \mathbb{R}$,*

$$\int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\,dx \le \frac{c_1}{\delta_1 - \beta} t^{-1} \Phi^{-1}(t^{-1})^{-\gamma-\beta}\,. \tag{2.17}$$

(b) For every $T \ge 1$, there exists $C_0 = C_0(T) = C_0(d, \delta_1, a_1, C_*, T, \Phi^{-1}(T^{-1})) > 0$ such that for all $\beta_1, \beta_2 \ge 0$ with $\beta_1 + \beta_2 < \delta_1$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $0 < s < t \le T$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_{\gamma_1}^{\beta_1}(t-s, x-z) \rho_{\gamma_2}^{\beta_2}(s,z)\,dz \\
&\le \frac{C_0}{\delta_1 - \beta_1 - \beta_2} \Big((t-s)^{-1} \Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1-\beta_2} \Phi^{-1}(s^{-1})^{-\gamma_2} \\
&\quad + \Phi^{-1}((t-s)^{-1})^{-\gamma_1} s^{-1} \Phi^{-1}(s^{-1})^{-\gamma_2-\beta_1-\beta_2} \Big) \rho(t,x) \\
&\quad + \frac{C_0}{\delta_1 - \beta_1 - \beta_2} (t-s)^{-1} \Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1} \Phi^{-1}(s^{-1})^{-\gamma_2} \rho_0^{\beta_2}(t,x) \\
&\quad + \frac{C_0}{\delta_1 - \beta_1 - \beta_2} \Phi^{-1}((t-s)^{-1})^{-\gamma_1} s^{-1} \Phi^{-1}(s^{-1})^{-\gamma_2-\beta_2} \rho_0^{\beta_1}(t,x)\,.
\end{aligned} \tag{2.18}$$

(c) Let $T \ge 1$. For all $\beta_1, \beta_2 \ge 0$ with $\beta_1 + \beta_2 < \delta_1$, and all $\theta, \eta \in [0,1], \gamma_1, \gamma_2 \in \mathbb{R}$ satisfying $\mathbf{1}_{\gamma_1 \ge 0}(\gamma_1/2) + \mathbf{1}_{\gamma_1 < 0}(\gamma_1/\delta_1) + \beta_1/2 + 1 - \theta > 0$ and $\mathbf{1}_{\gamma_2 \ge 0}(\gamma_2/2) + \mathbf{1}_{\gamma_2 < 0}(\gamma_2/\delta_1) + \beta_2/2 + 1 - \eta > 0$, there exists $c_2 > 0$ such that for all $0 < t \le T$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} (t-s)^{1-\theta} \rho_{\gamma_1}^{\beta_1}(t-s, x-z) s^{1-\eta} \rho_{\gamma_2}^{\beta_2}(s,z)\,dz\,ds \\
&\le c_2 t^{2-\theta-\eta} \left(\rho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2} + \rho^{\beta_1}_{\gamma_1+\gamma_2+\beta_2} + \rho^{\beta_2}_{\gamma_1+\gamma_2+\beta_1} \right)(t,x)\,.
\end{aligned} \tag{2.19}$$

Moreover, when we further assume that $\gamma_1, \gamma_2 \ge 0$, we can take that

$$c_2 = 4 \frac{C_0(T)}{\delta_1 - \beta_1 - \beta_2} B\left((\gamma_1 + \beta_1)/2 + 1 - \theta, \gamma_2 + \beta_2/2 + 1 - \eta \right). \tag{2.20}$$

Proof. (a) Let $c_1 = c_1(d) = d|B(0,1)|$ and $T_1 = \Phi^{-1}(T^{-1}) \le 1$. We have that for all $0 < t \le T$,

$$\begin{aligned}
&\Phi^{-1}(t^{-1})^\gamma \int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\,dx = \int_{\mathbb{R}^d} \left(|x|^\beta \wedge 1 \right) \rho(t,x)\,dx \\
&\le c_1 \int_0^{T_1/\Phi^{-1}(t^{-1})} r^{\beta+d-1} \frac{\Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} \right)^{-1} \right)}{\left(\frac{T_1}{\Phi^{-1}(t^{-1})} \right)^d} dr + \\
&\qquad + c_1 \int_{T_1/\Phi^{-1}(t^{-1})}^1 r^{\beta-1} \Phi(r^{-1}) dr + c_1 \int_1^\infty \frac{\Phi(r^{-1})}{r} dr \\
&\le \frac{c_1 T_1^\beta}{\beta + d} t^{-1} \Phi^{-1}(t^{-1})^d \Phi^{-1}(t^{-1})^{-\beta-d} + c_1 \int_{T_1/\Phi^{-1}(t^{-1})}^1 r^{\beta-1} \Phi(r^{-1}) dr + c_1 \int_0^1 \frac{\Phi(r)}{r} dr
\end{aligned} \tag{2.21}$$

$$\le c_1 d^{-1} t^{-1}\Phi^{-1}(t^{-1})^{-\beta} + \frac{c_1 T_1^{\beta-2}}{a_1(\delta_1-\beta)} t^{-1}\Phi^{-1}(t^{-1})^{-\beta} + c_1 C_*$$
$$\le c_1(d^{-1} + T_1^{-2} a_1^{-1}\delta_1^{-1}(\delta_1-\beta)^{-1} + C_* a_1^{-1/2} T) t^{-1}\Phi^{-1}(t^{-1})^{-\beta},$$

where in the second to last line we used (2.16) to estimate the second term in (2.21) and used (1.5) to estimate the last term in (2.21), and in the last line we used the assumption $\beta \in [0,\delta_1)$ and the inequality $t\Phi^{-1}(t^{-1})^{\beta} \le t(a_1^{-1/\delta_1}(T/t)^{1/\delta_1})^{\beta} \le a_1^{-\beta/\delta_1} T \le a_1^{-1} T$ which follows from (2.4) with $\lambda = T/t$ and $r_0 = r = T^{-1}$.
(b) Let $c_2 = (2(2/a_1)^{1/\delta_1}/\Phi^{-1}((2T)^{-1}))^{d+2}$. As in the display after [6, (2.5)], we have that

$$\left(|x-z|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right) \le (|x-z|^{\beta_1+\beta_2}\wedge 1) + \left(|x-z|^{\beta_1}\wedge 1\right)\left(|x|^{\beta_2}\wedge 1\right).$$

By using this and (2.8), we have

$$\begin{aligned}
&\rho^{\beta_1}_{\gamma_1}(t-s,x-z)\rho^{\beta_2}_{\gamma_2}(s,z)\\
=&\Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\left(|x-z|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right)\rho(t-s,x-z)\rho(s,z)\\
\le& c_2\Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\left(|x-z|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right)\big(\rho(t-s,x-z)+\rho(s,z)\big)\rho(t,x)\\
\le& c_2\Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\Big\{(|x-z|^{\beta_1+\beta_2}\wedge 1) + \left(|x-z|^{\beta_1}\wedge 1\right)\left(|x|^{\beta_2}\wedge 1\right)\Big\}\\
&\quad\times\rho(t-s,x-z)\rho(t,x)\\
&+ c_2\Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\Big\{(|z|^{\beta_1+\beta_2}\wedge 1) + \left(|x|^{\beta_1}\wedge 1\right)\left(|z|^{\beta_2}\wedge 1\right)\Big\}\rho(s,z)\rho(t,x)\\
=& c_2\Phi^{-1}(s^{-1})^{-\gamma_2}\Big\{\rho^{\beta_1+\beta_2}_{\gamma_1}(t-s,x-z)\rho(t,x) + \rho^{\beta_1}_{\gamma_1}(t-s,x-z)\rho^{\beta_2}_0(t,x)\Big\}\\
&+ c_2\Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Big\{\rho^{\beta_1+\beta_2}_{\gamma_2}(s,z)\rho(t,x) + \rho^{\beta_2}_{\gamma_2}(s,z)\rho^{\beta_1}_0(t,x)\Big\}.
\end{aligned}$$

Since $\beta_1+\beta_2<\delta_1$, now (2.18) follows by integrating the above and using (2.17).
(c) By integrating (2.18) and using Lemma 2.3, we get (2.19). When we further assume that $\gamma_1,\gamma_2\ge 0$, by integrating (2.18) and using the last part of Lemma 2.3, we get (2.19) with the constant

$$\begin{aligned}
&C_0\left(B\left(\frac{\gamma_1+\beta_1+\beta_2}{2}+1-\theta, \frac{\gamma_2+2}{2}+1-\eta\right) + B\left(\frac{\gamma_2+\beta_1+\beta_2}{2}+1-\eta, \frac{\gamma_1+2}{2}+1-\theta\right)\right.\\
&\left.+B\left(\frac{\gamma_1+\beta_1}{2}+1-\theta, \frac{\gamma_2+2}{2}+1-\eta\right) + B\left(\frac{\gamma_2+\beta_2}{2}+1-\eta, \frac{\gamma_1+2}{2}+1-\theta\right)\right),
\end{aligned}$$

which is, using that the beta function B is symmetric and non-increasing in each variable, less than or equal to $4C_0 B\left((\gamma_1+\beta_1)/2+1-\theta, \gamma_2+\beta_2/2+1-\eta\right)$. □

Lemma 2.7 *Suppose* $0<t_1\le t_2<\infty$. *For* $\beta\in(0,\delta_1/2)$,

$$\lim_{h\downarrow 0}\sup_{x,y\in\mathbb{R}^d, t\in[t_1,t_2]}\left(\int_0^h + \int_{t-h}^t\right)\int_{\mathbb{R}^d}\rho^{\beta}_0(t-s,x-z)(\rho^{\beta}_0(s,z-y)+\rho^0_{\beta}(s,z-y))dzds = 0.$$

Proof. We first apply Lemma 2.6(b) and then use Remark 2.5, to get that for $t \in [t_1, t_2]$,

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho_0^\beta(t-s, x-z)(\rho_0^\beta(s, z-y) + \rho_\beta^0(s, z-y))dz \\ &\le c_1((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta} + s^{-1}\Phi^{-1}(s^{-1})^{-\beta})\rho(t_1, 0).\end{aligned}$$

Now the conclusion of the lemma follows immediately from Lemma 2.4. □

3 Analysis of the heat kernel of $\mathcal{L}^{\mathfrak{K}}$

Throughout this paper, $Y = (Y_t, \mathbb{P}_x)$ is a subordinate Brownian motion via an independent subordinator with Laplace exponent ϕ and Lévy measure μ. The Lévy density of Y, denoted by j, is given by

$$j(x) = j(|x|) = \int_0^\infty (4\pi s)^{-d/2} e^{-|x|^2/4s} \, \mu(ds) \, .$$

It is well known that there exists $c = c(d)$ depending only on d such that

$$j(r) \le c\frac{\phi(r^{-2})}{r^d}, \quad r > 0 \tag{3.1}$$

(see [2, (15)]). The function $r \mapsto j(r)$ is non-decreasing. Recall that we have assumed that $r \mapsto \Phi(r)(= \phi(r^2))$, the radial part of the characteristic exponent Φ of Y, satisfies the weak lower scaling condition at infinity in (2.2).

Suppose that $Z = (Z_t, \mathbb{P}_x)$ is a purely discontinuous symmetric Lévy process with characteristic exponent ψ_Z such that its Lévy measure admits a density j_Z satisfying

$$\widehat{\gamma}_0^{-1} j(|x|) \le j_Z(x) \le \widehat{\gamma}_0 j(|x|) \, , \qquad x \in \mathbb{R}^d \, , \tag{3.2}$$

for some $\widehat{\gamma}_0 \ge 1$. Hence, $\int_{\mathbb{R}^d} j_Z(x)dx = \infty$. The characteristic exponents of Z, respectively Y, are given by

$$\psi_Z(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j_Z(y) \, dy \, , \qquad \Phi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j(|y|) \, dy \, ,$$

and satisfy

$$\widehat{\gamma}_0^{-1} \Phi(|\xi|) \le \psi_Z(\xi) \le \widehat{\gamma}_0 \Phi(|\xi|) \, , \quad \xi \in \mathbb{R}^d \, . \tag{3.3}$$

Let ψ denote the radial nondecreasing majorant of the characteristic exponent of Z, i.e., $\psi(r) := \sup_{|z| \le r} \psi_Z(z)$. Clearly

$$\widehat{\gamma}_0^{-1} \Phi(r) \le \psi(r) \le \widehat{\gamma}_0 \Phi(r) \, , \quad r > 0, \quad \text{and} \quad \widehat{\gamma}_0^{-2} \psi(|\xi|) \le \psi_Z(\xi) \le \psi(|\xi|) \, , \quad \xi \in \mathbb{R}^d,$$

and thus ψ also satisfies the weak lower scaling condition at infinity in (2.2).

By (3.1) and (3.2),

$$j_Z(x) \le \widehat{\gamma}_0 \frac{\Phi(|x|^{-1})}{|x|^d}. \tag{3.4}$$

Moreover, for every $n \in \mathbb{Z}_+$,

$$\begin{aligned}\int_{\mathbb{R}^d} \left|\mathbb{E}\left[e^{i\xi\cdot Z_t}\right]\right| |\xi|^n \, d\xi &= \int_{\mathbb{R}^d} e^{-t\psi_Z(\xi)}|\xi|^n \, d\xi \le \int_{\mathbb{R}^d} e^{-t\widehat{\gamma}_0^{-1}\Phi(|\xi|)}|\xi|^n \, d\xi \\ &\le c\left(\int_0^1 r^{d-1+n} dr + \int_1^\infty r^{d-1+n} e^{-t\widehat{\gamma}_0^{-1}a_1 r^{\delta_1}} \, dr\right) < \infty \,. \end{aligned} \tag{3.5}$$

It follows from [13, Proposition 2.5(xii) and Proposition 28.1] that Z_t has a density

$$p(t,x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\psi_Z(\xi)} \, d\xi = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \cos(x\cdot\xi) e^{-t\psi_Z(\xi)} \, d\xi,$$

which is infinitely differentiable in x. Let $\mathcal{L}$ be the infinitesimal generator of Z.

Lemma 3.1 *(a) For every $x \in \mathbb{R}^d$, the function $t \mapsto p(t,x)$ is differentiable and*

$$\frac{\partial p(t,x)}{\partial t} = -(2\pi)^{-d/2} \int_{\mathbb{R}^d} \cos(x\cdot\xi)\psi_Z(\xi) e^{-t\psi_Z(\xi)} \, d\xi = \mathcal{L}p(t,x) \,.$$

(b) For every $\varepsilon > 0$ there exists a constant $c = c(d, \delta_1, a_1, \widehat{\gamma}_0, \varepsilon) > 0$ such that for all $s, t \ge \varepsilon$ and all $x, y \in \mathbb{R}^d$,

$$|p(t,x) - p(s,y)| \le c\left(|t-s| + |x-y|\right) \,.$$

Proof. (a) Note that for any $t \ge 0$ and any $h \in \mathbb{R}$ such that $t + h \ge 0$,

$$\frac{p(t+h,x) - p(t,x)}{h} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \cos(x\cdot\xi) e^{-t\psi_Z(\xi)} \frac{e^{-h\psi_Z(\xi)} - 1}{h} d\xi.$$

The absolute value of the integrand is bounded by $2\widehat{\gamma}_0\Phi(|\xi|)e^{-\widehat{\gamma}_0^{-1}\Phi(|\xi|)}$ which is integrable since $\Phi(|\xi|) \le |\xi|^2$. The claim follows from the dominated convergence theorem by letting $h \to 0$. The last equality in the statement of the lemma follows from [9, Example 4.5.5].
(b) By the triangle inequality we have that

$$\begin{aligned}|p(t,x) - p(s,y)| &\le \int_{\mathbb{R}^d} \left|\cos(x\cdot\xi) - \cos(y\cdot\xi)\right| e^{-t\psi_Z(\xi)} \, d\xi \\ &+ \int_{\mathbb{R}^d} \left|\cos(y\cdot\xi)\right| \left|e^{-t\psi_Z(\xi)} - e^{-s\psi_Z(\xi)}\right| \, d\xi =: I_1 + I_2 \,.\end{aligned}$$

Clearly, $|\cos(x\cdot\xi) - \cos(y\cdot\xi)| \le |x\cdot\xi - y\cdot\xi| \le |x-y||\xi|$, which implies that, by (3.5),

$$I_1 \le |x-y| \int_{\mathbb{R}^d} |\xi| e^{-t\psi_Z(\xi)} \, d\xi \le |x-y| \int_{\mathbb{R}^d} |\xi| e^{-\varepsilon\widehat{\gamma}_0^{-1}\Phi(|\xi|)} \, d\xi = c_1(\widehat{\gamma}_0, \varepsilon)|x-y| \,.$$

In order to estimate I_2, without loss of generality we assume that $s \le t$. Then by the mean value theorem we have that

$$\left|e^{-t\psi_Z(\xi)} - e^{-s\psi_Z(\xi)}\right| \le |t-s|\psi_Z(\xi)e^{-s\psi_Z(\xi)} \le \widehat{\gamma}_0|t-s|\Phi(|\xi|)e^{-\varepsilon\widehat{\gamma}_0^{-1}\Phi(|\xi|)}.$$

Therefore, by (3.5),

$$I_2 \le \widehat{\gamma}_0|t-s|\int_{\mathbb{R}^d}|\xi|^2 e^{-\varepsilon\widehat{\gamma}_0^{-1}\Phi(|\xi|)}\,d\xi = c_2(\widehat{\gamma}_0,\varepsilon)|t-s|.$$

The claim follows by taking $c = c_1 \vee c_2$. □

Define the Pruitt function $\mathcal{P}$ by

$$\mathcal{P}(r) = \int_{\mathbb{R}^d}\left(\frac{|x|^2}{r^2} \wedge 1\right) j(x)dx. \qquad (3.6)$$

By [2, (6) and Lemma 1],

$$\frac{1}{2\widehat{\gamma}_0}\psi(r^{-1}) \le \frac{1}{2}\Phi(r^{-1}) \le \mathcal{P}(r) \le \frac{d\pi^2}{2}\Phi(r^{-1}) \le \frac{\widehat{\gamma}_0 d\pi^2}{2}\psi(r^{-1}). \qquad (3.7)$$

In this paper we will use (3.7) several times.

We next discuss the upper estimate of $p(t,x)$ and its derivatives for $0 < t \le T$ and all $x \in \mathbb{R}^d$ using [10, Theorem 3].

Proposition 3.2 *For each $T \ge 1$ and $k \in \mathbb{Z}_+$, there is a constant $c = c(k, T, \widehat{\gamma}_0, d, \delta_1, a_1) \ge 1$ such that*

$$|\nabla^k p(t,x)| \le c\,t\,(\Phi^{-1}(t^{-1}))^k \rho(t,x), \qquad 0 < t \le T, x \in \mathbb{R}^d,$$

where ∇^k stands for the k-th order gradient with respect to the spatial variable x.

Proof. First, we recall that $\int_{\mathbb{R}^d} j_Z(x)dx = \infty$. Let $f(s) := \frac{\Phi(s^{-1})}{s^d}$. Then by (3.4) we have $j_Z(x) \le C\widehat{\gamma}_0 f(|x|)$. Thus for $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\int_A j_Z(x)dx \le C\widehat{\gamma}_0\int_A \frac{\Phi(|x|^{-1})}{|x|^d}dx \le C\widehat{\gamma}_0\frac{\Phi(\mathrm{dist}(0,A)^{-1})}{\mathrm{dist}(0,A)^d}|A| \le C\widehat{\gamma}_0 f(\mathrm{dist}(0,A))(\mathrm{diam}(A))^d.$$

Therefore, [10, (1)] holds with $\gamma = d$ and $M_1 = C\widehat{\gamma}_0$.

Since $(s \vee |y|) - (|y|/2) \ge s/2$ for $s > 0$, using (3.7) in the last inequality we have that for $s, r > 0$,

$$\int_{|y|>r} f((s\vee|y|) - (|y|/2)) j_Z(y)dy \le 2^d\frac{\Phi((s/2)^{-1})}{s^d}\int_{|y|>r} j_Z(y)dy$$
$$= 2^d\frac{\Phi((s/2)^{-1})}{s^d}\int_{|y|>r}(\frac{|y|^2}{r^2}\wedge 1)j_Z(y)dy \le 2^{d+2}\frac{\Phi(s^{-1})}{s^d}\mathcal{P}(r) \le 2^{d+1}\widehat{\gamma}_0 d\pi^2 f(s)\psi(r^{-1}). \qquad (3.8)$$

Therefore, [10, (2)] holds with $M_1 = 2^{d+1}\widehat{\gamma}_0 d\pi^2$.

Furthermore, by (3.3) and (2.2), for $k \in \mathbb{Z}_+$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-t\psi_Z(\xi)}|\xi|^k\, d\xi \le \int_{\mathbb{R}^d} e^{-t\widehat{\gamma}_0^{-1}\Phi(|\xi|)}|\xi|^k\, d\xi = d|B(0,1)| \int_0^\infty r^{d+k-1} e^{-t\widehat{\gamma}_0^{-1}\Phi(r)}\, dr \\
&= d|B(0,1)| \int_0^\infty (\Phi^{-1}(s/t))^{d+k-1} e^{-\widehat{\gamma}_0^{-1}s} (\Phi^{-1})'(s/t) t^{-1}\, ds \\
&\le d|B(0,1)| \int_0^1 (\Phi^{-1}(s/t))^{d+k-1} (\Phi^{-1})'(s/t) t^{-1}\, ds \\
&\quad + d|B(0,1)| \sum_{n=1}^\infty e^{-\widehat{\gamma}_0^{-1}2^{n-1}} \int_{2^{n-1}}^{2^n} (\Phi^{-1}(s/t))^{d+k-1} (\Phi^{-1})'(s/t) t^{-1}\, ds \\
&= \frac{d|B(0,1)|}{d+k} \int_0^1 ((\Phi^{-1}(s/t))^{d+k})'\, ds + \frac{d|B(0,1)|}{d+k} \sum_{n=1}^\infty e^{-\widehat{\gamma}_0^{-1}2^{n-1}} \int_{2^{n-1}}^{2^n} ((\Phi^{-1}(s/t))^{d+k})'\, ds \\
&\le \frac{d|B(0,1)|}{d+k} \left((\Phi^{-1}(t^{-1}))^{d+k} + \sum_{n=1}^\infty e^{-\widehat{\gamma}_0^{-1}2^{n-1}} (\Phi^{-1}(2^n/t))^{d+k} \right).
\end{aligned}$$

Since $t \le T$, by (2.4) we have $\Phi^{-1}(2^n/t) \le c_0 2^{n/\delta_1}\Phi^{-1}(t^{-1})$. Thus we see that

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-t\psi_Z(\xi)}|\xi|^k\, d\xi \le \frac{d|B(0,1)|}{d+k} (\Phi^{-1}(t^{-1}))^{d+k} (1 + c_0 \sum_{n=1}^\infty 2^{n(d+k)/\delta_1} e^{-\widehat{\gamma}_0^{-1}2^{n-1}}) \\
&\le c_1 \Phi^{-1}(t^{-1})^{d+k} \le c_2 \psi^-(t^{-1})^{d+k},
\end{aligned}$$

where $c_2 = c_2(k) > 0$ and ψ^- is the generalized inverse of ψ: $\psi^-(s) = \inf\{u \ge 0 : \psi(u) \ge s\}$. Therefore, [10, (8)] holds with the set $(0, T]$.

We have checked that the conditions in [10, Theorem 3] hold for all $k \in \mathbb{Z}_+$. Thus by [10, Theorem 3] (with $n = d+2$ in [10, Theorem 3]), there exists $c_3(k) > 0$ such that for $t \le T$,

$$\begin{aligned}
&|\nabla^k p(t,x)| \le c_3 \psi^-(t^{-1})^k \left(\psi^-(t^{-1})^d \wedge \left(\frac{t\Phi(|x|^{-1})}{|x|^d} + \frac{\psi^-(t^{-1})^d}{(1+|x|\psi^-(t^{-1}))^{d+2}} \right) \right) \\
&\le c_4 \Phi^{-1}(t^{-1})^k \left(\Phi^{-1}(t^{-1})^d \wedge \left(\frac{t\Phi(|x|^{-1})}{|x|^d} + \frac{\Phi^{-1}(t^{-1})^d}{(1+|x|\Phi(t^{-1}))^{d+2}} \right) \right).
\end{aligned}$$

When $|x|\Phi^{-1}(t^{-1}) \ge 1$ (so that $t\Phi(|x|^{-1}) \le 1$),

$$\frac{\Phi^{-1}(t^{-1})^d}{(1+|x|\Phi(t^{-1}))^{d+2}} \le \frac{\Phi^{-1}(t^{-1})^d}{(|x|\Phi^{-1}(t^{-1}))^{d+2}} = |x|^{-d} \left(\frac{\Phi^{-1}(\Phi(|x|^{-1}))}{\Phi^{-1}(\frac{\Phi(|x|^{-1})}{t\Phi(|x|^{-1})})} \right)^2 \le |x|^{-d}(t\Phi(|x|^{-1})).$$

In the last inequality we have used (2.3). Therefore using Proposition 2.1 we conclude that for all $0 < t \le T$ and $x \in \mathbb{R}^d$,

$$|\nabla^k p(t,x)| \le c_4 \Phi^{-1}(t^{-1})^k \left(\Phi^{-1}(t^{-1})^d \wedge \frac{t\Phi(|x|^{-1})}{|x|^d} \right) \le c_4 2^{d+2} t \Phi^{-1}(t^{-1})^k \rho(t,x)\,.$$

□

3.1 Further properties of $p(t,x)$

We will need the following simple inequality, cf. [6, (2.9)]: Let $a>0$ and $x\in\mathbb{R}^d$. For every $z\in\mathbb{R}^d$ such that $|z|\le(2a)\vee(|x|/2)$, we have

$$(a+|x+z|)^{-1}\le 4(a+|x|)^{-1}. \tag{3.9}$$

Indeed, if $|z|\le 2a$, then $a+|x|\le a+|x+z|+|z|\le a+|x+z|+2a\le 4(a+|x+z|)$. If $|z|\le|x|/2$, then $4(a+|x+z|)\ge 4a+4|x|-4|z|\ge 4a+4|x|-2|x|\ge a+|x|$.

For a function $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$, we define

$$\delta_f(t,x;z):=f(t,x+z)+f(t,x-z)-2f(t,x). \tag{3.10}$$

Also, $f(x\pm z)$ is an abbreviation for $f(x+z)+f(x-z)$.

The following result is the counterpart of [6, Lemma 2.3].

Proposition 3.3 *For every $T\ge1$, there exists a constant $c=c(T,d,\widehat{\gamma}_0,d,\delta_1,a_1)>0$ such that for every $t\in(0,T]$ and $x,x',z\in\mathbb{R}^d$,*

$$|p(t,x)-p(t,x')|\le c\left((\Phi^{-1}(t^{-1})|x-x'|)\wedge1\right)t\left(\rho(t,x)+\rho(t,x')\right), \tag{3.11}$$

$$|\delta_p(t,x;z)|\le c\left((\Phi^{-1}(t^{-1})|z|)^2\wedge1\right)t\left(\rho(t,x\pm z)+\rho(t,x)\right), \tag{3.12}$$

and

$$\begin{aligned}|\delta_p(t,x;z)-\delta_p(t,x';z)|\le{}& c\left((\Phi^{-1}(t^{-1})|x-x'|)\wedge1\right)\left((\Phi^{-1}(t^{-1})|z|)^2\wedge1\right)\\ &\times t\left(\rho(t,x\pm z)+\rho(t,x)+\rho(t,x'\pm z)+\rho(t,x')\right).\end{aligned} \tag{3.13}$$

Proof. (1) Note that, by Proposition 3.2 with $k=0$, (3.11) is clearly true if $\Phi^{-1}(t^{-1})|x-y|\ge1$. Thus we assume that $\Phi^{-1}(t^{-1})|x-y|\le1$. We use Proposition 3.2 for $k=1$ and

$$p(t,x)-p(t,y)=(x-y)\cdot\int_0^1\nabla p(t,x+\theta(y-x))\,d\theta \tag{3.14}$$

to estimate $|p(t,x)-p(t,y)|\le c_1t\Phi^{-1}(t^{-1})|x-y|\int_0^1\rho(t,x+\theta(y-x))d\theta$. Since $\theta|y-x|\le 1/\Phi^{-1}(t^{-1})$, we get from (3.9) that

$$\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x+\theta(y-x)|\right)^{-1}\le4\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x|\right)^{-1}.$$

Therefore using (2.1) we have $|p(t,x)-p(t,y)|\le c_2|x-y|\Phi^{-1}(t^{-1})t\rho(t,x)$, $t\in(0,T]$.
(2) Note that (3.12) is clearly true if $\Phi^{-1}(t^{-1})|z|\ge1$. In order to prove (3.12) when $\Phi^{-1}(t^{-1})|z|\le1$ we use (3.14) twice to obtain

$$\delta_p(t,x;z)\;=\;z\cdot\int_0^1\left(\nabla p(t,x+\theta z)-\nabla p(t,x-\theta z)\right)d\theta$$

$$= \quad 2(z \otimes z) \cdot \int_0^1 \int_0^1 \theta \nabla^2 p(t, x + (1-2\theta')\theta z)\, d\theta'\, d\theta\,. \qquad (3.15)$$

Note that $|(1-2\theta')\theta z| \le |z| \le \frac{1}{\Phi^{-1}(t^{-1})}$. Hence, by Proposition 3.2 and (3.9) we get the estimate

$$\left|\theta \nabla^2 p(t, x + (1-2\theta')\theta z)\right| \le c_3 \left(\Phi^{-1}(t^{-1})\right)^2 t\rho(t,x)\,.$$

Therefore, $\delta_p(t,x;z) \le c_4 \left(\Phi^{-1}(t^{-1})|z|\right)^2 t\rho(t,x)$, $t \in (0,T]$.
(3) It follows from (3.12) that it suffices to prove (3.13) in the case when $\Phi^{-1}(t^{-1})|x-y| \le 1$. To do this, we start with the subcase when $\Phi^{-1}(t^{-1})|z| \le 1$ and $\Phi^{-1}(t^{-1})|x-y| \le 1$. Then by (3.15),

$$\begin{aligned} &|\delta_p(t,x;z) - \delta_p(t,y;z)| \\ &\le \quad c_5|x-y| \cdot |z|^2 \int_0^1 \int_0^1 \int_0^1 |\nabla^3 p(t, x + (1-2\theta')\theta z + \theta''(y-x))|\, d\theta d\theta' d\theta''. \end{aligned}$$

Note that $|(1-2\theta')\theta z + \theta''(y-x))| \le \frac{2}{\Phi^{-1}(t^{-1})}$. Hence, by Proposition 3.2 and (3.9) we get

$$|\delta_p(t,x;z) - \delta_p(t,y;z)| \quad \le \quad c_6 \Phi^{-1}(t^{-1})|x-y|(\Phi^{-1}(t^{-1})|z|)^2 t\rho(t,x)\,.$$

If $\Phi^{-1}(t^{-1})|z| \ge 1$ and $\Phi^{-1}(t^{-1})|x-y| \le 1$, then again by Proposition 3.2 and (3.9),

$$\begin{aligned} &|\delta_p(t,x;z) - \delta_p(t,y;z)| \\ &\le \quad c_7 \left(|x-y| \int_0^1 |\nabla p(t, x \pm z + \theta(y-x))|\, d\theta + |x-y| \int_0^1 |\nabla p(t, x + \theta(y-x))|\, d\theta \right) \\ &\le \quad c_8 \Phi^{-1}(t^{-1})|x-y| \left(t\rho(t, x \pm z) + t\rho(t,x) \right)\,, \quad t \in (0,T]. \end{aligned}$$

□

The following result is the counterpart of [6, Theorem 2.4].

Theorem 3.4 *For every $T \ge 1$, there exists a constant $c = c(T, d, \widehat{\gamma}_0, d, \delta_1, a_1) > 0$ such that for all $t \in (0,T]$ and all $x, x' \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |\delta_p(t,x;z)|\, j(|z|)\, dz \le c\rho(t,x) \qquad (3.16)$$

and

$$\int_{\mathbb{R}^d} |\delta_p(t,x;z) - \delta_p(t,x';z)|\, j(|z|)\, dz \le c \left((\Phi^{-1}(t^{-1})|x-x'|) \wedge 1 \right) (\rho(t,x) + \rho(t,x'))\,. \quad (3.17)$$

Proof. By (3.12) we have

$$\int_{\mathbb{R}^d} |\delta_p(t,x;z)|\, j(|z|)\, dz$$

$$
\begin{aligned}
&\le c_0 \int_{\mathbb{R}^d} \left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\right) t\,(\rho(t,x\pm z)+\rho(t,x))\, j(|z|)\,dz \qquad (3.18)\\
&= c_0 \left(\int_{\mathbb{R}^d} \left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\right) t\rho(t,x\pm z) j(|z|)\,dz + t\rho(t,x)\mathcal{P}(1/\Phi^{-1}(t^{-1}))\right)\\
&=: c_0\,(I_1+I_2)\,.
\end{aligned}
$$

Clearly by (3.7), $I_2 \le c_1 t\rho(t,x)\Phi(\Phi^{-1}(t^{-1})) = c_1\rho(t,x)$. Next,

$$
\begin{aligned}
I_1 &= \Phi^{-1}(t^{-1})^2 \int_{\Phi^{-1}(t^{-1})|z|\le 1} |z|^2 t\rho(t,x\pm z) j(|z|)\,dz + \int_{\Phi^{-1}(t^{-1})|z|>1} t\rho(t,x\pm z) j(|z|)\,dz\\
&=: I_{11}+I_{12}\,.
\end{aligned}
$$

By using (3.9) in the first inequality below and (3.7) in the third, we further have

$$
\begin{aligned}
I_{11} &\le 4^{d+1} t\rho(t,x) \int_{|z|\le \frac{1}{\Phi^{-1}(t^{-1})}} ((\Phi^{-1}(t^{-1})|z|)^2\wedge 1) j(|z|)\,dz\\
&\le 4^{d+1} t\rho(t,x)\mathcal{P}(1/\Phi^{-1}(t^{-1})) \le c_2\rho(t,x)\,.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
I_{12} &\le t\int_{|z|>\frac{1}{\Phi^{-1}(t^{-1})}} \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})}\right)^{-1}\right)\left(\frac{1}{\Phi(t^{-1})}\right)^{-d} j(|z|)\,dz\\
&= \Phi^{-1}(t^{-1})^d \int_{|z|>\frac{1}{\Phi^{-1}(t^{-1})}} ((\Phi^{-1}(t^{-1})|z|)^2\wedge 1) j(|z|)\,dz\\
&\le \Phi^{-1}(t^{-1})^d \mathcal{P}(1/\Phi^{-1}(t^{-1})) \le c_3\Phi^{-1}(t^{-1})^d \Phi(\Phi^{-1}(t^{-1})) = c_3\Phi^{-1}(t^{-1})^d t^{-1}\,,
\end{aligned}
$$

where in the last line we used (3.7). If $|x|\le 2/\Phi^{-1}(t^{-1})$, we have that

$$
\rho(t,x) \ge \Phi\left(\left(\frac{3}{\Phi^{-1}(t^{-1})}\right)^{-1}\right)\left(\frac{3}{\Phi(t^{-1})}\right)^{-d} \ge c_4 t^{-1}\Phi^{-1}(t^{-1})^d,
$$

implying that $I_{12}\le c_5\rho(t,x)$.

If $|x| > 2/\Phi^{-1}(t^{-1})$, then by (3.7),

$$
\begin{aligned}
I_{12} &= \left(\int_{\frac{|x|}{2}\ge|z|>\frac{1}{\Phi^{-1}(t^{-1})}} + \int_{|z|>\frac{|x|}{2}}\right) t\rho(t,x\pm z) j(|z|)\,dz\\
&\le c_6\left(t\rho(t,x)\int_{\frac{|x|}{2}\ge|z|>\frac{1}{\Phi^{-1}(t^{-1})}} j(|z|)\,dz + j(|x|/2)\int_{|z|>\frac{|x|}{2}} t\rho(t,x\pm z)\,dz\right)\\
&\le c_7\left(t\rho(t,x)\int_{|z|>\frac{1}{\Phi^{-1}(t^{-1})}} j(|z|)\,dz + \frac{\Phi(2|x|^{-1})}{|x|^d}\int_{\mathbb{R}^d} t\rho(t,x\pm z)\,dz\right)\\
&\le c_7\left(t\rho(t,x)\mathcal{P}(1/\Phi^{-1}(t^{-1})) + \frac{\Phi(|x|^{-1})}{|x|^d}\right)
\end{aligned}
$$

$$\leq \; c_8\left(\rho(t,x)+\frac{\Phi(|x|^{-1})}{|x|^d}\right)\leq c_9\rho(t,x)\,,$$

where in the last line the second term is estimated by a constant times the first term in view of the assumption that $|x|>2/\Phi^{-1}(t^{-1})$. This finishes the proof of (3.16).

Next, by (3.13) we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\delta_p(t,x;z)-\delta_p(t,x';z)|\,j(|z|)\,dz &\leq c_{10}\left((\Phi^{-1}(t^{-1})|x-x'|)\wedge 1\right)\\ &\times\left\{\int_{\mathbb{R}^d}\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)\big(t\rho(t,x\pm z)+t\rho(t,x'\pm z)\big)j(|z|)\,dz\right.\\ &\left.+\big(t\rho(t,x)+t\rho(t,x')\big)\int_{\mathbb{R}^d}\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)j(|z|)\,dz\right\}\\ &\leq c_{11}\left((\Phi^{-1}(t^{-1})|x-x'|)\wedge 1\right)t^{-1}\big(t\rho(t,x)+t\rho(t,x')\big)\,,\end{aligned}$$

where the last line follows by using the estimates of the integrals I_1 and I_2 from the first part of the proof. □

3.2 Continuous dependence of heat kernels with respect to $\mathfrak{K}$

Recall that $J:\mathbb{R}^d\to(0,\infty)$ is a symmetric function satisfying (1.7). We now specify the jumping kernel j_Z. Let $\mathfrak{K}:\mathbb{R}^d\to(0,\infty)$ be a symmetric function, that is, $\mathfrak{K}(z)=\mathfrak{K}(-z)$. Assume that there are $0<\kappa_0\leq\kappa_1<\infty$ such that

$$\kappa_0\leq\mathfrak{K}(z)\leq\kappa_1\,,\qquad\text{for all } z\in\mathbb{R}^d\,. \tag{3.19}$$

Let $j^{\mathfrak{K}}(z):=\mathfrak{K}(z)J(z)$, $z\in\mathbb{R}^d$. Then $j^{\mathfrak{K}}$ satisfies (3.2) with $\widehat{\gamma}_0=\gamma_0(\kappa_1\vee\kappa_0^{-1})$. The infinitesimal generator of the corresponding symmetric Lévy process $Z^{\mathfrak{K}}$ is given by

$$\begin{aligned}\mathcal{L}^{\mathfrak{K}}f(x) &= \text{p.v.}\int_{\mathbb{R}^d}(f(x+z)-f(x))\mathfrak{K}(z)J(z)\,dz\\ &= \frac{1}{2}\text{p.v.}\int_{\mathbb{R}^d}\delta_f(x;z)\mathfrak{K}(z)J(z)\,dz\,.\end{aligned} \tag{3.20}$$

We note in passing that, when $f\in C_b^2(\mathbb{R}^d)$, it is not necessary to take the principal value in the last line above. The transition density of $Z^{\mathfrak{K}}$ (i.e., the heat kernel of $\mathcal{L}^{\mathfrak{K}}$) will be denoted by $p^{\mathfrak{K}}(t,x)$. Then by Lemma 3.1,

$$\frac{\partial p^{\mathfrak{K}}(t,x)}{\partial t}=\mathcal{L}^{\mathfrak{K}}p^{\mathfrak{K}}(t,x)\,,\qquad\lim_{t\to 0}p^{\mathfrak{K}}(t,x)=\delta_0(x)\,. \tag{3.21}$$

We will need the following observation for the next result. The inequality (2.4) implies that there exists a constant $c(\kappa_0)\geq 1$ such that

$$\Phi^{-1}((\kappa_0 t/2)^{-1})\leq a_1^{-1/\delta_1}\Phi^{-1}(T^{-1})^{-1}(1\vee(\kappa_0/2))^{1/\delta_1}\Phi^{-1}(t^{-1})\qquad\text{for all } t\in(0,T].$$

Consequently, for all $z \in \mathbb{R}^d$ and $t \in (0, T]$,

$$\left(\Phi^{-1}((\kappa_0 t/2)^{-1})|z|\right) \wedge 1 \le a_1^{-1\delta_1} \Phi^{-1}(T^{-1})^{-1} (1 \vee (\kappa_0/2))^{1/\delta_1} \left(\left(\Phi^{-1}(t^{-1})|z|\right) \wedge 1\right). \tag{3.22}$$

The following result is the counterpart of [6, Theorem 2.5], and in its proof we follow the proof of [6, Theorem 2.5] with some modifications.

Theorem 3.5 *For every $T \ge 1$, there exists a constant $c > 0$ depending on $T, d, \kappa_0, \kappa_1, \gamma_0, a_1$ and δ_1 such that for any two symmetric functions $\mathfrak{K}_1$ and $\mathfrak{K}_2$ in $\mathbb{R}^d$ satisfying* (3.19)*, every $t \in (0, T]$ and $x \in \mathbb{R}^d$, we have*

$$\begin{aligned} \left|p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x)\right| &\le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \, t\rho(t,x)\,, & (3.23)\\ \left|\nabla p^{\mathfrak{K}_1}(t,x) - \nabla p^{\mathfrak{K}_2}(t,x)\right| &\le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \Phi^{-1}(t^{-1}) t\rho(t,x) & (3.24) \end{aligned}$$

and

$$\int_{\mathbb{R}^d} \left|\delta_{p^{\mathfrak{K}_1}}(t,x;z) - \delta_{p^{\mathfrak{K}_2}}(t,x;z)\right| j(|z|)\,dz \le c\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \rho(t,x)\,. \tag{3.25}$$

Proof. (i) Using (3.21) in the second line, the fact $\mathcal{L}^{\mathfrak{K}_1}$ is self-adjoint in the third and fourth lines, we have

$$\begin{aligned} p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x) &= \int_0^t \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y) p^{\mathfrak{K}_2}(t-s,y-x)\,dy\right) ds \\ =& \int_0^t \left(\int_{\mathbb{R}^d} \left(\mathcal{L}^{\mathfrak{K}_1} p^{\mathfrak{K}_1}(s,\cdot)(y) p^{\mathfrak{K}_2}(t-s,y-x) - p^{\mathfrak{K}_1}(s,y)\mathcal{L}^{\mathfrak{K}_2} p^{\mathfrak{K}_2}(t-s,\cdot)(y-x)\right) dy\right) ds \\ =& \int_0^{t/2} \left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y) \left(\mathcal{L}^{\mathfrak{K}_1} - \mathcal{L}^{\mathfrak{K}_2}\right) p^{\mathfrak{K}_2}(t-s,\cdot)(y-x) dy\right) ds \\ &+ \int_{t/2}^t \left(\int_{\mathbb{R}^d} \left(\mathcal{L}^{\mathfrak{K}_1} - \mathcal{L}^{\mathfrak{K}_2}\right) p^{\mathfrak{K}_1}(s,\cdot)(y) p^{\mathfrak{K}_2}(t-s,y-x) dy\right) ds \\ =& \frac{1}{2}\int_0^{t/2} \left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y) \left(\int_{\mathbb{R}^d} \delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)(\mathfrak{K}_1(z) - \mathfrak{K}_2(z))J(z)dz\right) dy\right) ds \\ &+ \frac{1}{2}\int_{t/2}^t \left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(t-s,x-y) \left(\int_{\mathbb{R}^d} \delta_{p^{\mathfrak{K}_1}}(s,y;z)(\mathfrak{K}_1(z) - \mathfrak{K}_2(z))J(z)dz\right) dy\right) ds. \end{aligned}$$

By using (3.16), Proposition 3.2 and the convolution inequality (2.19), we have

$$\begin{aligned} &\int_0^{t/2} \left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y) \left(\int_{\mathbb{R}^d} \delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)(\mathfrak{K}_1(z) - \mathfrak{K}_2(z))J(z)dz\right) dy\right) ds \\ &\quad + \int_0^{t/2} \left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(s,x-y) \left(\int_{\mathbb{R}^d} \delta_{p^{\mathfrak{K}_1}}(t-s,y;z)(\mathfrak{K}_1(z) - \mathfrak{K}_2(z))J(z)dz\right) dy\right) ds \\ &\le \widehat{\gamma}_0 \|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \Bigg(\int_0^{t/2} \left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y) \left(\int_{\mathbb{R}^d} \left|\delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)\right| j(|z|)dz\right) dy\right) ds \end{aligned}$$

$$+\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(s,x-y)\left(\int_{\mathbb{R}^d}\left|\delta_{p^{\mathfrak{K}_1}}(t-s,y;z)\right| j(|z|)dz\right)dy\right)ds\Bigg)$$

$$\le c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \int_0^{t/2}\int_{\mathbb{R}^d} s\left(\rho(s,y)\rho(t-s,x-y)+\rho(s,x-y)\rho(t-s,y)\right)dy\,ds$$

$$\le 2c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty t^{-1}\int_0^{t}\int_{\mathbb{R}^d} s(t-s)\left(\rho(s,y)\rho(t-s,x-y)+\rho(s,x-y)\rho(t-s,y)\right)dyds$$

$$\le c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\ t\rho(t,x), \qquad \text{for all } t\in(0,T],\ x\in\mathbb{R}^d.$$

(ii) Set $\widehat{\mathfrak{K}}_i(z) := \mathfrak{K}_i(z)-\kappa_0/2, \quad i=1,2$. It is straightforward to see that $p^{\kappa_0/2}(t,x) = p^1(\kappa_0 t/2,x)$. Thus, by the construction of the Lévy process we have that for $i=1,2$,

$$p^{\mathfrak{K}_i}(t,x)=\int_{\mathbb{R}^d} p^{\kappa_0/2}(t,x-y)p^{\widehat{\mathfrak{K}}_i}(t,y)\,dy=\int_{\mathbb{R}^d} p^1(\kappa_0 t/2,x-y)p^{\widehat{\mathfrak{K}}_i}(t,y)\,dy. \tag{3.26}$$

By (3.26), Proposition 3.2, (3.23), (2.18) in the penultimate line (with $t, 2t$ instead of s,t), and Lemma 2.2(b) in the last line, we have that for all $t\in(0,T]$ and $x\in\mathbb{R}^d$,

$$\left|\nabla p^{\mathfrak{K}_1}(t,x)-\nabla p^{\mathfrak{K}_2}(t,x)\right| = \left|\int_{\mathbb{R}^d}\nabla p^1\left(\kappa_0 t/2,x-y\right)\left(p^{\widehat{\mathfrak{K}}_1}(t,y)-p^{\widehat{\mathfrak{K}}_2}(t,y)\right)dy\right|$$

$$\le c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \Phi^{-1}(t^{-1})t^2\int_{\mathbb{R}^d}\rho(t,x-y)\rho(t,y)\,dy$$

$$\le c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \Phi^{-1}(t^{-1})t\rho(t,y).$$

(iii) By using (3.26), (3.12), Lemma 2.6(b) and (3.23), we have

$$\left|\delta_{p^{\mathfrak{K}_1}}(t,x;z)-\delta_{p^{\mathfrak{K}_2}}(t,x;z)\right|$$

$$=\left|\int_{\mathbb{R}^d}\delta_{p^1}\left(\kappa_0 t/2,x-y;z\right)\left(p^{\widehat{\mathfrak{K}}_1}(t,y)-p^{\widehat{\mathfrak{K}}_2}(t,y)\right)dy\right|$$

$$\le c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)t^2\int_{\mathbb{R}^d}(\rho(t,x-y\pm z)+\rho(t,x-y))\rho(t,y)dy$$

$$\le c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)t\left(\rho(t,x\pm z)+\rho(t,x)\right).$$

Now we have

$$\int_{\mathbb{R}^d}\left|\delta_{p^{\mathfrak{K}_1}}(t,x;z)-\delta_{p^{\mathfrak{K}_2}}(t,x;z)\right| j(|z|)\,dz \ \le\ c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty$$

$$\times\int_{\mathbb{R}^d}\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)t\left(\rho(t,x\pm z)+\rho(t,x)\right)\ j(|z|)\,dz$$

$$=c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\int_{\mathbb{R}^d}\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)t\left(\rho(t,x\pm z)+\rho(t,x)\right)\ j(|z|)\,dz,$$

which is the same as (3.18) and was estimated in the proof of Theorem 3.4 by $c_3\rho(t,x)$. This finishes the proof. □

4 Levi's construction of heat kernels

For the remainder of this paper, we always assume that $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ is a Borel function satisfying (1.1) and (1.2), that Φ satisfies (1.4) and (1.5) and that J satisfies (1.7). Throughout the remaining part of this paper, β is the constant in (1.2).

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y, z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \mathcal{L}^{\mathfrak{K}_y,0} f(x) = \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x), \text{ where } \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x) = \int_{|z|>\varepsilon} \delta_f(x; z)\kappa(y, z)J(z)dz. \tag{4.1}$$

Let $p_y(t, x) = p^{\mathfrak{K}_y}(t, x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$. Note that $x \mapsto p_y(t, x)$ is in $C_0^\infty(\mathbb{R}^d)$ and satisfies (3.21).

4.1 Estimates on $p_y(t, x - y)$

The following result is the counterpart of [6, Lemmas 3.2 and 3.3].

Lemma 4.1 *For every $T \geq 1$ and $\beta_1 \in (0, \delta_1) \cap (0, \beta]$, there exists a constant $c = c(T, d, \delta_1, \beta_1, \kappa_0, \kappa_1, \kappa_2, \gamma_0) > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in (0, T]$,*

$$\left| \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_y(t, \cdot)(x - y)\, dy \right| \leq c\, t^{-1}\, \Phi^{-1}(t^{-1})^{-\beta_1}\,, \ \textit{for all } \varepsilon \in [0, 1], \tag{4.2}$$

$$\left| \int_{\mathbb{R}^d} \partial_t p_y(t, x - y)\, dy \right| \leq c\, t^{-1}\, \Phi^{-1}(t^{-1})^{-\beta_1}, \tag{4.3}$$

$$\left| \int_{\mathbb{R}^d} \nabla p_y(t, \cdot)(x - y)\, dy \right| \leq c\, \Phi^{-1}(t^{-1})^{1-\beta_1}\,. \tag{4.4}$$

Furthermore, we have

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_y(t, x - y)\, dy - 1 \right| = 0\,. \tag{4.5}$$

Proof. Choose $\gamma \in (0, \delta_1 - \beta_1) \cap (0, 1]$. Since $\int_{\mathbb{R}^d} p_z(t, \xi - y)dy = 1$ for every $\xi, z \in \mathbb{R}^d$, by the definition of δ_{p_x} we have $\int_{\mathbb{R}^d} \delta_{p_x}(t, x - y; w)dy = 0$. Therefore, using this, (1.1), (1.7) and (3.25), for $\varepsilon \in [0, 1]$ and $t \in (0, T]$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_y(t, \cdot)(x - y)\, dy \right| \\
&= \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \left(\delta_{p_y}(t, x - y; w) - \delta_{p_x}(t, x - y; w)\right) \kappa(x, w)J(w)\, dw \right) dy \right| \\
&\leq \kappa_1\gamma_0 \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \left|\delta_{p_y}(t, x - y; w) - \delta_{p_x}(t, x - y; w)\right| j(|w|)\, dw \right) dy \\
&\leq c_1 \int_{\mathbb{R}^d} \|\kappa(y, \cdot) - \kappa(x, \cdot)\|_\infty \rho(t, x - y)\, dy
\end{aligned}$$

$$\leq \ c_1\kappa_2 \int_{\mathbb{R}^d} \left(|x-y|^{\beta_1}\wedge 1\right)\rho(t,x-y)\,dy \leq c_2 t^{-1}\Phi^{-1}(t^{-1})^{-\beta_1}.$$

Here the last line follows from (1.2) and (2.17) since $\beta_1+\gamma \in (0,\delta_1)$.

For (4.3), by using (3.16) and (4.2) in the third line, we get, for $t\in(0,T]$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \partial_t p_y(t,x-y)\,dy\right| = \left|\int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_y} p_y(t,\cdot)(x-y)\,dy\right| \\
&\leq \ \left|\int_{\mathbb{R}^d}\left(\mathcal{L}^{\mathfrak{K}_x}-\mathcal{L}^{\mathfrak{K}_y}\right)p_y(t,\cdot)(x-y)\,dy\right| + \left|\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_x}p_y(t,\cdot)(x-y)\,dy\right| \\
&\leq \ c_3\int_{\mathbb{R}^d}\rho_0^{\beta_1}(t,x-y)\,dy + c_2t^{-1}\Phi^{-1}(t^{-1})^{-\beta_1} \ \leq \ c_4t^{-1}\Phi^{-1}(t^{-1})^{-\beta_1}\,.
\end{aligned}$$

Here we have used (2.17) in the last inequality.

For (4.4), by (3.24) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\nabla p_y(t,\cdot)(x-y)\,dy\right| = \left|\int_{\mathbb{R}^d}\left(\nabla p_y(t,\cdot)-\nabla p_x(t,\cdot)\right)(x-y)\,dy\right| \\
&\leq c_5\int_{\mathbb{R}^d}\|\kappa(x,\cdot)-\kappa(y,\cdot)\|_\infty t\Phi^{-1}(t^{-1})\rho(t,x-y)\,dy \\
&\leq c_6\int_{\mathbb{R}^d}\left(|x-y|^{\beta_1}\wedge 1\right)t\Phi^{-1}(t^{-1})\rho(t,x-y)\,dy \\
&= t\Phi^{-1}(t^{-1})\int_{\mathbb{R}^d}\rho_0^{\beta_1}(t,x-y)\,dy \\
&\leq c_7t\Phi^{-1}(t^{-1})t^{-1}\Phi^{-1}(t^{-1})^{-\beta_1} = \Phi^{-1}(t^{-1})^{1-\beta_1}\,.
\end{aligned}$$

In the last inequality we used Lemma 2.6(a) which requires that $\beta_1+\gamma\in(0,\delta_1)$.

Finally, by using (3.23) in the second line and (2.17) in the last inequality, we get

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}p_y(t,x-y)\,dy-1\right| \leq \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|p_y(t,x-y)-p_x(t,x-y)|\,dy \\
&\leq c_8\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\|\kappa(y,\cdot)-\kappa(x,\cdot)\|_\infty t\rho(t,x-y)\,dy \\
&\leq c_9t\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_0^{\beta_1}(t,x-y)\,dy \leq c_{10}\Phi^{-1}(t^{-1})^{-\beta_1}, \quad t\in(0,T]\,.
\end{aligned}$$

□

Lemma 4.2 *The function $p_y(t,x)$ is jointly continuous in (t,x,y).*

Proof. By the triangle inequality, we have

$$|p_{y_1}(t_1,x_1)-p_{y_2}(t_2,x_2)| \leq |p_{y_1}(t_1,x_1)-p_{y_2}(t_1,x_1)| + |p_{y_2}(t_1,x_1)-p_{y_2}(t_2,x_2)|.$$

Applying (3.23) and (1.2) to the first term on the right hand side and Lemma 3.1(b) to the second term on the right hand side, we immediately get the desired joint continuity. □

4.2 Construction of $q(t,x,y)$

For $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ define

$$\begin{aligned} q_0(t,x,y) &:= \frac{1}{2}\int_{\mathbb{R}^d} \delta_{p_y}(t,x-y;z)\,(\kappa(x,z)-\kappa(y,z))\,J(z)\,dz \\ &= \big(\mathcal{L}^{\mathfrak{K}_x}-\mathcal{L}^{\mathfrak{K}_y}\big)p_y(t,\cdot)(x-y)\,. \end{aligned} \tag{4.6}$$

In the next lemma we collect several estimates on q_0 that will be needed later on.

Lemma 4.3 *For every $T\ge 1$ and $\beta_0\in(0,\beta]$, there is a constant $C_1\ge 1$ depending on $d,\delta_1,\kappa_0,\kappa_1,\kappa_2,\gamma,T$ and $\Phi^{-1}(T^{-1})$ such that for $t\in(0,T]$ and $x,x',y,y'\in\mathbb{R}^d$,*

$$|q_0(t,x,y)|\le C_1(|x-y|^{\beta_0}\wedge 1)\rho(t,x-y)=C_1\rho_0^{\beta_0}(t,x-y), \tag{4.7}$$

and for all $\gamma\in(0,\beta_0)$,

$$\begin{aligned} &|q_0(t,x,y)-q_0(t,x',y)| \\ &\le C_1\left(|x-x'|^{\beta_0-\gamma}\wedge 1\right)\left\{\left(\rho_\gamma^0+\rho_{\gamma-\beta_0}^{\beta_0}\right)(t,x-y)+\left(\rho_\gamma^0+\rho_{\gamma-\beta_0}^{\beta_0}\right)(t,x'-y)\right\} \end{aligned} \tag{4.8}$$

and

$$|q_0(t,x,y)-q_0(t,x,y')|\le C_1\Phi^{-1}(t^{-1})^{\beta_0}\left(|y-y'|^{\beta_0}\wedge 1\right)\left(\rho(t,x-y)+\rho(t,x-y')\right). \tag{4.9}$$

Proof. (a) (4.7) follows from (3.16) and (1.2).
(b) By (4.7) and (2.14), we have that for $t\in(0,T]$,

$$|q_0(t,x,y)|\le c_0\rho_0^{\beta_0}(t,x-y)\le c_0\Phi^{-1}(T^{-1})^{\gamma-\beta_0}\rho_{\gamma-\beta_0}^{\beta_0}(t,x-y),$$

which proves (4.8) for $|x-x'|\ge 1$. Now suppose that $1\ge|x-x'|\ge\Phi^{-1}(t^{-1})^{-1}$. Then, by (4.7), for $t\in(0,T]$,

$$|q_0(t,x,y)|\le c_1\left(\Phi^{-1}(t^{-1})\right)^{-(\beta_0-\gamma)}\rho_{\gamma-\beta_0}^{\beta_0}(t,x-y)\le c_1|x-x'|^{\beta_0-\gamma}\rho_{\gamma-\beta_0}^{\beta_0}(t,x-y),$$

and the same estimate is valid for $|q_0(t,x',y)|$. By adding we get (4.8) for this case. Finally, assume that $|x-x'|\le 1\wedge\Phi^{-1}(t^{-1})^{-1}$. Then, by (1.7), (1.2) and (3.17), for $t\in(0,T]$,

$$\begin{aligned} |q_0(t,x,y)-q_0(t,x',y)| &= \bigg|\int_{\mathbb{R}^d}\delta_{p_y}(t,x-y;z)(\kappa(x,z)-\kappa(y,z))J(z)\,dz \\ &\quad -\int_{\mathbb{R}^d}\delta_{p_y}(t,x'-y;z)(\kappa(x',z)-\kappa(y,z))J(z)\,dz\bigg| \\ \le&\gamma_0\int_{\mathbb{R}^d}|\delta_{p_y}(t,x-y;z)-\delta_{p_y}(t,x'-y;z)|\;|\kappa(x,z)-\kappa(y,z)|j(|z|)\,dz \\ &+\gamma_0\int_{\mathbb{R}^d}|\delta_{p_y}(t,x'-y;z)||\kappa(x,z)-\kappa(x',z)|j(|z|)\,dz \end{aligned}$$

$$
\begin{aligned}
\leq & \gamma_0\kappa_2\left(|x-y|^{\beta_0}\wedge 1\right)\int_{\mathbb{R}^d}|\delta_{p_y}(t,x-y;z)-\delta_{p_y}(t,x'-y;z)|j(|z|)\,dz \\
&\quad + \gamma_0\kappa_2\left(|x-x'|^{\beta_0}\wedge 1\right)\int_{\mathbb{R}^d}|\delta_{p_y}(t,x'-y;z)|j(|z|)\,dz \\
\leq & c_2\left(|x-y|^{\beta_0}\wedge 1\right)\big(\rho(t,x-y)+\rho(t,x'-y)\big)+c_2|x-x'|^{\beta_0}\rho(t,x'-y).
\end{aligned}
$$

By using the definition of $\rho(t,x'-y)$, the obvious equality $x'-y=(x-y)+(x'-x)$, the assumption that $|x-x'|\leq \Phi^{-1}(t^{-1})^{-1}$ and (3.9), we conclude that $\rho_0^{\beta}(t,x'-y)\leq 4\rho_0^{\beta}(t,x-y)$. Thus, it follows that for $t\in(0,T]$,

$$
\begin{aligned}
|q_0(t,x,y)-q_0(t,x',y)| &\leq 5\,c_2\,\rho_0^{\beta_0}(t,x-y)+c_2|x-x'|^{\beta_0}\rho(t,x'-y) \\
&\leq 5\,c_2\,|x-x'|^{\beta_0-\gamma}\rho_{\gamma-\beta_0}^{\beta_0}(t,x-y)+c_2|x-x'|^{\beta_0-\gamma}\rho_{\gamma}^{0}(t,x'-y)\,.
\end{aligned}
$$

(c) First note that

$$
\begin{aligned}
& q_0(t,x,y)-q_0(t,x,y') \\
=& \frac{1}{2}\int_{\mathbb{R}^d}\delta_{p^y}(t,x-y;z)\left(\kappa(y',z)-\kappa(y,z)\right)J(z)\,dz \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^d}\left(\delta_{p^y}(t,x-y;z)-\delta_{p^y}(t,x-y';z)\right)\left(\kappa(x,z)-\kappa(y',z)\right)J(z)\,dz \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^d}\big(\delta_{p^y}(t,x-y';z)-\delta_{p^{y'}}(t,x-y';z)\big)\left(\kappa(x,z)-\kappa(y',z)\right)J(z)\,dz \\
=:& I_1+I_2+I_3\,.
\end{aligned}
$$

It follows from (1.2), (1.7) and (3.16) that for $t\in(0,T]$,

$$
|I_1|\leq c_1\left(|y-y'|^{\beta_0}\wedge 1\right)\int_{R^d}|\delta_{p^y}(t,x-y;z)|\,j(|z|)\,dz\leq c_2\left(|y-y'|^{\beta_0}\wedge 1\right)\rho(t,x-y)\,,
$$

which is smaller than or equal to the right-hand side in (4.9) since $\Phi^{-1}(t^{-1})\geq\Phi^{-1}(T^{-1})$. By (1.1), (1.7) and (3.17) we get that

$$
\begin{aligned}
|I_2| &\leq c_1\int_{\mathbb{R}^d}|\delta_{p^y}(t,x-y;z)-\delta_{p^y}(t,x-y';z)|\,j(|z|)\,dz \\
&\leq c_2\,\big((\Phi^{-1}(t^{-1})|y-y'|)\wedge 1\big)\left(\rho(t,x-y)+\rho(t,x-y')\right) \\
&\leq c_2\,\Phi^{-1}(T^{-1})^{-\beta_0}\Phi^{-1}(t^{-1})^{\beta_0}\left(|y-y'|^{\beta_0}\wedge 1\right)\left(\rho(t,x-y)+\rho(t,x-y')\right).
\end{aligned}
$$

Finally, by (1.1), (1.2), (1.7) and (3.25), for $t\in(0,T]$,

$$
\begin{aligned}
|I_3| &\leq c_1\int_{\mathbb{R}^d}\left|\delta_{p^y}(t,x-y';z)-\delta_{p^{y'}}(t,x-y';z)\right|j(|z|)\,dz \\
&\leq c_3\,\|\kappa(y,\cdot)-\kappa(y',\cdot)\|_\infty\rho(t,x-y')\leq\; c_4\left(|y-y'|^{\beta_0}\wedge 1\right)\rho(t,x-y')\,.
\end{aligned}
$$

□

Lemma 4.4 *The function $q_0(t,x,y)$ is jointly continuous in (t,x,y).*

Proof. It follows from Lemma 4.2 that $(t,x,y) \mapsto p_y(t,x-y)$ is jointly continuous and hence also that $\delta_{p_y}(t,x-y;z)$ is jointly continuous in (t,x,y). To prove the joint continuity of $q_0(t,x,y)$, let $(t_n,x_n,y_n) \to (t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ and assume that $0 < \varepsilon \le t_n \le T$. The integrands will converge because of the joint continuity of δ_{p_y} and continuity of κ in the first variable. Moreover, by (3.12),

$$\begin{aligned}&\left|\delta_{p_{y_n}}(t_n,x_n-y_n;z)\right| |\kappa(x_n,z)-\kappa(y_n,z)| j(|z|)\\ &\le c_1\left((\Phi^{-1}(t_n^{-1})|z|^2)\wedge 1\right) T\left(\rho(t_n,x_n-y_n\pm z)+\rho(t_n,x_n,y_n)\right) j(|z|)\\ &\le c_2\rho(\varepsilon,0)\left((\Phi^{-1}(\varepsilon^{-1})|z|^2)\wedge 1\right) j(|z|).\end{aligned}$$

Since the right-hand side is integrable on $\mathbb{R}^d$, the joint continuity follows by use of the dominated convergence theorem. □

For $n \in \mathbb{N}$, we inductively define

$$q_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_{n-1}(s,z,y)\, dz\, ds, \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \quad (4.10)$$

The following result is the counterpart of [6, Theorem 3.1].

Theorem 4.5 *The series $q(t,x,y) := \sum_{n=0}^\infty q_n(t,x,y)$ is absolutely and locally uniformly convergent on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and solves the integral equation*

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q(s,z,y)\, dz\, ds. \quad (4.11)$$

Moreover, $q(t,x,y)$ is jointly continuous in $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and has the following estimates: for every $T \ge 1$ and $\beta_2 \in (0,\beta] \cap (0,\delta_1/2)$ there is a constant $C_2 = C_2(T,d,\delta_1,\kappa_0,\kappa_1,\kappa_2,\beta_2,\gamma_0) > 0$ such that

$$|q(t,x,y)| \le C_2\big(\rho_0^{\beta_2} + \rho_{\beta_2}^0\big)(t,x-y), \quad (t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d, \quad (4.12)$$

and for any $\gamma \in (0,\beta_2)$ and $T \ge 1$ there is a constant $C_3 = C_3(T,d,\delta_1,\gamma,\kappa_0,\kappa_1,\kappa_2,\gamma_0,\beta_2) > 0$ such that for all $(0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned}&|q(t,x,y) - q(t,x',y)|\\ &\le C_3\left(|x-x'|^{\beta_2-\gamma} \wedge 1\right)\Big(\big(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\big)(t,x-y) + \big(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\big)(t,x'-y)\Big). \quad (4.13)\end{aligned}$$

Proof. This proof follows the main idea of the proof of [6, Theorem 3.1], except that we give a full proof of the joint continuity in Step 2. We give the details for the readers' convenience. In this proof, $T \ge 1$ is arbitrary.

Step 1: By (4.7), (2.19) and (2.20), we have that

$$\begin{aligned}|q_1(t,x,y)| &\le C_1^2\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t-s,x-y-u)\rho_0^{\beta_2}(s,u)\,du\,ds\\ &\le 8C_0C_1^2B\left(\beta_2/2,\beta_2/2\right)\left(\rho^0_{2\beta_2}+\rho^{\beta_2}_{\beta_2}\right)(t,x-y),\quad t\le T\,.\end{aligned}$$

Let $C=2^4C_0C_1^2$ and we claim that for $n\ge 1$ and $t\le T$,

$$|q_n(t,x,y)|\le \gamma_n\left(\rho^0_{(n+1)\beta_2}+\rho^{\beta_2}_{n\beta_2}\right)(t,x-y) \tag{4.14}$$

with

$$\gamma_n=C^{n+1}\prod_{j=1}^n B\left(\beta_2/2,j\beta_2/2\right)\,.$$

We have seen that (4.14) is valid for $n=1$. Suppose that it is valid for n. Then by using (2.19), (2.20), (2.14) and (2.15), we have that for $t\le T$,

$$\begin{aligned}|q_{n+1}(t,x,y)|\le&\int_0^t\int_{\mathbb{R}^d}|q_0(t-s,x,z)|\;|q_n(s,z,y)|\,dz\,ds\\ \le&C_1\gamma_n\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t-s,x-z)\left(\rho^0_{(n+1)\beta_2}+\rho^{\beta_2}_{n\beta_2}\right)(s,z-y)\,dz\,ds\\ \le&2^4C_0C_1\gamma_nB\left(\frac{\beta_2}{2},\frac{(n+1)\beta_2}{2}\right)\left(\rho^0_{(n+2)\beta_2}+\rho^{\beta_2}_{(n+1)\beta_2}\right)(t,x-y)\\ \le&\gamma_{n+1}\left(\rho^0_{(n+2)\beta_2}+\rho^{\beta_2}_{(n+1)\beta_2}\right)(t,x-y)\,.\end{aligned}$$

Thus (4.14) is valid. Since

$$\sum_{n=0}^\infty\gamma_n\Phi^{-1}(T^{-1})^{-(n+1)\beta_2}=\sum_{n=0}^\infty\frac{\left(\Phi^{-1}(T^{-1})^{-\beta_2}C\Gamma\left(\frac{\beta_2}{2}\right)\right)^{n+1}}{\Gamma\left(\frac{(n+1)\beta_2}{2}\right)}=:C_2<\infty\,,$$

by using (2.14) and (2.15) in the second line, it follows that for $t\le T$,

$$\begin{aligned}\sum_{n=0}^\infty|q_n(t,x,y)|\;&\le\;\sum_{n=0}^\infty\gamma_n\left(\rho^0_{(n+1)\beta_2}+\rho^{\beta_2}_{n\beta_2}\right)(t,x-y)\\ &\le\;\sum_{n=0}^\infty\gamma_n\Phi^{-1}(T^{-1})^{-(n+1)\beta_2}\left(\rho^0_{\beta_2}+\rho_0^{\beta_2}\right)(t,x-y)=C_2\left(\rho^0_{\beta_2}+\rho_0^{\beta_2}\right)(t,x-y)\,.\end{aligned}$$

This proves that $\sum_{n=0}^\infty q_n(t,x,y)$ is absolutely and uniformly convergent on $[\varepsilon,T]\times\mathbb{R}^d\times\mathbb{R}^d$ for all $\varepsilon\in(0,1)$ and $T\ge1$, hence $q(t,x,y)$ is well defined. Further, by (4.10),

$$\sum_{n=0}^{m+1}q_n(t,x,y)=q_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d}q_0(t-s,x,z)\sum_{n=0}^m q_n(s,z,y)\,dz\,ds\,,$$

and (4.11) follows by taking the limit of both sides as $m\to\infty$.

Step 2: The joint continuity of $q_0(t,x,y)$ was shown in Lemma 4.4. We now prove the joint continuity of $q_1(t,x,y)$. For any $x,y \in \mathbb{R}^d$ and $t,h>0$, we have

$$\begin{aligned} &q_1(t+h,x,y)-q_1(t,x,y)\\ &=\int_t^{t+h}\int_{\mathbb{R}^d} q_0(t+h-s,x,z)q_0(s,z,y)dzds\\ &\quad+\int_0^t\int_{\mathbb{R}^d}\left(q_0(t+h-s,x,z)-q_0(t-s,x,z)\right)q_0(s,z,y)dzds. \end{aligned} \tag{4.15}$$

It follows from (4.7) that, there exists $c_1=c_1(T)>0$ such that, for $0<h\le t/4$ and $t+h\le T$,

$$\begin{aligned} &\sup_{x,y\in\mathbb{R}^d}\left|\int_t^{t+h}\int_{\mathbb{R}^d} q_0(t+h-s,x,z)q_0(s,z,y)dzds\right|\\ &\le c_1\sup_{x,y\in\mathbb{R}^d}\int_t^{t+h}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t+h-s,x-z)\rho_0^{\beta_2}(s,z-y)dzds\\ &=c_1\sup_{x,y\in\mathbb{R}^d}\int_0^{h}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(r,x-z)\rho_0^{\beta_2}(t+h-r,z-y)dzdr\\ &\le c_1\int_0^{h}\sup_{x,y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(r,x-z)\rho_0^{\beta_2}(t-r,z-y)dzdr. \end{aligned}$$

Now applying Lemma 2.6(b), we get

$$\begin{aligned} &\sup_{x,y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(r,x-z)\rho_0^{\beta_2}(t-r,z-y)dz\\ &\le c_2((t-r)^{-1}\Phi^{-1}((t-r)^{-1})^{-\beta_2}+r^{-1}\Phi^{-1}(r^{-1})^{-\beta_2})\rho(t,0). \end{aligned}$$

It follows from Lemma 2.3 that the right-hand side of the display above is integrable in $r\in(0,t)$, so by the dominated convergence theorem, we get

$$\lim_{h\downarrow 0}\sup_{x,y\in\mathbb{R}^d}\left|\int_t^{t+h}\int_{\mathbb{R}^d} q_0(t+h-s,x,z)q_0(s,z,y)dzds\right|=0. \tag{4.16}$$

Using (4.7) again, we get that for $s\in(0,t]$,

$$\begin{aligned} &\left|\left(q_0(t+h-s,x,z)-q_0(t-s,x,z)\right)q_0(s,z,y)\right|\\ &\le c_3\left(\rho_0^{\beta_2}(t+h-s,x-z)+\rho_0^{\beta_2}(t-s,x-z)\right)\rho_0^{\beta_2}(s,z-y)\\ &\le c_4\rho_0^{\beta_2}(t-s,x-z)\rho_0^{\beta_2}(s,z-y). \end{aligned}$$

It follows from Lemma 2.6(c) that

$$\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t-s,x-z)\rho_0^{\beta_2}(s,z-y)dzds\le c_5(\rho_0^{2\beta_2}(t,0)+\rho_{\beta_2}^{\beta_2}(t,0))<\infty,$$

thus we can use the dominated convergence theorem to get that, by the continuity of q_0,

$$\lim_{h\downarrow 0}\int_0^t\int_{\mathbb{R}^d}\left(q_0(t+h-s,x,z)-q_0(t-s,x,z)\right)q_0(s,z,y)dzds=0. \tag{4.17}$$

It follows from (4.9) that for $s\in(0,T]$,

$$\begin{aligned}&|q_0(s,z,y)-q_0(s,z,y')|\\ \le& c_6\left((\Phi^{-1}(s^{-1})|y-y'|)^{\beta_2}\wedge 1\right)\left(\rho(s,z-y)+\rho(s,z-y')\right).\end{aligned}$$

Now we fix $0<t_1\le t_2\le T$. Then for any $\varepsilon\in(0,t_1/4)$, $t\in[t_1,t_2]$ and $s\in[\varepsilon,t]$,

$$\begin{aligned}&|q_0(t-s,x,z)\left(q_0(s,z,y)-q_0(s,z,y')\right)|\\ \le\ & c_7\left((\Phi^{-1}(\varepsilon^{-1})|y-y'|)^{\beta_2}\wedge 1\right)\rho_0^{\beta_2}(t-s,x,z)\left(\rho(s,z-y)+\rho(s,z-y')\right).\end{aligned}$$

By Lemma 2.6(c), we have

$$\sup_{x,y,y'\in\mathbb{R}^d,t\in[t_1,t_2]}\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t-s,x,z)\left(\rho(s,z-y)+\rho(s,z-y')\right)dzds<\infty.$$

Thus

$$\lim_{y'\to y}\sup_{x\in\mathbb{R}^d,t\in[t_1,t_2]}\int_\varepsilon^t\int_{\mathbb{R}^d}|q_0(t-s,x,z)\left(q_0(s,z,y)-q_0(s,z,y')\right)|dzds=0.$$

Consequently, for each $0<t_1<t_2\le T$ and $\varepsilon\in(0,t_1/4)$, the family of functions

$$\left\{\int_\varepsilon^t\int_{\mathbb{R}^d}q_0(t-s,x,z)q_0(s,z,\cdot)dzds: x\in\mathbb{R}^d, t\in[t_1,t_2]\right\}$$

is equi-continuous. By combining (4.7) and Lemma 2.7, we get that

$$\lim_{\varepsilon\to 0}\sup_{x,y\in\mathbb{R}^d,t\in[t_1,t_2]}\left(\int_0^\varepsilon+\int_{t-\varepsilon}^t\right)\int_{\mathbb{R}^d}q_0(t-s,x,z)q_0(s,z,y)dzds=0. \tag{4.18}$$

Therefore the family

$$\left\{\int_0^t\int_{\mathbb{R}^d}q_0(t-s,x,z)q_0(s,z,\cdot)dzds: x\in\mathbb{R}^d, t\in[t_1,t_2]\right\} \tag{4.19}$$

is equi-continuous.

Similarly, by using (4.8), we can show that, for each $0<t_1<t_2\le T$ and $\varepsilon\in(0,t_1/4)$, the family of functions

$$\left\{\int_0^{t-\varepsilon}\int_{\mathbb{R}^d}q_0(t-s,\cdot,z)q_0(s,z,y)dzds: y\in\mathbb{R}^d, t\in[t_1,t_2]\right\}$$

is equi-continuous. Combining this with (4.18), we get the family of functions

$$\left\{\int_0^t\int_{\mathbb{R}^d}q_0(t-s,\cdot,z)q_0(s,z,y)dzds: y\in\mathbb{R}^d, t\in[t_1,t_2]\right\} \tag{4.20}$$

is equi-continuous.

Now combining the continuity of $t \to q_1(t,x,y)$ (by (4.16) and (4.17)) and the equi-continuities of the families (4.19) and (4.20), we immediately get the joint continuity of q_1.

The joint continuity of $q_n(t,x,y)$ can be proved by induction by using the estimate (4.14) of q_n and Lemma 2.7. The joint continuity of $q(t,x,y)$ follows immediately.
Step 3: By replacing α by 2 and β by β_2, this step is exactly the same as Step 4 in [6]. □

4.3 Properties of $\phi_y(t,x)$

Let

$$\phi_y(t,x,s) := \int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz, \quad x\in\mathbb{R}^d,\ 0<s<t \tag{4.21}$$

and

$$\phi_y(t,x) := \int_0^t \phi_y(t,x,s)\,ds = \int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,. \tag{4.22}$$

The following result is the counterpart of [6, Lemma 3.5].

Lemma 4.6 *For all $x,y\in\mathbb{R}^d$, $x\neq y$, the mapping $t\mapsto\phi_y(t,x)$ is absolutely continuous on $(0,\infty)$ and*

$$\partial_t\phi_y(t,x) = q(t,x,y) + \int_0^t\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_z}p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds, \quad t\in(0,\infty)\,. \tag{4.23}$$

Proof. *Step 1:* Here we prove that for any $T\geq 1$, $t\in(0,T]$ and $s\in(0,t)$,

$$\partial_t\phi_y(t,x,s) = \int_{\mathbb{R}^d}\partial_t p_z(t-s,x-z)q(s,z,y)\,dz\,. \tag{4.24}$$

Let $|\varepsilon|<(t-s)/2$. We have that

$$\frac{\phi_y(t+\varepsilon,x,s)-\phi_y(t,x,s)}{\varepsilon} = \int_{\mathbb{R}^d}\left(\int_0^1\partial_t p_z(t+\theta\varepsilon-s,x,z)\,d\theta\right)q(s,z,y)\,dz\,.$$

By using (1.7), (3.21), (3.16) and (3.20), we have,

$$\begin{aligned}
|\partial_t p_z(t+\theta\varepsilon-s,x-z)| &= \left|\mathcal{L}^{\mathfrak{K}_z}p_z(t+\theta\varepsilon-s,\cdot)(x-z)\right| \\
&\leq \frac{1}{2}\gamma_0\int_{\mathbb{R}^d}|\delta_{p_z}(t+\theta\varepsilon-s,x-z;w)|\kappa(z,w)j(|w|)\,dw \\
&\leq c_1\rho(t+\theta\varepsilon-s,x-z) \leq c_2\rho(t-s,x-z)\,.
\end{aligned}$$

In the last inequality we used that $|\varepsilon|<(t-s)/2$ and applied Lemma 2.2(b). Together with (4.12) this gives that for any $\beta_2\in(0,\beta)\cap(0,\delta_1/2)$ and $t\in(0,T]$

$$|\partial_t p_z(t+\theta\varepsilon-s,x-z)q(s,z,y)| \leq c_3(T)\rho(t-s,x-z)\big(\rho^0_{\beta_2}+\rho^{\beta_2}_0\big)(s,z-y) =: g(z)\,.$$

By (2.18), we see that $\int_{\mathbb{R}^d} g(z)\,dz < \infty$. Thus, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\frac{\phi_y(t+\varepsilon,x,s)-\phi_y(t,x,s)}{\varepsilon}=\int_{\mathbb{R}^d}\partial_t p_z(t-s,x-z)q(s,z,y)\,dz\,,$$

proving (4.24).
Step 2: Here we prove that for all $x\neq y$ and $t\in(0,T]$, $T\geq 1$,

$$\int_0^t\int_0^r|\partial_r\phi_y(r,x,s)|\,ds\,dr\;\leq\;c_1(T)\,t\frac{\Phi(|x-y|^{-1})}{|x-y|^d}<+\infty\,. \tag{4.25}$$

By (4.24) we have

$$\begin{aligned}|\partial_r\phi_y(r,x,s)|\;&\leq\;\int_{\mathbb{R}^d}|\partial_r p_z(r-s,x-z)|\;|q(s,z,y)-q(s,x,y)|\;dz\\&\quad+|q(s,x,y)|\left|\int_{\mathbb{R}^d}\partial_r p_z(r-s,x-z)\,dz\right|=:Q_y^{(1)}(r,x,s)+Q_y^{(2)}(r,x,s)\,.\end{aligned}$$

For $Q_y^{(1)}(r,x,s)$, by (4.13), (3.20), (3.16) and Lemma 2.6(a) and (c), for $\beta_2\in(0,\delta_1/2)\cap(0,\beta]$ and $\gamma\in((2-\delta_1)\beta_2/2,\beta_2)$,

$$\begin{aligned}&\int_0^t\int_0^r Q_y^{(1)}(r,x,s)\,ds\,dr\\&\leq\;c_2\int_0^t\int_0^r\int_{\mathbb{R}^d}\left|\mathcal{L}^{\mathfrak{K}_z}p_z(r-s,x-z)\right|\left(|x-z|^{\beta_2-\gamma}\wedge 1\right)\\&\quad\times\left\{\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y)+\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\right\}dz\,ds\,dr\\&\leq\;c_3\int_0^t\int_0^r\left(\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(r-s,x-z)dz\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y)\,ds\,dr\\&\quad+c_3\int_0^t\int_0^r\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(r-s,x-z)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz\,ds\,dr\\&\leq\;c_4\int_0^t\int_0^r(r-s)^{-1}\Phi^{-1}((r-s)^{-1})^{\gamma-\beta_2}\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y)\,ds\,dr\\&\quad+c_4\int_0^t\left(\rho_{\beta_2}^0+\rho_0^{\beta_2}+\rho_\gamma^{\beta_2-\gamma}\right)(r,x-y)\,dr\\&\leq c_4\frac{\Phi(|x-y|^{-1})}{|x-y|^d}\int_0^t\int_0^r(r-s)^{-1}\Phi^{-1}((r-s)^{-1})^{\gamma-\beta_2}\left(\Phi^{-1}(s^{-1})^{-\gamma}+\Phi^{-1}(s^{-1})^{\beta_2-\gamma}\right)ds\,dr\\&\quad+c_4\frac{\Phi(|x-y|^{-1})}{|x-y|^d}\int_0^t\left(\Phi^{-1}(r^{-1})^{-\beta_2}+1+\Phi^{-1}(r^{-1})^{-\gamma}\right)dr\\&\leq c_5\frac{\Phi(|x-y|^{-1})}{|x-y|^d}\int_0^t\left(\Phi^{-1}(r^{-1})^{-\beta_2}+1+\Phi^{-1}(r^{-1})^{-\gamma}\right)dr\leq c_6 t\frac{\Phi(|x-y|^{-1})}{|x-y|^d}<+\infty\,.\end{aligned}$$

The second to last inequality follows from Lemma 2.3.

For $Q_y^{(2)}$, by (4.3), (4.12) and Lemma 2.3 we have

$$\int_0^t \int_0^r Q_y^{(2)}(r,x,s)\,dr\,ds \le c_7 \int_0^t \int_0^r \Big(\rho_{\beta_2}^0 + \rho_0^{\beta_2}\Big)(s,x-y)(r-s)^{-1}\Phi^{-1}((r-s)^{-1})^{-\beta_2}\,ds\,dr$$
$$\le 2c_7 \frac{\Phi(|x-y|^{-1})}{|x-y|^d} \int_0^t \left(\int_0^r (r-s)^{-1}\Phi^{-1}((r-s)^{-1})^{-\beta_2}\,ds\right) dr \le c_8 t \frac{\Phi(|x-y|^{-1})}{|x-y|^d} < +\infty\,.$$

Step 3: We claim that for fixed $s>0$ and $x,y\in\mathbb{R}^d$,

$$\lim_{t\downarrow s} \phi_y(t,x,s) = q(s,x,y)\,. \tag{4.26}$$

Assume $t\le T$, $T\ge 1$. For any $\delta>0$ we have

$$\left|\int_{\mathbb{R}^d} p_z(t-s,x-z)\,(q(s,z,y)-q(s,x,y))\,dz\right|$$
$$\le \int_{|x-z|\le\delta} p_z(t-s,x-z)\,|q(s,z,y)-q(s,x,y)|\,dz$$
$$+\int_{|x-z|>\delta} p_z(t-s,x-z)\,(|q(s,z,y)|+|q(s,x,y)|)\,dz =: J_1(\delta,t,s)+J_2(\delta,t,s)\,.$$

By (4.13), for any $\varepsilon>0$ there exists $\delta=\delta(s,x,y,T)>0$ such that if $|z-x|\le\delta$, then $|q(s,z,y)-q(s,x,y)|\le\varepsilon$. Therefore, by Proposition 3.2 and Lemma 2.6(a),

$$J_1(\delta,t,s) \le \varepsilon \int_{\mathbb{R}^d} p_z(t-s,x-z)\,dz \le \varepsilon(t-s)\int_{\mathbb{R}^d} \rho(t-s,z)\,dz \le c_1\varepsilon\,.$$

For $J_2(\delta,t,s)$, since $p_z(t-s,x-z)\le c_2(t-s)\rho(t-s,x-z)\le c_2(t-s)\rho(0,x-z)$, by (4.12) we have

$$J_2(\delta,t,s)\le c_3(t-s)\left(\frac{\Phi(\delta^{-1})}{\delta^d}\int_{\mathbb{R}^d}\rho(s,z-y)\,dz + \rho(s,x-y)\int_{|x-z|>\delta}\frac{\Phi(|x-z|^{-1})}{|x-z|^d}\,dz\right)$$

where $c_3=c_3(T)>0$ is independent of t. By (2.17), the term in parenthesis is finite. Hence, the last line converges to 0 as $t\downarrow s$. This and (4.5) prove (4.26).
Step 4: By (4.26), we have that

$$\phi_y(t,x,s)-q(s,x,y)=\int_s^t \partial_r\phi_y(r,x,s)\,dr\,.$$

Integrating both sides with respect to s from 0 to t, using first (4.25) and Fubini's theorem, and then (4.24) and (3.21), we get

$$\begin{aligned}\phi_y(t,x)-\int_0^t q(s,x,y)\,ds &= \int_0^t\int_s^t \partial_r\phi_y(r,x,s)\,dr\,ds = \int_0^t\int_0^r \partial_r\phi_y(r,x,s)\,ds\,dr\\ &= \int_0^t\int_0^r\int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z}p_z(r-s,\cdot)(x-z)q(s,z,y)\,dz\,ds\,dr\,.\end{aligned}$$

This proves that $t \mapsto \phi_y(t,x)$ is absolutely continuous and gives its Radon-Nykodim derivative (4.23). □

The following result is the counterpart of [6, Lemma 3.6].

Lemma 4.7 *For all $t > 0$, $x \neq y$ and $\varepsilon \in [0,1]$, we have*

$$\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds \tag{4.27}$$

and

$$t \mapsto \mathcal{L}^{\mathfrak{K}_x} p_y(t,x-y) \text{ and } t \mapsto \mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x) \text{ are continuous on } (0,\infty)\,. \tag{4.28}$$

Furthermore, if $\beta + \delta_1 > 1$ and $\delta_1 \in (2/3, 2)$ we also have

$$\nabla_x \phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds. \tag{4.29}$$

Proof. Fix $x \neq y$ and $T \geq 1$. In this proof we assume $0 < t < T$ and all the constants will depend on T, but independent of s and t.
(a) By (1.7), (1.1), (3.16), (4.12) and Lemma 2.6(b), for each $s \in (0,t)$,

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\delta_{p_z}(t-s,x-z;w)|\kappa(x,w)J(w)dw|q(s,z,y)|dz \\ &\leq c_1 \int_{\mathbb{R}^d} \rho(t-s,x-z)\rho(s,z-y)dz < \infty. \end{aligned} \tag{4.30}$$

Thus we can use Fubini's theorem so that from (4.21) we have that for $s \in (0,t)$,

$$\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x) = \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz, \quad \varepsilon \in [0,1]\,. \tag{4.31}$$

Let $\beta_2 \in (0,\delta_1/2) \cap (0,\beta]$ and $\gamma \in (0,\beta_2)$. By the definition of ϕ_y, (4.21), and Fubini's theorem, using the notation (3.10) we have for $\varepsilon \in (0,1]$ and $s \in (0,t)$,

$$\begin{aligned} &\left|\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\right| \\ =\ & \frac{1}{2}\left|\int_{|w|>\varepsilon}\left(\int_{\mathbb{R}^d} \delta_{p_z}(t-s,x-z;w)q(s,z,y)\,dz\right)\kappa(x,w)J(w)\,dw\right| \\ =\ & \frac{1}{2}\left|\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon} \delta_{p_z}(t-s,x-z;w)\kappa(x,w)J(w)\,dw\right)q(s,z,y)\,dz\right| \\ \leq\ & \frac{1}{2}\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon} |\delta_{p_z}(t-s,x-z;w)|\kappa(x,w)J(w)\,dw\right)|q(s,z,y)-q(s,x,y)|\,dz \\ & +\frac{1}{2}\left|\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon} \delta_{p_z}(t-s,x-z;w)\kappa(x,w)J(w)\,dw\right)\,dz\right|\,|q(s,x,y)|\,. \end{aligned}$$

By using (1.7), (3.16), (4.2), (4.12) and (4.13) first and then using Lemma 2.6(a)–(b), we have that for $\varepsilon \in (0,1]$ and $s \in (0,t)$,

$$
\begin{aligned}
&\left|\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\right| \\
\le\ & c_2 \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz \\
&+c_2\left(\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\,dz\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y) \\
&+c_2(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\left(\rho_0^{\beta_2}(s,x-y)+\rho_{\beta_2}^0(s,x-y)\right) \\
\le\ & c_2\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\rho_\gamma^0(s,z-y)\,dz + c_2\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\rho_{\gamma-\beta_2}^{\beta_2}(s,z-y)\,dz \\
&+c_3(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y) \\
&+c_3(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\left(\rho_0^{\beta_2}(s,x-y)+\rho_{\beta_2}^0(s,x-y)\right) \\
\le\ & c_4\big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-2\beta_2}\Phi^{-1}(s^{-1})^{\beta_2-\gamma} \\
&+(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}\Phi^{-1}(s^{-1})^{\beta_2-\gamma} \\
&\quad+(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}\Phi^{-1}(s^{-1})^{-\gamma}+(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2} \\
&\quad+s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}+s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\big)\rho(0,x-y) \\
\le\ & c_5(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\rho(0,x-y). \qquad (4.32)
\end{aligned}
$$

In the last inequality above we have used the inequality

$$\Phi^{-1}(s^{-1})^{\beta_2} \le a_1^{-\beta_2/\delta_1}\Phi^{-1}(T^{-1})^{-\beta_2}s^{-\beta_2/\delta_1} \le a_1^{-\beta_2/\delta_1}\Phi^{-1}(T^{-1})^{-\beta_2}T^{1-\beta_2/\delta_1}s^{-1}.$$

Using the fact that $x \neq y$ and Lemma 2.3 we see that the term on the right hand side of (4.32) is integrable in $s \in (0,t)$. Moreover, by (1.1), (1.7), (4.12) and Proposition 3.2,

$$
\begin{aligned}
&\int_{|w|>\varepsilon}\int_0^t |\delta_{\phi_y}(t,x,s;w)|\kappa(x,w)J(w)\,ds\,dw \\
\le& 2\kappa_1\gamma_0 C_2\int_{|w|>\varepsilon}\int_0^t\int_{\mathbb{R}^d}p_z(t-s,x-z)(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz j(|w|)\,ds\,dw \\
&+\kappa_1\gamma_0 C_2\int_{|w|>\varepsilon}\int_0^t\int_{\mathbb{R}^d}p_z(t-s,x\pm w-z)(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz j(|w|)\,ds\,dw \\
\le& c_6\int_{|w|>\varepsilon}j(|w|)dw\int_0^t(t-s)\left(\int_{\mathbb{R}^d}\rho(t-s,x-z)(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz\right)ds \\
&+c_6 j(\varepsilon)\int_0^t\int_{\mathbb{R}^d}(t-s)\left(\int_{\mathbb{R}^d}\rho(t-s,x\pm w-z)dw\right)(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz\,ds\,,
\end{aligned}
\qquad (4.33)
$$

which is, by Lemma 2.6(a)–(b), less than or equal to

$$c_7(\varepsilon)\left(\int_0^t s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\rho(t,x-y)\,ds+\int_0^t\int_{\mathbb{R}^d}(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz\,ds\right)$$

$$\le c_8(\varepsilon)\left(\int_0^t s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}ds\rho(t,x-y)+\int_0^t s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}ds\right)<\infty. \tag{4.34}$$

Thus we can apply Fubini's theorem to see that, by (4.31), (4.27) holds for $\varepsilon\in(0,1]$. Moreover, by Fubini's theorem and the dominated convergence theorem in the first equality and the second equality below respectively:

$$\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x)=\lim_{\varepsilon\downarrow 0}\int_0^t \mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\,ds=\int_0^t \lim_{\varepsilon\downarrow 0}\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\,ds\,,$$

which together with (4.31) yields (4.27) for $\varepsilon=0$.
(b) Now we prove (4.28). Note that, by Lemma 3.1(b), $t\mapsto\delta_{p_y}(t,x-y;z)=p_y(t,x-y+z)+p_y(t,x-y-z)-2p_y(t,x-y)$ is continuous. Let $\varepsilon\in(0,t)$. By (3.12),

$$\begin{aligned}|\delta_{p_y}(t,x-y;z)| &\le c_{11}\left(\Phi^{-1}(t^{-1})|z|^2\wedge 1\right)t\left(\rho(t,x-y\pm z)+\rho(t,x-y)\right)\\ &\le c_{12}\frac{t}{\varepsilon}\left(\Phi^{-1}(\varepsilon^{-1})|z|^2\wedge 1\right)\varepsilon\left(\rho(\varepsilon,x-y\pm z)+\rho(\varepsilon,x-y)\right).\end{aligned}$$

By (1.7) and the proof of (3.16) we see that the right-hand side multiplied by $\kappa(x,z)J(z)$ is integrable with respect to dz. This shows that the family $\{\delta_{p_y}(t,x-y;z)\kappa(x,z)J(z): t\in(\varepsilon,T)\}$ is dominated by an integrable function. Now by the dominated convergence theorem we see that $t\mapsto\mathcal{L}^{\mathfrak{K}_x}p_y(t,x-y)$ is continuous on $(0,T]$.

Let $\beta_2\in(0,\delta_1/2)\cap(0,\beta]$ and $\gamma\in(0,\beta_2)$. By (4.32),

$$\left|\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s)\right|\le c_5(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\rho(0,x-y)\,. \tag{4.35}$$

Note that for $0<t\le t+h\le T$,

$$\begin{aligned}&\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x)-\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x)\\ &=\int_t^{t+h}\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s)ds+\int_0^t\left(\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s)-\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s)\right)ds.\end{aligned} \tag{4.36}$$

When $h\le t/2$, by (2.3) and (2.4), we have

$$\begin{aligned}&\int_t^{t+h}(t+h-s)^{-1}\Phi^{-1}((t+h-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}ds\\ &=\int_0^h r^{-1}\Phi^{-1}(r^{-1})^{\gamma-\beta_2}(t+h-r)^{-1}\Phi^{-1}((t+h-r)^{-1})^{-\gamma}dr\\ &\le c_{13}\int_0^h r^{-1}\Phi^{-1}(r^{-1})^{\gamma-\beta_2}(t-r)^{-1}\Phi^{-1}((t-r)^{-1})^{-\gamma}dr\,,\end{aligned}$$

and so by Lemma 2.4 and (4.35) we get

$$\lim_{h\to 0}\int_t^{t+h}\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s)ds=0. \tag{4.37}$$

Note that, by (4.30) we can apply the dominated convergence theorem and use the continuity of $t \mapsto \mathcal{L}^{\mathfrak{K}_x} p_y(t, x-y)$ so that for each $s \in (0,t)$,

$$
\begin{aligned}
&\lim_{h\to 0}(\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s) - \mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s)) \\
&= \int_{\mathbb{R}^d} \lim_{h\to 0}(\mathcal{L}^{\mathfrak{K}_x} p_z(t+h-s,\cdot)(x-z) - \mathcal{L}^{\mathfrak{K}_x} p_z(t-s,\cdot)(x-z))q(s,z,y)\,dz = 0. \qquad (4.38)
\end{aligned}
$$

By Lemma 2.3, $s \mapsto (t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}$ is integrable in $(0,t)$, so using (4.35), we can apply the dominated convergence theorem and use (4.38) to get that

$$
\lim_{h\to 0}\int_0^t (\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s) - \mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s))ds = 0. \qquad (4.39)
$$

Combining (4.37)–(4.39) we get the desired continuity.
(c) Finally we show (4.29). Since $\beta+\delta_1 > 1$ and $\delta_1 \in (2/3, 2)$, we can and will choose $\beta_2 \in (0 \vee (1-\delta_1), \delta_1/2) \cap (0,\beta]$ and $\gamma \in (0, \beta_2 \wedge (\beta_2+\delta_1-1) \wedge (\delta_1 - 2\beta_2))$. For example, one can take $\beta_2 = \beta \wedge (1/3)$.

For each fixed $0 < s < t$ and $he_i = (0,\dots,h,\dots,0) \in \mathbb{R}^d$ with $|h| \le 1/(2\Phi^{-1}((t-s)^{-1}))$, by (3.11), (3.9), (2.1) and (4.12) we have

$$
\begin{aligned}
&\frac{1}{h}\big|p_z(t-s,x-z+he_i) - p_z(t-s,x-z)\big||q(s,z,y)| \\
&\le c\frac{1}{h}\left((\Phi^{-1}((t-s)^{-1})|h|)\wedge 1\right)(t-s)\left(\rho(t-s,x-z+he_i)+\rho(t-s,x-z)\right)|q(s,z,y)| \\
&\le 2^{d+2}c(t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x-z)(\rho_0^{\beta_2}+\rho_{\beta_2}^0)(s,z-y) \qquad (4.40)
\end{aligned}
$$

which is integrable in $z \in \mathbb{R}^d$ by Lemma 2.6(b). Thus we can use the dominated convergence theorem and (4.21) to get that for $s \in (0,t)$,

$$
\partial_i\phi_y(t,\cdot,s)(x) = \int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,. \qquad (4.41)
$$

Let

$$
\begin{aligned}
\partial_i\phi_y(t,\cdot,s)(w) &= \int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)q(s,z,y)\,dz \\
&= \mathbf{1}_{[t/2,t)}(s)\int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)\left(q(s,z,y)-q(s,w,y)\right)dz \\
&\quad + \mathbf{1}_{[t/2,t)}(s)\int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)\,q(s,w,y)dz \\
&\quad + \mathbf{1}_{(0,t/2)}(s)\int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)\,q(s,z,y)dz \\
&=: \mathbf{1}_{[t/2,t)}(s)R_1(t,s,w,y) + \mathbf{1}_{[t/2,t)}(s)R_2(t,s,w,y) \\
&\quad + \mathbf{1}_{(0,t/2)}(s)R_3(t,s,w,y)\,. \qquad (4.42)
\end{aligned}
$$

Let $x' \in B(x, |x-y|/4)$. Then it follows from Proposition 3.2 and (4.13) that for $s \in [t/2, t)$,

$$
\begin{aligned}
&\big|R_1(t,s,x',y)\big| \\
&\le \int_{\mathbb{R}^d} |\partial_i p_z(t-s,\cdot)(x'-z)||q(s,z,y)-q(s,x',y)|dz \\
&\le \int_{\mathbb{R}^d} \Big((t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x'-z)\left(|x'-z|^{\beta_2-\gamma}\wedge 1\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x'-y) \\
&\quad +(t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x'-z)\left(|x'-z|^{\beta_2-\gamma}\wedge 1\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\Big)\ dz \\
&= (t-s)\left(\int_{\mathbb{R}^d}\rho_{-1}^{\beta_2-\gamma}(t-s,x'-z)dz\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x'-y). \\
&\quad + (t-s)\int_{\mathbb{R}^d}\rho_{-1}^{\beta_2-\gamma}(t-s,x'-z)\rho_\gamma^0(s,z-y)dz \\
&\quad + (t-s)\int_{\mathbb{R}^d}\rho_{-1}^{\beta_2-\gamma}(t-s,x'-z)\rho_{\gamma-\beta_2}^{\beta_2}(s,z-y)dz \\
&\le c_9\Big(\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x'-y) \\
&\quad +\big(\Phi^{-1}((t-s)^{-1})^{1-2\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2}+\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma} \\
&\quad +(t-s)s^{-1}\Phi^{-1}(s^{-1})(\Phi^{-1}(s^{-1})^{-\gamma}+\Phi^{-1}(s^{-1})^{-\beta_2})\big)\,\rho(t,x'-y)\Big)\Big) \\
&\le c_{10}\big(\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2} \\
&\quad +\Phi^{-1}((t-s)^{-1})^{1-2\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2}+\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma} \\
&\quad +(t-s)s^{-1}\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\gamma}\big)\,\rho(t,(x-y)/2).
\end{aligned}
\tag{4.43}
$$

Here the third inequality follows from Lemma 2.6(a)–(b). Since $\delta_1 > 2/3 > 1/2$ and $\gamma < \delta_1+\beta_2-1$, using Lemma 2.3 (so that $\int_{t/2}^t \Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}ds$ and $\int_{t/2}^t (t-s)\Phi^{-1}((t-s)^{-1})ds$ are finite) it is straightforward to see that the function on the right-hand side above is integrable in s over $[t/2, t)$.

Next, for $s \in [t/2, t)$, using (4.12) in the second and (4.4) in the third line below,

$$
\begin{aligned}
\big|R_2(t,s,x',y)\big| &= \left|\int_{\mathbb{R}^d}\partial_i p_z(t-s,\cdot)(x'-z)\,dz\right| q(s,x',y) \\
&\le \left|\int_{\mathbb{R}^d}\partial_i p_z(t-s,\cdot)(x'-z)\,dz\right|\left(\rho_0^{\beta_2}+\rho_{\beta_2}^0\right)(s,x',y) \\
&\le c\Phi^{-1}((t-s)^{-1})^{1-\beta_2}\rho(t,x'-y) \\
&\le c\Phi^{-1}((t-s)^{-1})^{1-\beta_2}\rho(t,(x-y)/2).
\end{aligned}
\tag{4.44}
$$

Since $\int_{t/2}^t \Phi^{-1}((t-s)^{-1})^{1-\beta_2}\,ds < \infty$ because $\beta_2+\delta_1 > 1$, the right-hand side above is integrable in s over $[t/2, t)$.

Finally for $s \in (0, t/2]$, since $\beta_2 < \delta_1/2$,

$$
\big|R_3(t,s,x',y)\big| \le \int_{\mathbb{R}^d}|\partial_i p_z(t-s,\cdot)(x'-z)|q(s,z,y)|dz
$$

$$
\begin{aligned}
&\le c\int_{\mathbb{R}^d}(t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x'-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y)dz\\
&= c(t-s)\int_{\mathbb{R}^d}\rho_{-1}(t-s,x'-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y)dz\\
&\le c(t-s)\Big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{1-\beta_2}+(t-s)^{-1}\Phi^{-1}((t-s)^{-1})\\
&\quad+(t-s)^{-1}\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\beta_2}+\Phi^{-1}((t-s)^{-1})s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\Big)\rho(t,x'-y)\\
&\le c\Big(\Phi^{-1}((t-s)^{-1})+\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\beta_2}\\
&\quad+(t-s)\Phi^{-1}((t-s)^{-1})s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\Big)\rho(t,x'-y),
\end{aligned} \tag{4.45}
$$

which is integrable using Lemma 2.3.

Hence we can use the dominated convergence theorem and (4.41) to conclude that

$$
\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\big(\phi_y(t,x+w)-\phi_y(t,x)\big) &= \lim_{h\to 0}\int_0^t\int_0^1 \partial_i\phi_y(t,\cdot,s)(x+\theta w)\,d\theta ds ds\\
&= \int_0^t \partial_i\phi_y(t,\cdot,s)(x)ds = \int_0^t\int_{\mathbb{R}^d}\partial_i p_z(t-s,\cdot)(x-z)q(s,z,y)\,dzds,
\end{aligned}
$$

which gives (4.29). □

4.4 Estimates and Smoothness of $p^\kappa(t,x,y)$

Now we define and study the function

$$
p^\kappa(t,x,y) := p_y(t,x-y)+\phi_y(t,x) = p_y(t,x-y)+\int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,. \tag{4.46}
$$

Lemma 4.8 *(1) For every $T\ge 1$ and $\beta_2\in(0,\beta]\cap(0,\delta_1/2)$, there is a constant $c_1 = c_1(T,d,\delta_1,\beta_2,\gamma,\kappa_0,\kappa_1,\kappa_2)>0$ so that for all $t\in(0,T]$ and $x,y\in\mathbb{R}^d$, $p^\kappa(t,x,y)\le c_1 t\rho(t,x-y)$. (2) For any $\gamma\in(0,\delta_1)\cap(0,1]$ and $T\ge 1$ there exists $c_2=c_2(T,d,\delta_1,\beta_2,\gamma,\kappa_0,\kappa_1,\kappa_2)>0$ such that for all $x,x',y\in\mathbb{R}^d$ and $t\in(0,T]$,*

$$
|p^\kappa(t,x,y)-p^\kappa(t,x',y)|\le c_2|x-x'|^\gamma\; t\Big(\rho^0_{-\gamma}(t,x-y)+\rho^0_{-\gamma}(t,x'-y)\Big).
$$

Proof. Throughout this proof we assume that $x,x',y\in\mathbb{R}^d$ and $t\in(0,T]$.
(1) By the estimate of p_z (Proposition 3.2), (4.12), Lemma 2.6(c), (2.14) and (2.15), we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)|q(s,z,y)|\,dz\,ds\\
&\le\; c_1\int_0^t\int_{\mathbb{R}^d}(t-s)\rho(t-s,x-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y)\,dz\,ds\\
&\le\; c_2 t\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(t,x-y)
\end{aligned} \tag{4.47}
$$

$$\leq \ 2\Phi^{-1}(T^{-1})^{-\beta_2} c_2 t\rho(t, x-y), \qquad \text{for all } t \in (0, T]\,.$$

Therefore, $p^\kappa(t, x, y) \leq p_y(t, x-y) + |\phi_y(t, x)| \leq c_4 t\rho(t, x-y)$.
(2) We have by (3.11) and the fact that $\gamma \leq 1$,

$$\begin{aligned} |p_z(t, x-z) - p_z(t, x'-z)| &\leq c_1|x-x'|^\gamma t\Phi^{-1}(t^{-1})^\gamma \big(\rho(t, x-z) + \rho(t, x'-z)\big) \\ &= c_1|x-x'|^\gamma t\big(\rho^0_{-\gamma}(t, x-z) + \rho^0_{-\gamma}(t, x'-z)\big)\,. \end{aligned}$$

Thus, by (4.12) and a change of the variables, we further have

$$\begin{aligned} &|\phi_y(t, x) - \phi_y(t, x')| \leq \int_0^t \int_{\mathbb{R}^d} |p_z(t-s, x-z) - p_z(t-s, x'-z)|\,|q(s, z, y)|\,dz\,ds \\ &\leq c_2|x-x'|^\gamma \int_0^t \int_{\mathbb{R}^d} (t-s)\Big(\rho^0_{-\gamma}(t-s, x-z) + \rho^0_{-\gamma}(t-s, x'-z)\Big)\big(\rho^{\beta_2}_0 + \rho^0_{\beta_2}\big)(s, z-y)\,dz\,ds \\ &\leq c_3|x-x'|^\gamma t\Big(\rho^0_{-\gamma+\beta_2}(t, x-y) + \rho^{\beta_2}_{-\gamma}(t, x-y) + \rho^0_{-\gamma+\beta_2}(t, x'-y) + \rho^{\beta_2}_{-\gamma}(t, x'-y)\Big) \\ &\leq 2c_3\Phi^{-1}(T^{-1})^{-\beta_2}|x-x'|^\gamma t\big(\rho^0_{-\gamma}(t, x-y) + \rho^0_{-\gamma}(t, x'-y)\big), \qquad \text{for all } t \in (0, T]\,. \end{aligned}$$

Since $\gamma \in (0, \delta_1)$, the penultimate line follows from (2.19) (with $\theta = 0$), and the last line by (2.14) and (2.15). The claim of the lemma follows by combining the two estimates. □

The following result is the counterpart of [6, Lemma 3.7].

Lemma 4.9 *The function $p^\kappa(t, x, y)$ defined in* (4.46) *is jointly continuous on* $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. The joint continuity of $p_y(t, x-y)$ was shown in Lemma 4.2. For $\phi_y(t, x)$ we use (4.22) and the joint continuity of $q(s, z, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ together with the dominated convergence theorem. This is justified by the estimates $p_z(t-s, x-z) \leq c_1(t-s)\rho(t-s, x-z)$ and (4.12) which yield that $|p_z(t-s, x-z)q(s, z, y)| \leq c_2(t-s)\rho(t-s)\left(\rho^{\beta_2}_0 + \rho^0_{\beta_2}\right)(s, z-y)$ for $\beta_2 \in (0, \beta] \cap (0, \delta_1/2)$. The latter function is integrable over $(0, t] \times \mathbb{R}^d$ with respect to $ds\,dz$ by Lemma 2.6. □

Now we define the operator $\mathcal{L}^\kappa$ as in (1.8) which can be rewritten as

$$\mathcal{L}^\kappa f(x) = \mathcal{L}^{\kappa,0} f(x) = \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x), \quad \text{where } \mathcal{L}^{\kappa,\varepsilon} f(x) = \frac{1}{2}\int_{|z|>\varepsilon} \delta_f(x; z)\kappa(x, z)J(z)\,dz. \tag{4.48}$$

Note that for a fixed $x \in \mathbb{R}^d$, it holds that $\mathcal{L}^\kappa f(x) = \mathcal{L}^{\mathfrak{K}_x} f(x)$. This will be used later on.

The following result is the counterpart of [6, Lemma 4.2].

Lemma 4.10 *For every $T \geq 1$, there is a constant $c_1 = c_1(T, d, \delta_1, a_1, \beta, C_*, \gamma_0, \kappa_0, \kappa_1, \kappa_2) > 0$ such that for all $\varepsilon \in [0, 1]$,*

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_1\rho(t, x-y), \qquad \textit{for all } t \in (0, T] \textit{ and } x, y \in \mathbb{R}^d, x \neq y \tag{4.49}$$

and if $\beta+\delta_1>1$ *and* $\delta_1\in(2/3,2)$ *we also have*

$$|\nabla_x p^\kappa(t,x,y)|\le c_1 t\Phi^{-1}(t^{-1})\rho(t,x-y)\quad \text{for all } t\in(0,T] \text{ and } x,y\in\mathbb{R}^d, x\neq y\,. \tag{4.50}$$

Proof. By (3.16) and the fact that for fixed x, $\mathcal{L}^{\kappa,\varepsilon}f(x)=\mathcal{L}^{\mathfrak{K}_x,\varepsilon}f(x)$ for $\varepsilon\in[0,1]$, we see that

$$|\mathcal{L}^\kappa p_y(t,\cdot)(x-y)|\le c_1\rho(t,x-y),\qquad \text{for all } t\in(0,T] \text{ and } \varepsilon\in[0,1].$$

Let $\varepsilon\in[0,1]$. By recalling the definition (4.22) of ϕ_y and using (4.27), we have

$$\begin{aligned}
\mathcal{L}^{\kappa,\varepsilon}\phi_y(t,x) &= \int_{t/2}^t\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_x,\varepsilon}p_z(t-s,\cdot)(x-z)\left(q(s,z,y)-q(s,x,y)\right)dz\,ds\\
&\quad+\int_{t/2}^t\left(\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_x,\varepsilon}p_z(t-s,\cdot)(x-z)\,dz\right)q(s,x,y)\,ds\\
&\quad+\int_0^{t/2}\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_x,\varepsilon}p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds\\
&=: Q_1(t,x,y)+Q_2(t,x,y)+Q_3(t,x,y)\,.
\end{aligned}$$

Let $\beta_2\in(0,\delta_1/2)\cap(0,\beta]$. For $Q_1(t,x,y)$ we use (3.16), Lemmas 2.2(b), 2.3 and 2.6(a) and (c) to get that for any $\gamma\in((2-\delta_1)\beta_2/2,\beta_2)$,

$$\begin{aligned}
|Q_1(t,x,y)| &\le c_1\int_{t/2}^t\left(\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\,dz\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y)\,ds\\
&\quad+c_1\int_{t/2}^t\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz\,ds\\
&\le c_2\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(t,x-y)\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\,dzds\\
&\quad+c_1\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2-\gamma}(t-s,x-z)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz\,ds\\
&\le c_3\rho_{\gamma-\beta_2}^0(t,x-y)\Phi^{-1}(t^{-1})^{-\beta_2-\gamma}+c_3\left(\rho_{\beta_2}^0+\rho_\gamma^{\beta_2-\gamma}+\rho_0^{\beta_2}\right)(t,x-y)\\
&\le c_4\rho(t,x-y),\qquad \text{for all } t\in(0,T]\,,
\end{aligned}$$

where the last two lines follow from (2.14) and (2.15).

For $Q_2(t,x,y)$, by (4.2), (4.12), Lemmas 2.2(b), 2.3, (2.14) and (2.15),

$$\begin{aligned}
&|Q_2(t,x,y)|\le c_5\int_{t/2}^t(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\left(\rho_{\beta_2}^0+\rho_0^{\beta_2}\right)(s,x-y)\,ds\\
&\le c_6\left(\rho_{\beta_2}^0+\rho_0^{\beta_2}\right)(t,x-y)\int_0^t(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\,ds\\
&\le c_7\rho(t,x-y)\Phi^{-1}(t^{-1})^{-\beta_2}\le c_7\Phi^{-1}(T^{-1})^{-\beta_2}\rho(t,x-y),\qquad \text{for all } t\in(0,T]\,.
\end{aligned}$$

For $Q_3(t,x,y)$, by (3.16), (4.12), Lemma 2.6(c), (2.14) and (2.15),

$$|Q_3(t,x,y)| \le c_7\int_0^{t/2}\int_{\mathbb{R}^d}\rho(t-s,x-z)\left(\rho_{\beta_2}^0+\rho_0^{\beta_2}\right)(s,z-y)\,dz\,ds$$

$$\leq \; 2\frac{c_7}{t}\int_0^t\int_{\mathbb{R}^d}(t-s)\rho(t-s,x-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y)\,dz\,ds$$
$$\leq \; c_8\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(t,x-y)\leq 2c_8\Phi^{-1}(T^{-1})^{-\beta}\rho(t,x-y)\,.$$

Combining the above calculations and (4.46) we obtain (4.49).
(ii) Since $\beta+\delta_1>1$ and $\delta_1\in(2/3,2)$, we can and will choose $\beta_2\in(0\vee(1-\delta_1),\delta_1/2)\cap(0,\beta]$ and $\gamma\in(0,\beta_2\wedge(\beta_2+\delta_1-1)\wedge(\delta_1-2\beta_2))$. By (4.29) and (4.42)–(4.45) we have

$$\begin{aligned}|\nabla_x\phi_y(t,x)|\leq c_1\rho(t,x-y)\Bigg(&\int_0^{t/2}\Phi^{-1}((t-s)^{-1})+\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\beta_2}\\ &+(t-s)\Phi^{-1}((t-s)^{-1})s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}ds\\ +&\int_{t/2}^t\Phi^{-1}((t-s)^{-1})^{1-\beta_2}+\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2}\\ &+\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\beta_2}+(t-s)s^{-1}\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\gamma}ds\Bigg).\end{aligned}\tag{4.51}$$

Since $\beta+\delta_1>1$, $\delta_1>2/3>1/2$ and $\gamma<\delta_1+\beta_2-1$, using Lemma 2.3 we see that $\int_{t/2}^t\Phi^{-1}((t-s)^{-1})^{1-\beta_2}\,ds\leq c_2t\Phi^{-1}(t^{-1})^{1-\beta_2}$, $\int_{t/2}^t\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}ds\leq c_3t\Phi^{-1}(t^{-1})^{1-\beta_2+\gamma}$ and $\int_0^t(t-s)\Phi^{-1}((t-s)^{-1})ds\leq c_4t^2\Phi^{-1}(t^{-1})$. Thus, by Lemma 2.3, (4.51) is bounded above by $c_5t\Phi^{-1}(t^{-1})\rho(t,x-y)$. Now, (4.50) follows immediately from this, (4.46), (4.29) and Proposition 3.2. □

We will also need the following corollary, which follows from (4.28).

Corollary 4.11 *For $x\neq y$, the function $t\mapsto\mathcal{L}^\kappa p^\kappa(t,x,y)$ is continuous on $(0,\infty)$.*

5 Proofs of main results

5.1 A nonlocal maximum principle

We first establish a somewhat different version of [6, Theorem 4.1].

Theorem 5.1 *Suppose there exists a function $g:\mathbb{R}^d\to(0,\infty)$ such that (1.9) holds. Let $T>0$ and $u\in C_b([0,T]\times\mathbb{R}^d)$ be such that*

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|u(t,x)-u(0,x)|=0\,,\tag{5.1}$$

and for each $x\in\mathbb{R}^d$,

$$t\mapsto\mathcal{L}^\kappa u(t,x)\text{ is continuous on }(0,T].\tag{5.2}$$

Suppose that $u(t,x)$ satisfies the following inequality: for all $(t,x)\in(0,T]\times\mathbb{R}^d$,

$$\partial_t u(t,x)\leq\mathcal{L}^\kappa u(t,x)\,.\tag{5.3}$$

Then for all $t \in (0,T)$,

$$\sup_{x\in\mathbb{R}^d} u(t,x) \le \sup_{x\in\mathbb{R}^d} u(0,x)\,. \tag{5.4}$$

Proof. Choose $a > 0$ such that

$$\mathcal{L}^\kappa g(x) \le a g(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{5.5}$$

Let $\delta, \varepsilon > 0$ and $u^\delta_\varepsilon(t,x) := u(t,x) - \delta(t - \varepsilon + e^{at} g(x))$. Then by (5.3) and (5.5), for all $(t,x) \in (0,T] \times \mathbb{R}^d$, we have

$$\begin{aligned}\partial_t u^\delta_\varepsilon(t,x) &= \partial_t u(t,x) - \delta(1 + a e^{at} g(x)) \le \mathcal{L}^\kappa u(t,x) - \delta - \delta a e^{at} g(x)\\ &= \mathcal{L}^\kappa u^\delta_\varepsilon(t,x) - \delta + \delta e^{at}(\mathcal{L}^\kappa g(x) - a g(x)) \le \mathcal{L}^\kappa u^\delta_\varepsilon(t,x) - \delta.\end{aligned} \tag{5.6}$$

Since $u \in C_b([0,T]\times\mathbb{R}^d)$, by letting $\delta \to 0$ and $\varepsilon \to 0$, it suffices to show that

$$\sup_{x\in\mathbb{R}^d} u^\delta_\varepsilon(t,x) \le \sup_{x\in\mathbb{R}^d} u^\delta_\varepsilon(\varepsilon,x), \quad t \in (\varepsilon, T]\,. \tag{5.7}$$

Fix $\delta, \varepsilon > 0$ and suppose that (5.7) does not hold. Then, by the continuity of u^δ_ε and the fact that $\lim_{x\to\infty} u^\delta_\varepsilon(t,x) = -\infty$ (which is a consequence of (1.9)), there exist $t_0 \in (\varepsilon, T]$ and $x_0 \in \mathbb{R}^d$ such that

$$\sup_{t\in(\varepsilon,T], x\in\mathbb{R}^d} u^\delta_\varepsilon(t,x) = u^\delta_\varepsilon(t_0,x_0). \tag{5.8}$$

Thus by (5.6), for $h \in (0, t_0 - \varepsilon)$,

$$0 \le \frac{1}{h}(u^\delta_\varepsilon(t_0,x_0) - u^\delta_\varepsilon(t_0-h,x_0)) = \frac{1}{h}\int_{t_0-h}^{t_0} \partial_t u^\delta_\varepsilon(s,x_0)ds \le \frac{1}{h}\int_{t_0-h}^{t_0} \mathcal{L}^\kappa u^\delta_\varepsilon(s,x_0)ds - \delta.$$

Letting $h \to 0$ and using (5.2) and (5.8) we get

$$\begin{aligned}0 &\le \mathcal{L}^\kappa u^\delta_\varepsilon(t_0,x_0) - \delta\\ &= \text{p.v.}\int_{\mathbb{R}^d} \left(u^\delta_\varepsilon(t_0,x_0+z) - u^\delta_\varepsilon(t_0,x_0)\right)\kappa(x_0,z)J(z)\,dz - \delta \le -\delta,\end{aligned}$$

which gives a contradiction. Therefore (5.7) holds. □

Remark 5.2 Suppose that $\int_{|z|>1} |z|^\varepsilon j(|z|)dz < \infty$ for some $\varepsilon > 0$. Let $g(x) = (1+|x|^2)^{\varepsilon/2}$. Note that

$$|\partial_{i,j} g(x)| \le c_1(1+|x|)^{\varepsilon-2}, \quad i,j = 1,\ldots,d. \tag{5.9}$$

By (5.9) and (3.7), we have that for $|x| \le 1$,

$$|\mathcal{L}^\kappa g(x)| \le \gamma_0 \int_{|z|\le1} |\delta_g(x;z)| j(|z|)dz + \gamma_0 g(x)\int_{|z|>1} j(|z|)dz + \gamma_0 \int_{|z|>1} g(x\pm z) j(|z|)dz$$

$$\leq c_2\left(\int_{|z|\leq 1}|z|^2 j(|z|)dz+\int_{|z|>1}j(|z|)dz+\int_{|z|>1}|z|^{\varepsilon}j(|z|)dz\right)\leq c_3\leq c_3 g(x). \tag{5.10}$$

If $|x|>1$, then by (5.9) and (3.7),

$$\begin{aligned}|\mathcal{L}^{\kappa}g(x)| &\leq \gamma_0\int_{|z|\leq|x|}|\delta_g(x;z)|j(|z|)dz+\gamma_0 g(x)\int_{|z|>|x|}j(|z|)dz+\gamma_0\int_{|z|>|x|}g(x\pm z)j(|z|)dz\\ &\leq c_3\left(\int_{|z|\leq|x|}|x|^{\varepsilon-2}|z|^2 j(|z|)dz+g(x)\int_{|z|>1}j(|z|)dz+\int_{|z|>|x|}|z|^{\varepsilon}j(|z|)dz\right)\\ &\leq c_4\left(|x|^{\varepsilon}\int_{\mathbb{R}^d}((|z|/|x|)^2\wedge 1)j(|z|)dz+g(x)+1\right)\leq c_5 g(x).\end{aligned} \tag{5.11}$$

Therefore g satisfies (1.9).

5.2 Properties of the semigroup $(P^{\kappa}_t)_{t\geq 0}$

Define

$$P^{\kappa}_t f(x)=\int_{\mathbb{R}^d}p^{\kappa}(t,x,y)f(y)dy.$$

Lemma 5.3 *For any bounded function f, we have*

$$\mathcal{L}^{\kappa}P^{\kappa}_t f(x)=\int_{\mathbb{R}^d}\mathcal{L}^{\kappa}p^{\kappa}(t,\cdot,y)(x)f(y)dy\,. \tag{5.12}$$

Proof. By the same computation as in the proof of (3.16) we have that for all $t\leq T$, $T\geq 1$, and $\varepsilon>0$,

$$\begin{aligned}&t\int_{|z|>\varepsilon}\rho(t,x\pm z)j(|z|)\,dz\\ &\leq\int_{\Phi^{-1}(t^{-1})|z|\leq 1,|z|>\varepsilon}t\rho(t,x\pm z)j(|z|)\,dz+\int_{\Phi^{-1}(t^{-1})|z|>1}t\rho(t,x\pm z)j(|z|)\,dz\\ &\leq c_1 4^{d+1}t\rho(t,x)\int_{|z|>\varepsilon}j(|z|)\,dz+c_1\rho(t.x),\end{aligned}$$

thus by Lemma 4.8(1),

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon}|p^{\kappa}(t,x\pm w,y)-2p^{\kappa}(t,x,y)|\,\kappa(x,w)J(w)\,dw\right)dy\\ &\leq 2\gamma_0\kappa_1\int_{\mathbb{R}^d}\int_{|w|>\varepsilon}|p^{\kappa}(t,x,y)|j(|w|)\,dwdy+\gamma_0\kappa_1\int_{\mathbb{R}^d}\int_{|w|>\varepsilon}|p^{\kappa}(t,x\pm w,y)|j(|w|)\,dwdy\\ &\leq c_2 t\left(\int_{|w|>\varepsilon}j(|w|)\,dw\right)\int_{\mathbb{R}^d}\rho(t,x-y)dy+c_2 t\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon}\rho(t,x\pm w-y)j(|w|)\,dw\right)dy\\ &<\infty.\end{aligned}$$

Thus by Fubini's theorem, for all for bounded function f and $\varepsilon \in (0,1]$,

$$\mathcal{L}^{\kappa,\varepsilon} P_t^\kappa f(x) = \int_{\mathbb{R}^d} \mathcal{L}^{\kappa,\varepsilon} p^\kappa(t,\cdot,y)(x) f(y) dy.$$

Now, (5.12) follows from this, (4.49) and the dominated convergence theorem. □

The following result is the counterpart of [6, Lemma 4.4].

Lemma 5.4 *(a) For any $p \in [1,\infty]$, there exists a constant $c = c(p,d,\delta_1,\beta,\kappa_0,\kappa_1,\kappa_2) > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ and $t > 0$,*

$$\|\mathcal{L}^\kappa P_t^\kappa f\|_p \le ct^{-1}\|f\|_p\,. \tag{5.13}$$

(b) If $f \in L^\infty(\mathbb{R}^d)$, $t \mapsto \mathcal{L}^\kappa P_t^\kappa f$ is a continuous function on $(0,\infty)$.
(c) For any $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^d)$, $t \mapsto \mathcal{L}^\kappa P_t^\kappa f$ is continuous from $(0,\infty)$ into $L^p(\mathbb{R}^d)$.

Proof. (a) Let $p \in [1,\infty]$. By (5.12), Lemma 4.10, Young's inequality and Lemma 2.6(a), we have that for all $f \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \|\mathcal{L}^\kappa P_t^\kappa f\|_p &\le c_1 \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho(t,x-y)|f(y)|\,dy \right|^p dx \right)^{1/p} \\ &\le c_1 \|\rho(t,\cdot)\|_1 \,\|f\|_p \le c_2 t^{-1}\|f\|_p\,. \end{aligned}$$

Inequality (5.13) for $f \in L^p(\mathbb{R}^d)$ now follows by a standard density argument.
(b) For any $\varepsilon \in (0,1)$, by Lemma 4.10 we have for $x \neq y$,

$$\sup_{t\in(\varepsilon,T)} |\mathcal{L}^\kappa p^\kappa(t,x,y)| \le c \sup_{t\in(\varepsilon,T)} \rho(t,x-y) \le c\rho(\varepsilon,x-y)\,.$$

Assume that f is bounded and measurable. By Corollary 4.11, $t \mapsto \mathcal{L}^\kappa p^\kappa(t,x,y)f(y)$ is continuous for $x \neq y$. By the above display, the family $\{\mathcal{L}^\kappa p^\kappa(t,x,y)f(y) : t \in (\varepsilon,1)\}$ is bounded by the integrable function $\rho(\varepsilon,x-y)|f(y)|$. Now it follows from the dominated convergence theorem and (5.12) that $t \mapsto \mathcal{L}^\kappa P_t^\kappa f(x)$ is continuous.
(c) Let $p \in [1,\infty)$. When $f \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the claim follows similarly as (b) by using (5.12) and the domination by the L^p-function $\int_{\mathbb{R}^d} \rho(\varepsilon,x-y)f(y)\,dy$. The claim for $f \in L^p(\mathbb{R}^d)$ now follows by standard density argument and (5.13). □

Remark 5.5 Note that Lemma 5.4 uses only the following properties of $p^\kappa(t,x,y)$: (5.12), $|\mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)| \le c_1(T)\rho(t,x-y)$ for $t \in (0,T]$ and $t \mapsto \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ is continuous on $(0,T]$. Moreover, Lemma 5.3 uses only the following properties of $p^\kappa(t,x,y)$: $p^\kappa(t,\cdot,y)(x) \le c_2(T)t\rho(t,x-y)$ and $|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t,\cdot,y)(x)| \le c_3(T)\rho(t,x-y)$ for $\varepsilon \in [0,1]$ and $t \in (0,T]$.

The following result is the counterpart of [6, Lemma 4.3].

Lemma 5.6 *For any bounded Hölder continuous function $f \in C_b^\eta(\mathbb{R}^d)$, we have*

$$\mathcal{L}^\kappa \left(\int_0^t P_s^\kappa f(\cdot) ds \right)(x) = \int_0^t \mathcal{L}^\kappa P_s^\kappa f(x) ds \,, \quad x \in \mathbb{R}^d \,. \tag{5.14}$$

Proof. Define

$$T_t f(x) = \int_{\mathbb{R}^d} p_y(t, x-y) f(y) dy, \quad S_t f(x) = \int_{\mathbb{R}^d} q(t,x,y) f(y) dy$$

and

$$R_t f(x) = \int_0^t T_{t-s} S_s f(x) ds.$$

Then, by Fubini's theorem and (4.12), for all for bounded function f,

$$P_t^\kappa f(x) = T_t f(x) + R_t f(x). \tag{5.15}$$

We now assume $\varepsilon \in (0,1]$ and $0 < s < t \le T$, $T \ge 1$. Suppose that $|f(x) - f(y)| \le c_1(|x-y|^\eta \wedge 1)$. Without loss of generality we may and will assume that $\eta < \beta$. By Fubini's theorem, (1.7), (1.1) and (3.16),

$$\mathcal{L}^{\kappa,\varepsilon} T_t f(x) = \int_{\mathbb{R}^d} \mathcal{L}^{\kappa,\varepsilon} p_z(s,\cdot)(x-z) f(z)\, dz.$$

Thus,

$$\begin{aligned} |\mathcal{L}^{\kappa,\varepsilon} T_s f(x)| \;\le\; & \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} |\delta_{p_z}(s, x-z; w)| \kappa(x,w) J(w)\, dw \right) |f(z) - f(x)|\, dz \\ & + \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \delta_{p_z}(s, x-z; w) \kappa(x,w) J(w)\, dw \right) dz \right| |f(x)| \,. \end{aligned}$$

By using (1.7), (3.16), (4.2) and (2.17), for any $\beta_1 \in (0, \delta_1) \cap (0, \beta]$, $|\mathcal{L}^{\kappa,\varepsilon} T_s f(x)|$ is bounded by

$$\begin{aligned} & c_1 \int_{\mathbb{R}^d} \rho(s, x-z) \left(|x-z|^\eta \wedge 1 \right) dz + c_1\, s^{-1} \Phi^{-1}(s^{-1})^{-\beta_1} \\ \le\; & c_2\, s^{-1} \Phi^{-1}(s^{-1})^{-\eta} + c_1\, s^{-1} \Phi^{-1}(s^{-1})^{-\beta_1} \,, \end{aligned}$$

and the right hand side is integrable by Lemma 2.3. Thus by the dominated convergence theorem and Fubini's theorem,

$$\mathcal{L}^\kappa \int_0^t T_s f(x)\, ds = \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} \int_0^t T_s f(x)\, ds = \int_0^t \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} T_s f(x) ds = \int_0^t \mathcal{L}^\kappa T_s f(x)\, ds \,. \tag{5.16}$$

It follows from (4.13), (2.17) and the boundedness of f that for any $\beta_2 \in (0, \beta] \cap (0, \delta_1/2)$ and $\gamma \in (0, \beta_2)$, we have

$$|S_s f(x) - S_s f(x')| \le c_3 s^{-1} \Phi^{-1}(s^{-1})^{-\gamma} \left(|x-x'|^{\beta_2 - \gamma} \wedge 1 \right). \tag{5.17}$$

It follows from (4.12), (2.17) and the boundedness of f that

$$|S_s f(x)| \le c_4 s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}. \tag{5.18}$$

We use Lemma 4.8(1) and Fubini's theorem in the first line below, which can be justified by an argument similar to (4.33) and (4.34):

$$\begin{aligned}
&|\mathcal{L}^{\kappa,\varepsilon} R_s f(x)| \\
\le\ & \int_0^s \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \delta_{p_z}(s-r, x-z; w)\kappa(x,w)J(w)\,dw \right) S_r f(z)\,dz \right| dr \\
\le\ & \int_0^s \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} |\delta_{p_z}(s-r, x-z; w)|\kappa(x,w)J(w)\,dw \right) |S_r f(z) - S_r f(x)|\,dzdr \\
& + \int_0^s \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \delta_{p_z}(s-r, x-z; w)\kappa(x,w)J(w)\,dw \right) dz \right| |S_r f(x)| dr\,.
\end{aligned}$$

By using (1.7), (3.16), (4.2), (2.17), (5.17), (5.18) and Lemma 2.3, we further have that

$$\begin{aligned}
|\mathcal{L}^{\kappa,\varepsilon} R_s f(x)| \ \le\ & c_5 \int_0^s \int_{\mathbb{R}^d} \rho(s-r, x-z) r^{-1}\Phi^{-1}(r^{-1})^{-\gamma} \left(|x-z|^{\beta_2-\gamma} \wedge 1\right) dzdr \\
& + c_5 \int_0^s r^{-1}\Phi^{-1}(r^{-1})^{-\beta_2} dr \\
\le\ & c_6 \int_0^s (s-r)^{-1}\Phi^{-1}((s-r)^{-1})^{-(\beta_2-\gamma)} r^{-1}\Phi^{-1}(r^{-1})^{-\gamma} dr + c_5 \int_0^s r^{-1}\Phi^{-1}(r^{-1})^{-\beta_2} dr \\
\le\ & c_7 s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2} + c_5 \Phi^{-1}(s^{-1})^{-\beta_2} = 2c_7 s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\,,
\end{aligned}$$

and the right hand side is integrable by Lemma 2.3. This justifies the use of the dominated convergence theorem in the second line of the following calculation:

$$\mathcal{L}^{\kappa} \int_0^t R_s f(x)\,ds = \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} \int_0^t R_s f(x)\,ds = \int_0^t \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} R_s f(x)\,ds = \int_0^t \mathcal{L}^{\kappa} R_s f(x)\,ds\,. \tag{5.19}$$

Combining (5.19) with (5.16) and (5.15), we arrive at the conclusion of this lemma. □

5.3 Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1. By using Lemma 4.6 in the second equality, (4.6) in the third, (4.11) in the fourth, (4.6) in the fifth, and Lemma 4.7 in the sixth equality, we have

$$\begin{aligned}
\partial_t p^{\kappa}(t,x,y) &= \partial_t p_y(t, x-y) + \partial_t \phi_y(t,x) \\
&= \mathcal{L}^{\mathfrak{K}_y} p_y(t, x-y) + \left(q(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z} p_z(t-s, \cdot)(x-z) q(s,z,y)\,dz\,ds \right) \\
&= \left(\mathcal{L}^{\mathfrak{K}_x} p_y(t, x-y) - q_0(t,x,y) \right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(q(t,x,y)+\int_0^t\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_z}p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds\right)\\
=\;&\mathcal{L}^{\mathfrak{K}_x}p_y(t,x-y)+\int_0^t\int_{\mathbb{R}^d}q_0(t-s,x-z)q(s,z,y)\,dz\,ds\\
&+\int_0^t\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_z}p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds\\
=\;&\mathcal{L}^{\mathfrak{K}_x}p_y(t,x-y)+\int_0^t\int_{\mathbb{R}^d}\mathcal{L}^{\mathfrak{K}_x}p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds\\
=\;&\mathcal{L}^{\kappa}p^{\kappa}(t,x,y)\,.
\end{aligned}
$$

Thus (1.10) holds. The joint continuity of $p^\kappa(t,x,y)$ is proved in Lemma 4.9. Further, if we apply the maximum principle, Theorem 5.1, to $u_f(t,x) := P_t^\kappa f(x)$ with $f \in C_c^\infty(\mathbb{R}^d)$ and $f \le 0$, we get $u_f(t,x) \le 0$ for all $t \in (0,T]$ and all $x \in \mathbb{R}^d$. This implies that $p^\kappa(t,x,y) \ge 0$.
(i) (1.11) is proved in Lemma 4.8(1).
(ii) The estimate (1.12) is given in (4.49), while continuity of $t \mapsto \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ is proven in Corollary 4.11.
(iii) Let f be a bounded and uniformly continuous function. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ for all $|x-y| < \delta$. By (4.5), (1.5), (2.17) and the estimate for $p_y(t,x-y)$ in Proposition 3.2 we have

$$
\begin{aligned}
&\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}p_y(t,x-y)f(y)\,dy-f(x)\right|\\
&\quad=\;\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}p_y(t,x-y)f(y)\,dy-\int_{\mathbb{R}^d}p_y(t,x-y)f(x)\,dy\right|\\
&\quad\le\;c_1\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}t\rho(t,x-y)\,|f(y)-f(x)|\,dy\\
&\quad\le\;\varepsilon c_1\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<\delta}t\rho(t,x-y)dy+2c_1\|f\|_\infty\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\ge\delta}t\rho(t,x-y)dy\\
&\quad\le\;c_2\varepsilon\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}t\rho(t,x-y)dy+2c_1\|f\|_\infty\lim_{t\downarrow 0}t\sup_{x\in\mathbb{R}^d}\int_{|x-y|\ge\delta}\frac{\Phi(|x-y|^{-1})}{|x-y|^d}dy\\
&\quad\le\;c_2\varepsilon+2c_1\|f\|_\infty\lim_{t\downarrow 0}t\int_{|z|\ge\delta}\frac{\Phi(|z|^{-1})}{|z|^d}dz=c_2\varepsilon\,.
\end{aligned}
$$

This implies that

$$
\limsup_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}p_y(t,x-y)f(y)\,dy-f(x)\right|=0\,. \tag{5.20}
$$

Further, by (4.47) and (2.17), for any $\beta_2 \in (0,\beta] \cap (0,\delta_1)$, we have

$$
\left|\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,f(y)dy\right|
$$

$$\leq \; c_3 \|f\|_\infty \, t \int_{\mathbb{R}^d} \left(\rho_0^{\beta_2} + \rho_{\beta_2}^0\right)(t, x-y)\, dy \;\leq\; c_4 \, \Phi^{-1}(t^{-1})^{-\beta_2} \longrightarrow 0\,, \quad t \downarrow 0\,.$$

The claim now follows from this, (4.46) and (5.20).

Uniqueness of the kernel satisfying (1.10)-(1.13): Let $\widetilde{p}^\kappa(t,x,y)$ be another non-negative jointly continuous kernel satisfying (1.10)–(1.13). For any function $f \in C_c^\infty(\mathbb{R}^d)$, define $\widetilde{u}_f(t,x) := \int_{\mathbb{R}^d} \widetilde{p}^\kappa(t,x,y) f(y)\, dy$. By the joint continuity of $\widetilde{p}^\kappa(t,x,y)$, (i) and (iii) we have that

$$\widetilde{u}_f \in C_b([0,T]\times\mathbb{R}^d), \qquad \lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} |\widetilde{u}_f(t,x) - f(x)| = 0\,.$$

By Lemma 5.3 and Remark 5.5,

$$\mathcal{L}^\kappa \widetilde{u}_f(t,x) = \int_{\mathbb{R}^d} \mathcal{L}^\kappa \widetilde{p}^\kappa(t,x,y) f(y)\, dy \quad \text{and} \quad \mathcal{L}^\kappa u_f(t,x) = \int_{\mathbb{R}^d} \mathcal{L}^\kappa p^\kappa(t,x,y) f(y)\, dy. \tag{5.21}$$

Moreover, by Lemma 5.4 and Remark 5.5, $t \mapsto \mathcal{L}^\kappa u_f(t,x)$ and $t \mapsto \mathcal{L}^\kappa \widetilde{u}_f(t,x)$ are continuous on $(0,T]$. Here and in (5.21) we use that $\widetilde{p}^\kappa$ satisfies (i)–(ii).

Let $w(t,x) := u_f(t,x) - \widetilde{u}_f(t,x)$. Then $w(0,x) = 0$, $\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} |w(t,x) - w(0,x)| = 0$, and $t \mapsto \mathcal{L}^\kappa w(t,x)$ is continuous on $(0,T]$. Note that by (1.12) and (1.10),

$$|\partial_t p^\kappa(t,x,y)| + |\partial_t \widetilde{p}^\kappa(t,x,y)| \leq c_5 \rho(t, x-y), \quad t \in (0,T]\,.$$

Thus, by the dominated convergence theorem,

$$\partial_t \widetilde{u}_f(t,x) = \int_{\mathbb{R}^d} \partial_t \widetilde{p}^\kappa(t,x,y) f(y)\, dy \quad \text{and} \quad \partial_t u_f(t,x) = \int_{\mathbb{R}^d} \partial_t p^\kappa(t,x,y) f(y)\, dy.$$

By this, (1.10) and (5.21), we have $\partial_t w(t,x) = \mathcal{L}^\kappa w(t,x)$. Hence, all the assumptions of Theorem 5.1 are satisfied and we can conclude that for every $t \in (0,T]$, $\sup_{x\in\mathbb{R}^d} w(t,x) \leq \sup_{x\in\mathbb{R}^d} w(0,x) = 0$. By applying the theorem to $-w$ we get that $w(t,x) = 0$ for all $t \in (0,T]$ and every $x \in \mathbb{R}^d$. Hence, $u_f = \widetilde{u}_f$ for every $f \in C_c^\infty(\mathbb{R}^d)$, which implies that $\widetilde{p}^\kappa(t,x,y) = p^\kappa(t,x,y)$.

The last statement of the theorem about the dependence of constants c_1 and c_2 has been already proved in the results above. □

Proof of Theorem 1.2. (1) The constant function $u(t,x) = 1$ solves $\partial_t u(t,x) = \mathcal{L}^\kappa u(t,x)$, hence applying Theorem 5.1 to $\pm(P_t^\kappa 1(x) - 1)$ we get that $P_t^\kappa 1(x) \equiv 1$ proving (1.14).
(2) Same as the proof of [6, Theorem 1.1(3)].
(3) By (1.10) and (1.12) we see that $|\partial_t p^\kappa(t,x,y)| \leq c_2 \rho(t,x-y)$ for $t \in (0,T]$ and $x \neq y$. Hence by the mean value theorem, for $0 < s \leq t \leq T$ and $x \neq y$,

$$|p^\kappa(s,x,y) - p^\kappa(t,x,y)| \leq c_2 |t-s| \rho(s, x-y)\,. \tag{5.22}$$

Let $\gamma \in (0,\delta_1) \cap (0,1]$. By Lemma 4.8 and by the definition of ρ_{-1}^0, we have that for every $t \in (0,T]$,

$$|p^\kappa(t,x,y) - p^\kappa(t,x',y)| \;\; \leq \;\; c_1 |x-x'|^\gamma \Phi^{-1}(t^{-1}) t \left(\rho(t,x-y) + \rho(t,x'-y)\right)$$

$$\leq \; 2c_1|x-x'|^\gamma \Phi^{-1}(t^{-1})t\left(\rho(t,x-y)\vee\rho(t,x'-y)\right) . \quad (5.23)$$

By use of the triangle inequality, this together with (5.22) implies the first claim.

By (1.11), if $\Phi^{-1}(t^{-1})|x-x'|\geq 1$,

$$\begin{aligned}&|p^\kappa(t,x,y)-p^\kappa(t,x',y)| \leq p^\kappa(t,x,y)+p^\kappa(t,x',y) \leq c_1 t\left(\rho(t,x-y)+\rho(t,x'-y)\right)\\ &\leq 2c_1|x-x'|\Phi^{-1}(t^{-1})t\left(\rho(t,x-y)\vee\rho(t,x'-y)\right) . \qquad (5.24)\end{aligned}$$

Suppose $\Phi^{-1}(t^{-1})|x-x'|\geq 1$, $\beta+\delta_1>1$ and $\delta_1\in(2/3,2)$. Then by (4.50)

$$\begin{aligned}&|p^\kappa(t,x,y)-p^\kappa(t,x',y)| \leq |x-x'|\cdot\int_0^1|\nabla p(t,x+\theta(x'-x),y)|\,d\theta\\ &\leq ct\Phi^{-1}(t^{-1})|x-x'|\int_0^1\rho(t,(x-y)+\theta(x'-x))d\theta. \qquad (5.25)\end{aligned}$$

Since $\theta|x'-x|\leq 1/\Phi^{-1}(t^{-1})$, from (5.25) we have

$$\begin{aligned}&|p^\kappa(t,x,y)-p^\kappa(t,x',y)| \leq ct\Phi^{-1}(t^{-1})|x-x'|\rho(t,x-y)\\ &\leq ct\Phi^{-1}(t^{-1})|x-x'|\left(\rho(t,x-y)\vee\rho(t,x'-y)\right). \qquad (5.26)\end{aligned}$$

(5.22), (5.24) and (5.26) imply the second claim.

(4) This follows immediately from the second part of Lemma 4.10. □

Proof of Theorem 1.3. (1) We first claim that for $f\in C_b^{2,\varepsilon}(\mathbb{R}^d)$, $\mathcal{L}^\kappa f$ is bounded Hölder continuous. We will use results from [1]. For $f\in C_b^{2,\varepsilon}(\mathbb{R}^d)$ and $x,z\in\mathbb{R}^d$, let

$$E_z f(x)=f(x+z)-f(x) \quad \text{and} \quad F_z f(x)=f(x+z)-f(x)-\nabla f(x)\cdot z.$$

Using the assumption that $\kappa(y,z)=\kappa(y,-z)$, we have

$$\mathcal{L}^{\mathfrak{K}_y}f(x)=\int_{|z|<1}F_z f(x)\kappa(y,z)J(z)dz+\int_{|z|\geq 1}E_z f(x)\kappa(y,z)J(z)dz.$$

Thus, $\mathcal{L}^\kappa f$ is bounded by (1.7) and (1.1). Moreover, using (1.2), (1.7) and [1, Theorem 5.1 (b) and (e)] with $\gamma=2+\varepsilon$,

$$\begin{aligned}&|\mathcal{L}^\kappa f(x)-\mathcal{L}^\kappa f(y)|\\ \leq&\left|\int_{\mathbb{R}^d}\delta_f(x;z)(\kappa(x,z)-\kappa(y,z))J(z)dz\right|+|\mathcal{L}^{\mathfrak{K}_y}f(x)-\mathcal{L}^{\mathfrak{K}_y}f(y)|\\ \leq&c_1(|x-y|^\beta\wedge 1)\int_{\mathbb{R}^d}(|z|^2\wedge 1)j(|z|)dz+c_1\int_{|z|<1}|F_zf(x)-F_zf(y)|\kappa(y,z)j(|z|)dz\\ &+c_1\int_{|z|\geq 1}|E_zf(x)-E_zf(y)|\kappa(y,z)j(|z|)dz\\ \leq&c_2|x-y|^\beta+c_2\left(\int_{|z|<1}|z|^2j(|z|)dz\right)|x-y|^\varepsilon+c_2\left(\int_{|z|\geq 1}j(|z|)dz\right)|x-y|.\end{aligned}$$

Thus we have proved the claim.

For $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we define $u(t,x) := f(x) + \int_0^t P_s^\kappa \mathcal{L}^\kappa f(x)\, ds$. Note that

$$|u(t,x) - u(0,x)| \le \int_0^t |P_s^\kappa \mathcal{L}^\kappa f(x)| ds \le t \|\mathcal{L}^\kappa f\|_\infty.$$

Thus (5.1) holds. Since $\mathcal{L}^\kappa f$ is bounded Hölder continuous, we can use (5.14) (together with (1.12), (1.10) and (5.21)) to get $\mathcal{L}^\kappa P_s^\kappa \mathcal{L}^\kappa f(x) = \partial_s \left(P_s^\kappa \mathcal{L}^\kappa f\right)(x))$ and obtain

$$\begin{aligned} \mathcal{L}^\kappa u(t,x) &= \mathcal{L}^\kappa f(x) + \int_0^t \mathcal{L}^\kappa P_s^\kappa \mathcal{L}^\kappa f(x)\, ds \\ &= \mathcal{L}^\kappa f(x) + \int_0^t \partial_s \left(P_s^\kappa \mathcal{L}^\kappa f\right)(x)\, ds = P_t \mathcal{L}^\kappa f(x) = \partial_t u(t,x)\,. \end{aligned}$$

Therefore $u(t,x)$ satisfies the assumptions of Theorem 5.1. Since $u(0,x) = f(x)$, it follows from the maximum principle that

$$P_t^\kappa f(x) = u(t,x) = f(x) + \int_0^t P_s^\kappa \mathcal{L}^\kappa f(x)\, ds\,. \tag{5.27}$$

Since $\mathcal{L}^\kappa f$ is bounded and uniformly continuous, we can use (1.13) to get

$$\lim_{t\downarrow 0} \frac{1}{t}\left(P_t^\kappa f(x) - f(x)\right) = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t P_s^\kappa \mathcal{L}^\kappa f(x) ds = \mathcal{L}^\kappa f(x)$$

and the convergence is uniform.

(2) Using our Theorem 1.1(iii), Theorem 1.2(1) and Lemma 5.4, the proof of this part is the same as in [6]. □

5.4 Lower bound estimate of $p^\kappa(t,x,y)$

By Theorem 1.3, we have that $(P_t^\kappa)_{t\ge 0}$ is a Feller semigroup and there exists a Feller process $X = (X_t, \mathbb{P}_x)$ corresponding to $(P_t^\kappa)_{t\ge 0}$. Moreover, by (5.27) for $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$,

$$f(X_t) - f(x) - \int_0^t \mathcal{L}^\kappa f(X_s)\, ds \tag{5.28}$$

is a martingale with respect to the filtration $\sigma(X_s, s \le t)$. Therefore by the same argument as that in [6, Section 4.4], we have the following Lévy system formula: for every function $f : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ vanishing on the diagonal and every stopping time S,

$$\mathbb{E}_x \sum_{0<s\le S} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^S f(X_s, y) J_X(X_s, dy) ds\,, \tag{5.29}$$

where $J_X(x,y) := \kappa(x, y-x) J(x-y)$.

For $A \in \mathcal{B}(\mathbb{R}^d)$ we define $\tau_A := \inf\{t \ge 0 : X_t \notin A\}$.

The following result is the counterpart of [6, Lemma 4.6].

Lemma 5.7 *For each $\gamma \in (0,1)$ there exists $A = A(\gamma) > 0$ such that for every $r > 0$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}_x\left(\tau_{B(x,r)} \le (A\Phi(1/(4r)))^{-1}\right) \le \gamma\,. \tag{5.30}$$

Proof. Without loss of generality, we take $x = 0$. The constant A will be chosen later. Let $f \in C_b^\infty(\mathbb{R}^d)$ with $f(0) = 0$ and $f(y) = 1$ for $|y| \ge 1$. For any $r > 0$ set $f_r(y) = f(y/r)$. By the definition of f_r and the martingale property in (5.28) we have

$$\begin{aligned}\mathbb{P}_0\left(\tau_{B(0,r)} \le (A\Phi(1/(4r)))^{-1}\right) &\le \mathbb{E}_0\left[f_r\left(X_{\tau_{B(0,r)}\wedge(A\Phi(1/(4r)))^{-1}}\right)\right]\\ &= \mathbb{E}_0\left(\int_0^{\tau_{B(0,r)}\wedge(A\Phi(1/(4r)))^{-1}} \mathcal{L}^\kappa f_r(X_s)\,ds\right). \end{aligned}\tag{5.31}$$

By the definition of $\mathcal{L}^\kappa$, (1.1) and (1.7) we have

$$\begin{aligned}|\mathcal{L}^\kappa f_r(y)| &= \frac{1}{2}\left|\int_{\mathbb{R}^d}(f_r(y+z)+f_r(y-z)-2f_r(y))\,\kappa(y,z)J(z)\,dz\right|\\ &\le \frac{\kappa_1\gamma_0\|\nabla^2 f_r\|_\infty}{2}\int_{|z|\le r}|z|^2 j(|z|)\,dz + 2\kappa_1\gamma_0\|f_r\|_\infty\int_{|z|>r} j(|z|)\,dz\\ &\le c_1\left(\frac{\|\nabla^2 f\|_\infty}{r^2}r^2\mathcal{P}(r) + \|f\|_\infty\mathcal{P}(r)\right) \le c_2\,\Phi(r^{-1})\,,\end{aligned}$$

where $c_2 = c_2(\kappa_1, \gamma_0, f)$. Here the last inequality is a consequence of (3.7). Substituting in (5.31) we get that

$$\mathbb{P}_0\left(\tau_{B(0,r)} \le (A\Phi(1/(4r)))^{-1}\right) \le c_2\Phi(r^{-1})(A\Phi(1/(4r)))^{-1} \le 4c_2A^{-1}\,.$$

With $A = 4c_2/\gamma$ the lemma is proved. □

Proof of Theorem 1.4. Throughout the proof, we fix $T, M \ge 1$ and, without loss of generality, we assume that $\Phi^{-1}(T^{-1})^{-1} = M$.

By [4, Theorem 2.4] and the same argument as the one in [5, Proposition 2.2] (see also [7, Proposition 6.4(1)] or [3, Proposition 6.2]), (1.4), (1.20), (1.1) and (1.7) imply that there exists a constant $c_0 > 0$ such that

$$p_y(t,x) \ge c_0\left(\Phi^{-1}(t^{-1})^d \wedge tj(|x|)\right) \quad (t,x,y) \in (0,T]\times B(0,4M)\times\mathbb{R}^d\,. \tag{5.32}$$

Since by [11, Lemma 3.2(a)],

$$j\,(|x|) \ge c_1|x|^{-d}\Phi(|x|^{-1}), \quad |x| \le 4M \tag{5.33}$$

for some $c_1 \in (0,1)$, by Proposition 2.1 we have

$$p_y(t,x) \ge c_0c_1t\rho(t,x) \quad (t,x,y) \in (0,T]\times B(0,4M)\times\mathbb{R}^d\,. \tag{5.34}$$

(1) Let $\lambda = 1/A$ where A is the constant from Lemma 5.7 for $\gamma = 1/2$. Then for every $t > 0$,

$$\sup_{z\in\mathbb{R}^d} \mathbb{P}_z\big(\tau_{B(z,2^{-2}\Phi^{-1}(t^{-1})^{-1})} \le \lambda t\big) \le \frac{1}{2}\,. \tag{5.35}$$

Let $t \in (0,T]$ and $|x-y| \le 3\Phi^{-1}(t^{-1})^{-1}$(so that $|x-y| \le 3M$). By (4.47) we have that there exists a constant $c_2 > 0$ such that

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds &\ge -c_2 t\left(\rho^0_\beta + \rho^\beta_0\right)(t,x-y)\\ &= -c_2 t\left(\Phi^{-1}(t^{-1})^{-\beta} + |x-y|^\beta \wedge 1\right)\rho(t,x-y)\\ &\ge -c_2 t\left(\Phi^{-1}(t^{-1})^{-\beta} + 3^\beta\Phi^{-1}(t^{-1})^{-\beta}\right)\rho(t,x-y)\,.\end{aligned}$$

We choose $t_0 \in (0,1)$ so that for all $t \in (0,t_0)$, $c_2(1+3^\beta)\Phi^{-1}(t^{-1})^{-\beta} \le c_1/2$. Together with (5.34) and (4.46) we conclude that for all $t \in (0,t_0)$ and all $x,y \in \mathbb{R}^d$ satisfying $|x-y| \le 3\Phi^{-1}(t^{-1})^{-1}$ we have

$$p^\kappa(t,x,y) \ge \frac{c_1}{2} t\rho(t,x-y) \ge c_3 t \frac{\Phi\left(\frac{1}{\Phi^{-1}(t^{-1})} + \frac{3}{\Phi^{-1}(t^{-1})}\right)}{\left(\frac{1}{\Phi^{-1}(t^{-1})} + \frac{3}{\Phi^{-1}(t^{-1})}\right)^d} \ge c_4 \Phi^{-1}(t^{-1})^d\,.$$

By (1.15) and iterating $\lfloor T/t_0 \rfloor + 1$ times, we obtain the following near-diagonal lower bound

$$p^\kappa(t,x,y) \ge c_5\Phi^{-1}(t^{-1})^d \quad \text{for all } t \in (0,T] \text{ and } |x-y| \le 3\Phi^{-1}(t^{-1})^{-1}\,. \tag{5.36}$$

Now we assume $|x-y| > 3\Phi^{-1}(t^{-1})^{-1}$ and let $\sigma = \inf\{t \ge 0 : X_t \in B(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1})\}$. By the strong Markov property and (5.35) we have

$$\begin{aligned}&\mathbb{P}_x\left(X_{\lambda t} \in B(y,\Phi^{-1}(t^{-1})^{-1})\right) \ge \mathbb{P}_x\left(\sigma \le \lambda t, \sup_{s\in[\sigma,\sigma+\lambda t]} |X_s - X_\sigma| < 2^{-1}\Phi^{-1}(t^{-1})^{-1}\right)\\ =\;& \mathbb{E}_x\left(\mathbb{P}_{X_\sigma}\Big(\sup_{s\in[0,\lambda t]} |X_s - X_0| < 2^{-1}\Phi^{-1}(t^{-1})^{-1}\Big); \sigma \le \lambda t\right)\\ \ge\;& \inf_{z\in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} \mathbb{P}_z\big(\tau_{B(z,2^{-1}\Phi^{-1}(t^{-1})^{-1})} > \lambda t\big)\mathbb{P}_x\big(\sigma \le \lambda t\big)\\ \ge\;& \frac{1}{2}\mathbb{P}_x\big(\sigma \le \lambda t\big) \;\ge\; \frac{1}{2}\mathbb{P}_x\left(X_{\lambda t\wedge\tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})\right)\,.\end{aligned} \tag{5.37}$$

Since

$$X_s \notin B\left(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1}\right) \subset B\left(x,\Phi^{-1}(t^{-1})^{-1}\right)^c\,, \quad s < \lambda t \wedge \tau_{B(x,\Phi^{-1}(t^{-1})^{-1})},$$

we have

$$\mathbf{1}_{X_{\lambda t\wedge\tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} = \sum_{s\le\lambda t\wedge\tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \mathbf{1}_{X_s\in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1}}\,.$$

Thus, by the Lévy system formula in (5.29) we have

$$
\begin{aligned}
&\mathbb{P}_x\left(X_{\lambda t\wedge\tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})\right)\\
&= \mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \int_{B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} J_X(X_s,u)\,du\,ds\right]\\
&\geq \mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}} \int_{B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} \kappa_0 j(|X_s-u|)\mathbf{1}_{\{u:|X_s-u|<|x-y|\}}\,du\,ds\right]. \quad (5.38)
\end{aligned}
$$

Let w be the point on the line connecting x and y (i.e., $|x-y| = |x-w|+|w-y|$) such that $|w-y| = 7\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1}$. Then $B(w,2^{-4}\Phi^{-1}(t^{-1})^{-1}) \subset B(y,\,2^{-1}\Phi^{-1}(t^{-1})^{-1})$. Moreover, for every $(z,u)\in B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})\times B(w,2^{-4}\Phi^{-1}(t^{-1})^{-1})$, we have

$$
\begin{aligned}
|z-u| &\leq |z-x|+|w-u|+|x-w| = |z-x|+|w-u|+|x-y|-|w-y|\\
&< (6\cdot 2^{-4}+2^{-4})\Phi^{-1}(t^{-1})^{-1} + |x-y| - 7\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1} = |x-y|.
\end{aligned}
$$

Thus

$$
B(w,2^{-4}\Phi^{-1}(t^{-1})^{-1}) \subset \{u : |z-u|<|x-y|\} \quad \text{for } z\in B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1}). \quad (5.39)
$$

(5.39) and (5.35) imply that

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\lambda t\wedge\tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}} \int_{B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} j(|X_s-u|)\mathbf{1}_{\{u:|X_s-u|<|x-y|\}}\,du\,ds\right]\\
\geq&\mathbb{E}_x\left[\lambda t\wedge\tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}\right]\int_{B(w,2^{-4}\Phi^{-1}(t^{-1})^{-1})} j\left(|x-y|\right)\,du\\
\geq&\lambda t\mathbb{P}_x\left(\tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}\geq\lambda t\right)\left|B(w,2^{-4}\Phi^{-1}(t^{-1})^{-1})\right| j\left(|x-y|\right)\\
\geq&c_6 t\Phi^{-1}(t^{-1})^{-d}\, j\left(|x-y|\right). \quad (5.40)
\end{aligned}
$$

By combining (5.37), (5.38) and (5.40) we get that

$$
\mathbb{P}_x\left(X_{\lambda t}\in B(y,\Phi^{-1}(t^{-1})^{-1})\right) \geq \frac{1}{2}c_6 t\Phi^{-1}(t^{-1})^{-d}\, j\left(|x-y|\right) \quad (5.41)
$$

By (1.15), (5.36) and (5.41) we have

$$
\begin{aligned}
p^\kappa(t,x,y) &\geq \int_{B(y,\Phi^{-1}(t^{-1})^{-1})} p^\kappa(\lambda t,x,z)p^\kappa((1-\lambda)t,z,y)\,dz\\
&\geq \inf_{z\in B(y,\Phi^{-1}(t^{-1})^{-1})} p^\kappa((1-\lambda)t,z,y)\int_{B(y,\Phi^{-1}(t^{-1})^{-1})} p^\kappa(\lambda t,x,z)\,dz\\
&\geq c_7\Phi^{-1}(t^{-1})^d t\Phi^{-1}(t^{-1})^{-d}\, j\left(|x-y|\right) = c_7 t j\left(|x-y|\right).
\end{aligned}
$$

Combining this estimate with (5.36) we obtain (1.21). Inequality (1.22) follows from (1.21), Proposition 2.1 and (5.33). □

Acknowledgements: We are grateful to Xicheng Zhang for several valuable comments, in particular for suggesting the improvement of the gradient estimate (1.17). We also thank Karol Szczypkowski for pointing out some mistakes in an earlier version of this paper and Jaehoon Lee for reading the manuscript and giving helpful comments.

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