Two-sided Green function estimates for killed subordinate Brownian motions

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October 8, 2011

Abstract

A subordinate Brownian motion is a Lévy process which can be obtained by replacing the time of the Brownian motion by an independent subordinator. The infinitesimal generator of a subordinate Brownian motion is $-\phi(-\Delta)$, where ϕ is the Laplace exponent of the subordinator. In this paper, we consider a large class of subordinate Brownian motions without diffusion component and with ϕ comparable to a regularly varying function at infinity. This class of processes includes symmetric stable processes, relativistic stable processes, sums of independent symmetric stable processes, sums of independent relativistic stable processes, and much more. We give sharp two-sided estimates on the Green functions of these subordinate Brownian motions in any bounded κ-fat open set D. When D is a bounded $C^{1,1}$ open set, we establish an explicit form of the estimates in terms of the distance to the boundary. As a consequence of such sharp Green function estimates, we obtain a boundary Harnack principle in $C^{1,1}$ open sets with explicit rate of decay.

AMS 2010 Mathematics Subject Classification: Primary 60J45, Secondary 60J75, 60G51.
Keywords and phrases: Green function, Poisson kernel, subordinate Brownian motion, κ-fat open set, $C^{1,1}$ open set, symmetric stable process, relativistic stable process, harmonic functions, Harnack inequality, boundary Harnack principle, fluctuation theory, regularly varying function.

1 Introduction

The investigation of fine potential-theoretic properties of discontinuous Markov processes in the Euclidean space began in the late 1990's with the study of symmetric stable processes. One of the first results obtained in this area was sharp Green function estimates of symmetric α-stable processes in bounded $C^{1,1}$ domains in $\mathbb{R}^d$, $0<\alpha<2$, $d\geq 2$. Recall that if X is a symmetric Markov process in $\mathbb{R}^d$ and D is an open subset of $\mathbb{R}^d$, then the Green function $G_D(x,y)$ of X in D (if it exists) is the density of the mean occupation time for X before exiting D, that is, the density of the measure

$$U\mapsto \mathbb{E}_x\int_0^{\tau_D}\mathbf{1}_U(X_t)dt,\quad U\subset D,$$

where τ_D is the first time the process X exits D. Analytically speaking, if $\mathcal{L}$ is the infinitesimal generator of X and $\mathcal{L}|_D$ is the restriction of $\mathcal{L}$ to D with zero exterior condition, then $G_D(\cdot,y)$ is the solution of $(\mathcal{L}|_D)u=-\delta_y$.

A process $X=(X_t: t\geq 0)$ is called a (rotationally) symmetric α-stable (Lévy) process, $0<\alpha<2$, if it is a Lévy process whose characteristic exponent Φ, defined by $\mathbb{E}[\exp\{i\theta\cdot X_t\}]=\exp\{-t\Phi(\theta)\}$, is given

*Supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(0409-20110087).
†Supported in part by the MZOS grant 037-0372790-2801.

by $\Phi(\theta) = |\theta|^\alpha$. The infinitesimal generator of a symmetric α-stable process is $-(-\Delta)^{\alpha/2}$. The paths of the symmetric α-stable process X are purely discontinuous, as opposed to the case $\alpha = 2$ corresponding to Brownian motion which has continuous paths. It was independently shown in [11] and [22] that if D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ ($d \ge 2$), $G_D(x, y)$ the Green function of the symmetric α-stable process in D, and $\delta_D(x)$ the distance between the point x and the complement D^c of D, then there exists a constant $c > 1$ (depending on D and α) such that

$$c^{-1}\left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|^\alpha}\right)\frac{1}{|x-y|^{d-\alpha}} \le G_D(x,y) \le c\left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|^\alpha}\right)\frac{1}{|x-y|^{d-\alpha}}, \tag{1.1}$$

for all $x, y \in D$. Here and in the sequel, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The same form of the estimates in the case $\alpha = 2$ (and $d \ge 3$) were obtained much earlier in [33, 35] for the Brownian motion case.

The proofs of (1.1) for symmetric α-stable processes relied heavily on the explicit formulae for the Green functions and the Poisson kernels of the ball. Moving away from stable processes, such formulae were not available and new methods had to be developed. [25] studied the relativistic α-stable process (with relativistic mass $m > 0$) whose characteristic exponent is given by $\Phi(\theta) = (|\theta|^2+m^{2/\alpha})^{\alpha/2}-m$ and infinitesimal generator is given by $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$, and showed that the Green function of this process in any bounded $C^{1,1}$ domain D satisfies the same sharp estimates (1.1). Soon after, [12], using a perturbation method, established a general result which includes the main result of [25] as a special case. For different generalizations of the main result of [25], see the recent papers [14, 19].

Quite recently, [8] studied the Lévy process which is the sum of independent symmetric β-stable and α-stable processes, $0 < \beta < \alpha < 2$. The characteristic exponent of this process is given by $\Phi(\theta) = |\theta|^\alpha + |\theta|^\beta$ and the infinitesimal generator by $-(-\Delta)^{\alpha/2} - (-\Delta)^{\beta/2}$. Sharp two-sided estimates on the heat kernel of this process in $C^{1,1}$ open sets were established in [8]. As a by-product of the heat kernel estimates, sharp Green function estimates of the process in any bounded $C^{1,1}$ open set were obtained in [8]. The estimates obtained in [8] have the same form (1.1). In contrast with the relativistic stable processes, these Green function estimates cannot be obtained using the methods of [12, 14, 19, 25]. The case $\alpha = 2$ (i.e., one of the processes is a Brownian motion) was covered in [10] with analogous estimates.

The common feature of these Green function estimates is that both the distance between the points, $|x-y|$, and distances to D^c, $\delta_D(x), \delta_D(y)$, appear as arguments of *power functions*. However, it follows from [6, Chapter 5] that the asymptotic behavior of the free Green function $G(x, y)$ of many transient symmetric Lévy processes is of the form

$$G(x,y) \sim \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})} \quad \text{as } |x-y| \to 0$$

where $\alpha \in (0, 2)$ and ℓ is a nontrivial slowly varying function at infinity. (See also Theorem 2.9 below.) Therefore, Green function estimates of the form (1.1) cannot be true for these general symmetric Lévy processes. The purpose of this paper is to establish sharp two-sided Green function estimates for these more general Lévy processes in open sets of $\mathbb{R}^d$. In our estimates, $\delta_D(x), \delta_D(y)$ and $|x - y|$ appear as arguments of regularly varying functions, not necessarily power functions. In order to explain our setting and results, let us first note that stable processes, relativistic stable processes and sums of independent stable processes can be obtained as subordinate Brownian motions. Indeed, let $W = (W_t = (W_t^1, \dots, W_t^d) : t \ge 0)$ be a d-dimensional Brownian motion, and let $S = (S_t : t \ge 0)$ be an independent subordinator. Recall that a subordinator is an increasing Lévy process starting from 0, which can be characterized through its Laplace exponent ϕ: $\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}$, $\lambda > 0$. The process $X = (X_t : t \ge 0)$ defined by $X_t := W_{S_t}$ is called a subordinate Brownian motion. The infinitesimal generator of X is $-\phi(-\Delta)$. By choosing the Laplace exponent $\phi(\lambda)$ as $\lambda^{\alpha/2}$, $(\lambda + m^{2/\alpha})^{\alpha/2} - m$ and $\lambda^{\alpha/2} + \lambda^{\beta/2}$ respectively, the resulting subordinate Brownian motion turns out to be a symmetric α-stable process, a relativistic stable process and an independent sum of β and α-stable processes respectively. The Laplace exponent of a subordinator is a Bernstein function and hence has the representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)\,,$$

where $b \ge 0$ and μ is a measure (called the Lévy measure of ϕ) such that $\int_{(0,\infty)}(1 \wedge t)\, \mu(dt) < \infty$. If the measure μ has a completely monotone density, the Laplace exponent ϕ is called a complete Bernstein function. The common feature of the Laplace exponents $\phi(\lambda) = \lambda^{\alpha/2}$, $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$ and $\phi(\lambda) = \lambda^{\beta/2} + \lambda^{\alpha/2}$ is that all three of them are complete Bernstein functions whose behavior at infinity is given by $\lim_{\lambda\to\infty} \phi(\lambda)/\lambda^{\alpha/2} = 1$. We will see that those two properties (the latter slightly weakened) are the determining factors for the Green function estimates (1.1).

Recall that an open set D in $\mathbb{R}^d$ ($d \ge 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\psi = \psi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $\nabla\psi(0) = (0, \dots, 0)$, $\|\nabla\psi\|_\infty \le \Lambda$, $|\nabla\psi(x) - \nabla\psi(z)| \le \Lambda|x - z|$, and an orthonormal coordinate system CS_z: $y = (y_1, \cdots, y_{d-1}, y_d) := (\widetilde{y},\, y_d)$ with origin at z such that

$$B(z, R) \cap D = \{y = (\widetilde{y},\, y_d) \in B(0, R) \text{ in } CS_z : y_d > \psi(\widetilde{y})\}.$$

The pair (R, Λ) is called the characteristics of the $C^{1,1}$ open set D. We remark that in some literature, the $C^{1,1}$ open set defined above is called a *uniform* $C^{1,1}$ open set since (R, Λ) is universal for all $z \in \partial D$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a bounded $C^{1,1}$ open set can be disconnected.

The main result of this paper is the following sharp Green function estimates. In the statement and throughout the paper we use notation $f(t) \asymp g(t)$ as $t \to \infty$ (resp. $t \to 0+$) if the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to \infty$ (resp. $t \to 0+$).

Theorem 1.1 *Suppose that $X = (X_t : t \ge 0)$ is a Lévy process whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where*

$$\phi : (0, \infty) \to [0, \infty) \text{ is a complete Bernstein function such that } \phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda)\,, \quad \lambda \to \infty\,, \tag{1.2}$$

$\alpha \in (0, 2 \wedge d)$ and $\ell : (0, \infty) \to (0, \infty)$ is a measurable function, locally bounded from above and below by positive constants, and slowly varying at infinity. When $d \le 2$, assume additionally that $\liminf_{\lambda\to 0} \phi(\lambda)/\lambda^\gamma > 0$ for some $\gamma \in [0, d/2)$. Then for every bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $C_1 = C_1(diam(D), R, \Lambda, \phi, d) > 1$ such that the Green function $G_D(x, y)$ of X in D satisfies the following estimates:

$$\begin{aligned} &C_1^{-1} \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(x)^{-2})}}\right) \frac{1}{|x-y|^d\, \phi(|x-y|^{-2})} \\ &\le G_D(x, y) \le C_1 \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(x)^{-2})}}\right) \frac{1}{|x-y|^d\, \phi(|x-y|^{-2})}\,. \end{aligned} \tag{1.3}$$

Using (1.2), we can rewrite the conclusion of Theorem 1.1 in the following alternative form: There exists $c > 1$ such that

$$c^{-1} \left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}\ell(|x-y|^{-2})}{\sqrt{\ell((\delta_D(x))^{-2})\ell((\delta_D(y))^{-2})}\, |x-y|^\alpha}\right) \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}}$$

$$\leq G_D(x,y) \leq c \left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}\ell(|x-y|^{-2})}{\sqrt{\ell((\delta_D(x))^{-2})\ell((\delta_D(y))^{-2})}\,|x-y|^{\alpha}}\right) \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}} \tag{1.4}$$

(see (1.6) below for yet another alternative form of these estimates). To the best of our knowledge, the above Green function estimates include all known Green function estimates of purely discontinuous transient subordinate Brownian motions in bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ as special cases. The estimates above include a lot more processes: In the case $d \geq 3$, our estimates are valid for all subordinate Brownian motions satisfying (1.2). For more concrete examples, see Example 2.16.

Let us give the main ingredients of the proof of Theorem 1.1. The groundwork has been laid down in the recent paper [20] where a similar class of subordinate Brownian motions was studied. One difference to the current setting was that in [20] the Laplace exponent was assumed to be precisely regularly varying at infinity and not just comparable to a regularly varying function. Another difference is that [20] contains some additional assumptions that turned out to be redundant. The results of [20] are reproved in [21] under conditions valid in this paper (with the redundant assumptions removed). When referring to those results we will quote both sources. The main result of [20, 21] is the boundary Harnack principle for nonnegative harmonic functions of the subordinate Brownian motion X in bounded κ-fat open sets. Based on the boundary Harnack principle and using the well-established methodology of [4, 16], we will first obtain Green function estimates of the form (1.5) below (in the spirit of [4, 16]) in bounded κ-fat open sets.

Recall from [32] that an open set D in $\mathbb{R}^d$ is κ-fat if there exists $R_1 > 0$ such that for each $Q \in \partial D$ and $r \in (0, R_1)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (R_1, κ) is called the characteristics of the κ-fat open set D. All Lipschitz domains, non-tangentially accessible domains and John domains are κ-fat (cf. [18, 32] and the references therein). In general, the boundary of a κ-fat open set can be nonrectifiable.

Theorem 1.2 *Suppose that $X = (X_t : t \geq 0)$ is a Lévy process satisfying the same conditions as in Theorem 1.1 and that D is a bounded κ-fat open set with characteristics (R_1, κ). Then there exists $C_2 = C_2(diam(D), R_1, \kappa, \phi, d) > 1$ such that for every $x, y \in D$,*

$$C_2^{-1} \frac{g(x)g(y)}{g(A)^2|x-y|^d\phi(|x-y|^{-2})} \leq G_D(x,y) \leq C_2 \frac{g(x)g(y)}{g(A)^2|x-y|^d\phi(|x-y|^{-2})}, \quad A \in \mathcal{B}(x,y), \tag{1.5}$$

where g and $\mathcal{B}(x,y)$ are defined in (3.10) *and* (3.7) *respectively.*

Using the Harnack inequality and the boundary Harnack principle, the above form of Green function estimates has been established by several authors in some special cases where $0 < c_1 \leq \ell(\lambda) \leq c_2 < \infty$ for large λ. See [10, Theorem 1.1], [16, Theorem 2.4] and [17, Theorem 1].

To obtain the interior estimates in Theorem 1.2 (i.e. for points x, y away from the boundary), we use the asymptotic behavior of the Green function of X in $\mathbb{R}^d$ proved in [21, Theorem 3.2] (see also [20, Theorem 3.1]). Once we have the interior estimates, the full estimates in a bounded κ-fat open set follow easily from the now standard argument developed in [4, 16].

Even though some complications occur due to the fact the ϕ is not comparable to a power function near infinity, the proof of Theorem 1.2 is still routine. But the precise estimates (1.4) in bounded $C^{1,1}$ open sets are very delicate. One of the ingredients comes from the fluctuation theory of one-dimensional Lévy processes. Let $Z = (Z_t : t \geq 0)$ be the one-dimensional subordinate Brownian motion defined by $Z_t := W^d_{S_t}$, and let V be the renewal function of the ladder height process of Z. The function V is harmonic for the process Z killed upon exiting $(0, \infty)$, and the function $w(x) := V(x_d)\mathbf{1}_{\{x_d>0\}}$, $x \in \mathbb{R}^d$, is harmonic for the process X killed upon exiting the half space $\mathbb{R}^d_+ := \{x = (x_1, \dots, x_{d-1}, x_d) \in \mathbb{R}^d : x_d > 0\}$ (Theorem 4.1). Therefore, w gives the correct rate of decay of harmonic functions near the boundary of $\mathbb{R}^d_+$. This shows the importance of the fluctuation theory (of one-dimensional Lévy processes) in our approach.

The second ingredient is the "test function" method applied to the operator $\mathcal{A}$ defined by

$$\mathcal{A}f(x) = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |y-x| > \varepsilon\}} (f(y) - f(x))\, J(y-x)\, dy\,,$$

with the domain consisting of functions f for which the limit exists and is finite. Here J denotes the density of the Lévy measure of X. On the space of smooth functions with compact support, this operator coincides with the infinitesimal generator of X. We emphasize that, compared to the test function methods of [5, 9, 15], there are several differences in our approach. In [5, 9, 15], appropriate subharmonic and superharmonic functions of X (or the truncated version of X) are chosen as test functions, first in the case of half spaces and then for $C^{1,1}$ open sets, and the values of the generator acting on these test functions are computed in detail. Then suitable combinations of the test functions are used to find the correct exit distribution estimates. In [5, 9, 15], the test functions are power functions of the form $x \to (x_d)^p$ and the densities of the Lévy measures of the processes have closed forms. However, the density J of the Lévy measure of our process does not have a simple form. We do not even know the asymptotic behavior of J near infinity in general. Furthermore, in general, power functions of the form $x \to (x_d)^p$ are neither subharmonic nor superharmonic functions for our processes, and it is not clear what are the appropriate choices for the test functions.

Due to the above differences and difficulties, obtaining the correct boundary decay rate of the Green function in $C^{1,1}$ open set D requires new ideas and approaches. In this paper, we will use the function w which is smooth and harmonic on the half space, as our only test function. Using this and the characterization of harmonic functions recently established in [7], we show that $\mathcal{A}w \equiv 0$ on the half space (Theorem 4.3). With this, we prove the following fact in Lemma 4.4, which is the key to the proof of Theorem 1.1: If D is a $C^{1,1}$ open set with characteristics (R, Λ), $Q \in \partial D$ and $h(y) = V(\delta_D(y))\mathbf{1}_{D \cap B(Q,R)}$, then $\mathcal{A}h(y)$ is well defined and bounded for $y \in D$ close enough to the boundary point Q. Using this lemma, we give certain exit distribution estimates in Lemma 4.5, which provide the correct rate of decay of Green functions near the boundary of D. Unlike [5, 9, 15], in Lemma 4.5 we do not construct subharmonic and superharmonic functions on $C^{1,1}$ open set D. Instead we use Dynkin's formula on h to obtain the desired exit distribution estimates directly. In fact, our approach is simpler than the previous approaches and may be used for other types of discontinuous processes. We hope our approach will shed new light on the understanding of the boundary behavior of nonnegative harmonic functions of general Markov processes.

The estimates (1.3) can also be written in terms of the renewal function V which provides the exact rate of decay of G_D near the boundary. Let G be the Green function of X in the whole space $\mathbb{R}^d$. An equivalent form of (1.4) is given by

$$c^{-1}\left(1 \wedge \frac{V(\delta_D(x))V(\delta_D(y))}{V(|x-y|)^2}\right) G(x,y) \le G_D(x,y) \le c\left(1 \wedge \frac{V(\delta_D(x))V(\delta_D(y))}{V(|x-y|)^2}\right) G(x,y). \tag{1.6}$$

By combining the sharp estimates of the Green function in a bounded $C^{1,1}$ open set with the boundary Harnack principle proved in [20, 21] (see Theorem 2.15 below), we obtain a boundary Harnack principle with explicit decay rate. In the next theorem we give an extension to unbounded $C^{1,1}$ open sets. Recall that, given $Q \in \partial D$, a function $u : \mathbb{R}^d \to \mathbb{R}$ is said to vanish continuously on $D^c \cap B(Q,r)$ if $u = 0$ on $D^c \cap B(Q,r)$ and u is continuous at every point of $\partial D \cap B(Q,r)$.

Theorem 1.3 *Suppose that $X = (X_t : t \ge 0)$ is a Lévy process satisfying the same conditions as in Theorem 1.1 and that D is a (possibly unbounded) $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R, Λ). Then there exists $C_3 = C_3(R, \Lambda, \phi, d) > 0$ such that for $r \in (0, (R \wedge 1)/4]$, $Q \in \partial D$ and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q,r)$ with respect to X and vanishes continuously on $D^c \cap B(Q,r)$, we have*

$$\frac{u(x)}{(\phi((\delta_D(x))^{-2}))^{-1/2}} \le C_3 \frac{u(y)}{(\phi((\delta_D(y))^{-2}))^{-1/2}} \qquad \text{for every } x, y \in D \cap B(Q, r/2). \tag{1.7}$$

An alternative form of (1.7) reads as follows: There exists a constant $c > 0$ such that

$$\frac{u(x)}{V(\delta_D(x))} \le c \frac{u(y)}{V(\delta_D(y))} \qquad \text{for every } x, y \in D \cap B(Q, r/2). \tag{1.8}$$

Note that unlike the usual form of the boundary Harnack principle where one considers the ratio of two harmonic functions, the functions in the denominator of (1.7) and (1.8) are not harmonic. Instead, they provide the correct boundary decay of non-negative harmonic functions.

In this we paper, we will use a result from the theory of regular variations quite a few times. For the convenience of our readers, we recall the following result from [2, Theorem 1.5.6(i)]:

Theorem 1.4 (Potter's Theorem) *If ℓ is slowly varying at infinity, then for any $A > 1, \delta > 0$ there exists $T = T(A, \delta) > 0$ such that*

$$\frac{\ell(t)}{\ell(s)} \le A \max\left\{ \left(\frac{t}{s}\right)^{\delta}, \left(\frac{s}{t}\right)^{\delta} \right\}, \quad s, t \ge T.$$

This paper is organized as follows: In the next section we describe the setting and notation, prove several new results for complete Bernstein functions, and recall some of the known results from [20, 21]. In Section 3 we prove the Green function estimates in bounded κ-fat open sets. Section 4 is devoted to the Green function estimates in bounded $C^{1,1}$ open sets.

We will use the following conventions in this paper. The values of the constants $C_1, C_2, \dots, M$, ε_1 and $R, R_1, R_2, \cdots$ will remain the same throughout this paper, while $c, c_0, c_1, c_2, \cdots$ and $r, r_0, r_1, r_2, \dots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants $c_0, c_1, c_2, \cdots$ starts anew in the statement of each result. The dependence of the constants on dimension d may not be mentioned explicitly. We will use ":=" to denote a definition, which is read as "is defined to be". Further, $f(t) \sim g(t)$, $t \to 0$ ($f(t) \sim g(t)$, $t \to \infty$, respectively) means $\lim_{t\to 0} f(t)/g(t) = 1$ ($\lim_{t\to\infty} f(t)/g(t) = 1$, respectively). Simply, $f \asymp g$ means that the quotient $f(t)/g(t)$ stays bounded between two positive numbers on their common domain of definition. For any open set U, we denote by $\delta_U(x)$ the distance between x and the complement of U, i.e., $\delta_U(x) = \text{dist}(x, U^c)$. We will use ∂ to denote the cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. For every function f, let $f^+ := f \vee 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure and $\text{diam}(A)$ to denote the diameter of the set A.

2 Preliminaries

In this section we collect and explain preliminary results necessary for further development in Sections 3 and 4. Most of these results originate from [20] where they were proved under somewhat stronger conditions than in this paper. Their extensions to the current setting, in particular Theorems 2.9, 2.11 and 2.15, are given with full proofs in [21]. Here we prove only results that have not appeared in [20]. Lemma 2.1 and Propositions 2.4 and 2.6 about complete Bernstein functions may be of independent interest. The difference between the assumptions in [20] and this paper is discussed in Remark 2.2.

A C^∞ function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \le 0$ for every positive integer n. Every Bernstein function has a representation $\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$ where $a, b \ge 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$; a is called the killing coefficient, b the drift and μ the Lévy measure of the Bernstein function. A Bernstein function ϕ is called a complete Bernstein function if the Lévy measure μ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \ge 0$ for every non-negative integer n. Here and below, by abuse of notation we will denote the Lévy density by $\mu(t)$. For more on Bernstein and complete Bernstein functions we refer the readers to [27].

First, we show that ϕ being a complete Bernstein function implies that its Lévy density cannot decrease too fast in the following sense:

Lemma 2.1 *Suppose that ϕ is a complete Bernstein function with Lévy density μ. Then there exists $C_4 > 1$ such that $\mu(t) \le C_4 \mu(t+1)$ for every $t > 1$.*

Proof. Since μ is a completely monotone function, by Bernstein's theorem ([27, Theorem 1.4]), there exists a measure m on $[0,\infty)$ such that $\mu(t) = \int_{[0,\infty)} e^{-tx} m(dx)$. Choose $r > 0$ such that $\int_{[0,r]} e^{-x}\, m(dx) \ge \int_{(r,\infty)} e^{-x}\, m(dx)$. Then, for any $t > 1$, we have

$$\begin{aligned}
\int_{[0,r]} e^{-tx}\, m(dx) &\ge e^{-(t-1)r} \int_{[0,r]} e^{-x}\, m(dx) \\
&\ge e^{-(t-1)r} \int_{(r,\infty)} e^{-x}\, m(dx) \ge \int_{(r,\infty)} e^{-tx}\, m(dx).
\end{aligned}$$

Therefore, for any $t > 1$,

$$\mu(t+1) \ge \int_{[0,r]} e^{-(t+1)x}\, m(dx) \ge e^{-r} \int_{[0,r]} e^{-tx}\, m(dx) \ge \frac{1}{2}\, e^{-r} \int_{[0,\infty)} e^{-tx}\, m(dx) = \frac{1}{2}\, e^{-r} \mu(t).$$

□

Suppose that $S = (S_t : t \ge 0)$ is a subordinator with Laplace exponent ϕ, that is

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \qquad \forall\, t, \lambda > 0.$$

The Laplace exponent of a subordinator is always a Bernstein function. Let $U(A) := \mathbb{E} \int_0^\infty \mathbf{1}_{\{S_t \in A\}}\, dt$ denote the potential measure of S. If ϕ is a complete Bernstein function with infinite Lévy measure, then the potential measure U has a completely monotone density $u(t)$ (see, e.g., [27, Remark 10.6 and Corollary 10.7]).

Recall that a function $\ell : (0,\infty) \to (0,\infty)$ is slowly varying at infinity if

$$\lim_{t\to\infty} \frac{\ell(\lambda t)}{\ell(t)} = 1\,, \quad \text{for every } \lambda > 0\,.$$

In the remainder of this paper we assume that ϕ is a complete Bernstein function and we will always impose the following

Assumption (H): There exist $\alpha \in (0,2)$ and a function $\ell : (0,\infty) \to (0,\infty)$ which is measurable, locally bounded from above and below by positive constants, and slowly varying at infinity such that

$$\phi(\lambda) \asymp \lambda^{\alpha/2} \ell(\lambda)\,, \quad \lambda \to \infty\,. \tag{2.1}$$

Remark 2.2 (a) The precise interpretation of (2.1) will be as follows: There exists a positive constant $c > 1$ such that

$$c^{-1} \le \frac{\phi(\lambda)}{\lambda^{\alpha/2}\ell(\lambda)} \le c \qquad \text{for all } \lambda \in [1,\infty)\,.$$

The choice of the interval $[1,\infty)$ is, of course, arbitrary. Any interval $[a,\infty)$ would do, but with a different constant. This follows from the assumption that ℓ is locally bounded from above and below by positive constants. Moreover, by choosing $a > 0$ large enough, we could dispense with the local boundedness assumption. Indeed, by [2, Lemma 1.3.2], every slowly varying function at infinity is locally bounded on $[a,\infty)$ for a large enough. Although the choice of interval $[1,\infty)$ is arbitrary, it will have as a consequence the fact that all

relations of the type $f(t) \asymp g(t)$ as $t \to \infty$ (respectively $t \to 0+$) following from (2.1) will be interpreted as $\tilde{c}^{-1} \le f(t)/g(t) \le \tilde{c}$ for $t \ge 1$ (respectively $0 < t \le 1$) for an appropriate constant $\tilde{c}$.
(b) The assumption **(H)** is an assumption about the behavior of ϕ at infinity. We make no assumption on ϕ near zero. As a consequence, we will be able to obtain information about the small scale behavior of the subordinate process, but almost nothing can be inferred about its large scale behavior.
(c) The main assumption in [20] was that ϕ is a complete Bernstein function such that

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda), \quad \text{for all } \lambda > 0, \tag{2.2}$$

where $\alpha \in (0,2)$ and ℓ is a slowly varying function at infinity. This assumption allows us to obtain exact asymptotic behavior of various functions. More precisely, some of the results in [20] were of the form $f(t) \sim g(t)$, while with the assumption (2.1) we can obtain only the corresponding results in the weaker form $f(t) \asymp g(t)$. Proofs of these more general results can be found in [21]. We note that our current assumptions are indeed strictly weaker than the ones in [20]: There exists a complete Bernstein function satisfying (2.1) which is *not* regularly varying at infinity, see [21, Example 2.8].
(d) We briefly comment on the other assumptions from [20] which are now removed. The assumption **A1** in [20] needed for transience in case $d \le 2$ is replaced by (2.14) below. The assumptions **A2** and **A3** in [20] used for the technical lemma [6, Lemma 5.32] are redundant - see [21, Lemma 3.1]. Assumption **A4** in [20] is always valid for complete Bernstein function as proved here in Lemma 2.1. Finally, the assumption (2.5) in [20, Proposition 2.2] is no longer needed as [20, Proposition 2.2] is now replaced by Proposition 2.6 below.

It follows from (2.1) that $\lim_{\lambda\to\infty} \phi(\lambda)/\lambda = 0$ and $\lim_{\lambda\to\infty}\phi(\lambda) = \infty$, implying that ϕ has no drift and its Lévy measure is infinite. Therefore, the potential measure U of the corresponding subordinator S has a completely monotone density u.

The behavior of $u(t)$ and the density $\mu(t)$ of the Lévy measure can be inferred from the following result.

Proposition 2.3 ([34, Theorem 7]) *Suppose that ψ is a completely monotone function given by*

$$\psi(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt,$$

where f is a strictly positive decreasing function. Then

$$f(t) \le \left(1 - e^{-1}\right)^{-1} t^{-1}\psi(t^{-1}), \quad t > 0.$$

If, furthermore, there exist $\delta \in (0,1)$ and $a, t_0 > 0$ such that

$$\psi(r\lambda) \le a r^{-\delta}\psi(\lambda), \quad r \ge 1, t \ge 1/t_0, \tag{2.3}$$

then there exists $C_5 = C_5(\psi, f, a, t_0, \delta) > 0$ such that

$$f(t) \ge C_5 t^{-1}\psi(t^{-1}), \quad t \le t_0.$$

We first apply the above proposition to $\psi(\lambda) = \phi(\lambda)^{-1} = \int_0^\infty e^{-\lambda t} u(t)\, dt$ to obtain the behavior of u near zero:

$$u(t) \asymp t^{-1}\phi(t^{-1})^{-1} \asymp \frac{t^{\alpha/2-1}}{\ell(t^{-1})}, \quad t \to 0+ . \tag{2.4}$$

Condition (2.3) follows from (2.1) by use of Potter's theorem (Theorem 1.4). By applying (2.4) to the complete Bernstein function $\lambda \mapsto \lambda/\phi(\lambda)$ ([27, Proposition 7.1]) one obtains the following behavior of $\mu(t)$ near zero:

$$\mu(t) \asymp t^{-1}\phi(t^{-1}) \asymp t^{-\alpha/2-1}\ell(t^{-1}), \quad t \to 0+ . \tag{2.5}$$

We refer the reader to [21, Theorem 2.9, Theorem 2.10] for the detailed proofs of (2.4) and (2.5). The corresponding precise asymptotics are given in [20, p. 1603] under the assumption (2.2).

A consequence of the asymptotic behavior (2.5) of $\mu(t)$ is that for any $K > 0$ there exists $c = c(K) > 1$ such that

$$\mu(t) \le c\,\mu(2t), \qquad \forall t \in (0, K). \tag{2.6}$$

The behavior of $\mu(t)$ at infinity has already been determined in Lemma 2.1: There exists a constant $c > 1$ such that

$$\mu(t) \le c\,\mu(t+1), \qquad \forall\, t > 1. \tag{2.7}$$

This property of μ was assumed in [20] as **A4**, but we have shown in Lemma 2.1 that it always holds true.

We consider now one-dimensional subordinate Brownian motions. Let $B = (B_t : t \ge 0)$ be a Brownian motion in $\mathbb{R}$, independent of S, with

$$\mathbb{E}\left[e^{i\theta(B_t - B_0)}\right] = e^{-t\theta^2}, \qquad \forall \theta \in \mathbb{R}, t > 0.$$

The subordinate Brownian motion $Z = (Z_t : t \ge 0)$ in $\mathbb{R}$ defined by $Z_t = B_{S_t}$ is a symmetric Lévy process with the characteristic exponent $\Phi(\theta) = \phi(\theta^2)$ for all $\theta \in \mathbb{R}$.

Let $\overline{Z}_t := \sup\{0 \vee Z_s : 0 \le s \le t\}$ be the supremum process of Z and let $L = (L_t : t \ge 0)$ be a local time of $\overline{Z} - Z$ at 0. L is also called a local time of the process Z reflected at the supremum. The right continuous inverse L_t^{-1} of L is a subordinator and is called the ladder time process of Z. The process $\overline{Z}_{L_t^{-1}}$ is also a subordinator and is called the ladder height process of Z. (For the basic properties of the ladder time and ladder height processes, we refer our readers to [1, Chapter 6].)

Let χ be the Laplace exponent of the ladder height process of Z. It follows from [13, Corollary 9.7] that

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Phi(\lambda\theta))}{1+\theta^2}d\theta\right) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2}d\theta\right), \quad \forall \lambda > 0. \tag{2.8}$$

Using (2.8), it was proved in [20, Proposition 2.1] that, if ϕ is a special Bernstein function, so is χ, i.e., $\lambda \mapsto \lambda/\chi(\lambda)$ is also a Bernstein function. The next result tells us that such relation is also true for complete Bernstein functions. For the proof we will need the following fact, see [27, Theorem 6.10]: If ϕ is a complete Bernstein function, then there exist a real number γ and a $[0, 1]$-valued function η on $(0, \infty)$ such that

$$\log \phi(\lambda) = \gamma + \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\lambda + t}\right)\eta(t)dt. \tag{2.9}$$

Proposition 2.4 *Suppose ϕ, the Laplace exponent of the subordinator S, is a complete Bernstein function. Then the Laplace exponent χ of the ladder height process of the subordinate Brownian motion $Z_t = B_{S_t}$ is also a complete Bernstein function.*

Proof. By (2.8) and (2.9), we have

$$\log \chi(\lambda) = \frac{\gamma}{2} + \frac{1}{\pi}\int_0^\infty \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\lambda^2\theta^2 + t}\right)\eta(t)dt\frac{d\theta}{1+\theta^2}.$$

By using $0 \le \eta(t) \le 1$, we have

$$\begin{aligned} \eta(t)\left|\frac{t}{1+t^2} - \frac{1}{\lambda^2\theta^2+t}\right|\frac{1}{1+\theta^2} &\le \frac{1}{1+t^2}\frac{1}{1+\theta^2}\left(\frac{1}{\lambda^2\theta^2+t} + \frac{\lambda^2\theta^2 t}{\lambda^2\theta^2+t}\right) \\ &\le \frac{1}{1+t^2}\left(\frac{1}{\lambda^2\theta^2+t} + \frac{\lambda^2 t}{\lambda^2\theta^2+t}\right). \end{aligned}$$

Since

$$\int_0^\infty \frac{1}{\lambda^2\theta^2+t}\,d\theta = \frac{1}{t}\int_0^\infty \frac{1}{\frac{\lambda^2\theta^2}{t}+1}\,d\theta = \frac{1}{t}\frac{\sqrt{t}}{\lambda}\int_0^\infty \frac{1}{\gamma^2+1}\,d\gamma = \frac{\pi}{2\lambda\sqrt{t}},$$

we can use Fubini's theorem to get

$$\begin{aligned}\log\chi(\lambda) &= \frac{\gamma}{2}+\int_0^\infty\left(\frac{t}{2(1+t^2)}-\frac{1}{2\sqrt{t}(\lambda+\sqrt{t})}\right)\eta(t)dt \qquad (2.10)\\ &= \frac{\gamma}{2}+\int_0^\infty\left(\frac{t}{2(1+t^2)}-\frac{1}{2(1+t)}\right)\eta(t)dt+\int_0^\infty\left(\frac{1}{2(1+t)}-\frac{1}{2\sqrt{t}(\lambda+\sqrt{t})}\right)\eta(t)dt\\ &= \gamma_1+\int_0^\infty\left(\frac{s}{1+s^2}-\frac{1}{\lambda+s}\right)\eta(s^2)ds\,.\end{aligned}$$

Applying [27, Theorem 6.10] we get that χ is a complete Bernstein function. □

Remark 2.5 The above result has been independently proved in [23, Lemma 4].

The next result relates the behavior of χ to that of ϕ. It will be used to obtain the asymptotic behavior of χ at infinity.

Proposition 2.6 *Suppose that ϕ, the Laplace exponent of the subordinator S, is a complete Bernstein function. Then the Laplace exponent χ of the ladder height process of Z satisfies*

$$e^{-\pi/2}\sqrt{\phi(\lambda^2)}\le\chi(\lambda)\le e^{\pi/2}\sqrt{\phi(\lambda^2)}\,,\qquad \textit{for all } \lambda>0\,. \qquad (2.11)$$

Proof. By the representations (2.9) and (2.10), we get that for all $\lambda>0$

$$\begin{aligned}\left|\log\chi(\lambda)-\frac{1}{2}\log\phi(\lambda^2)\right| &= \frac{1}{2}\left|\int_0^\infty\left(\left(\frac{t}{1+t^2}-\frac{1}{\sqrt{t}(\lambda+\sqrt{t})}\right)-\left(\frac{t}{1+t^2}-\frac{1}{\lambda^2+t}\right)\right)\eta(t)\,dt\right|\\ &\le \frac{1}{2}\int_0^\infty\frac{\lambda(\sqrt{t}+\lambda)}{(\lambda^2+t)\sqrt{t}(\lambda+\sqrt{t})}\,dt=\frac{1}{2}\int_0^\infty\frac{\lambda}{(\lambda^2+t)\sqrt{t}}\,dt=\frac{\pi}{2}\,.\end{aligned}$$

This implies that $-\pi/2\le\log\chi(\lambda)-\frac{1}{2}\log\phi(\lambda^2)\le\pi/2$ for every $\lambda>0$, which is (2.11) □

Remark 2.7 We note that for the last two propositions we only need to assume that ϕ is a complete Bernstein function; the assumption **(H)** is not used.

Let V denote the potential measure of the ladder height process of Z. We will also use V to denote the corresponding renewal function, $V(t):=V((0,t))$. It follows from (2.1) and (2.11) that $\lim_{\lambda\to\infty}\chi(\lambda)/\lambda=0$ and $\lim_{\lambda\to\infty}\chi(\lambda)=\infty$. Therefore, the ladder height process of Z has no drift and has infinite Lévy measure. This suffices to conclude that the potential measure V has a density denoted by v, and the renewal function can be written as $V(t)=\int_0^t v(s)\,ds$. Since χ is a complete Bernstein function, v is completely monotone. We record these facts as

Corollary 2.8 *Suppose ϕ, the Laplace exponent of the subordinator S, is a complete Bernstein function satisfying the assumption* **(H)**. *Then the potential measure of the ladder height process of the subordinate Brownian motion $Z_t=B_{S_t}$ has a completely monotone density v. In particular, v and the renewal function V are C^∞ functions.*

The smoothness of the renewal function V of the ladder height process Z will be used later in this paper.

Similarly to the case of the density u of the potential measure U of the subordinator S in (2.4), by using Proposition 2.6, we can obtain the asymptotic behavior of the renewal function V and its density v:

$$V(t) \asymp \phi(t^{-2})^{-1/2} \asymp \frac{t^{\alpha/2}}{(\ell(t^{-2}))^{1/2}}\,, \quad t \to 0+\,, \tag{2.12}$$

$$v(t) \asymp t^{-1}\phi(t^{-2})^{-1/2} \asymp \frac{t^{\alpha/2-1}}{(\ell(t^{-2}))^{1/2}}\,, \quad t \to 0+\,, \tag{2.13}$$

see [21, Proposition 3.9]. The corresponding precise asymptotics are given in [20, Proposition 2.7] under the assumption (2.2).

We next consider multidimensional subordinate Brownian motions. Let $W = (W_t = (W_t^1, \dots, W_t^d) : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ with

$$\mathbb{E}\left[e^{i\theta\cdot(W_t - W_0)}\right] = e^{-t|\theta|^2}, \qquad \forall\, \theta \in \mathbb{R}^d, t > 0\,,$$

and let S be a subordinator independent of W with Laplace exponent ϕ. In the remainder of this paper we will use $X = (X_t : t \geq 0)$ to denote the subordinate Brownian motion defined by $X_t = W_{S_t}$. The process X is a (rotationally) symmetric Lévy process with the characteristic exponent given by $\Phi(\theta) = \phi(|\theta|^2)$. It is easy to check that when $d \geq 3$ the process X is transient. This follows from the criterion of Chung-Fuchs type (e.g., [26, p. 252]) which for the subordinate Brownian motion X translates to the following: X is transient if and only if

$$\int_{0+} \frac{\lambda^{d/2-1}}{\phi(\lambda)}\, d\lambda < +\infty\,.$$

Since transience is a global property of the process, it cannot be inferred from the behavior of ϕ at infinity. For example, $\phi(\lambda) = \log(1+\lambda) + \lambda^{\alpha/2}$, $\alpha \in (0, 2)$, is a complete Bernstein function satisfying (2.1), but the corresponding subordinate Brownian motion is recurrent in dimensions 1 and 2. To ensure transience, we will assume that in the case $d \leq 2$, there exists $\gamma \in [0, d/2)$ such that

$$\liminf_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^\gamma} > 0\,. \tag{2.14}$$

An immediate consequence of this assumption and [21, Corollary 2.6] is that the potential density u of S satisfies $u(t) \leq ct^{\gamma-1}$ for all $t \geq 1$, where $c > 0$ is some positive constant (cf. assumption **A1** from [20]).

Transience of the process X ensures that the Green function $G(x, y)$, $x, y \in \mathbb{R}^d$, is well defined. By spatial homogeneity we may write $G(x, y) = G(x - y)$, where the function G is radial and given by the following formula,

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} u(t) dt, \qquad x \in \mathbb{R}^d. \tag{2.15}$$

Note that G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

The Lévy measure of the process X has a density J, called the Lévy density, given by

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mu(t)\, dt, \qquad x \in \mathbb{R}^d.$$

Thus $J(x) = j(|x|)$ with

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)\, dt, \qquad r > 0. \tag{2.16}$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$. We will sometimes use the notation $J(x, y)$ for $J(x - y)$.

We discuss now the behavior of G and j near the origin. Under the assumption (2.2), the precise asymptotic behavior was obtained in [20, Theorem 3.1, Theorem 3.2] by using precise asymptotic behavior of the potential density u, and, respectively, Lévy density μ. These two results were proved by use of [6, Lemma 5.32], which required additional assumptions which were stated as **A2** and **A3** in [20]. It turned out that by using Potter's theorem (Theorem 1.4) one can circumvent those assumptions and still obtain the conclusion of the lemma. The details are provided in [21, Lemma 3.1]. This lemma combined with (2.4) (resp. (2.5)) and representation (2.15) (resp. (2.16)) gives the following asymptotic behavior of G (resp. J) under the assumption (2.1):

Theorem 2.9 ([21, Theorem 3.2]) *Suppose that the Laplace exponent ϕ is a complete Bernstein function satisfying the assumption* **(H)** *and that $\alpha \in (0, 2 \wedge d)$. In the case $d \le 2$, we further assume* (2.14)*. Then*

$$G(x) \asymp \frac{1}{|x|^d \phi(|x|^{-2})} \asymp \frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})}, \qquad |x| \to 0.$$

Remark 2.10 Since α is always assumed to be in $(0, 2)$, the assumption $\alpha \in (0, 2 \wedge d)$ in the theorem above makes a difference only in the case $d = 1$.

Theorem 2.11 ([21, Theorem 3.4]) *Suppose that the Laplace exponent ϕ is a complete Bernstein function satisfying the assumption* **(H)***. Then*

$$J(x) = j(|x|) \asymp \frac{\phi(|x|^{-2})}{|x|^d} \asymp \frac{\ell(|x|^{-2})}{|x|^{d+\alpha}}, \qquad |x| \to 0.$$

Using (2.6) and (2.7), and repeating the proof of [24, Lemma 4.2] we get that

(a) For any $K > 0$, there exists $c = c(K) > 1$ such that

$$j(r) \le c\, j(2r), \qquad \forall r \in (0, K). \tag{2.17}$$

(b) There exists $c > 1$ such that

$$j(r) \le c\, j(r + 1), \qquad \forall r > 1. \tag{2.18}$$

Therefore by [30, Theorem 2.2 and Section 3.1] (see also [6, Theorem 5.66], [21, Theorem 4.7, Corollary 4.8] and [24]) the Harnack inequality is valid for the process X. Before we state the Harnack inequality, we recall the definition of harmonic functions.

For any open set D, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$.

Definition 2.12 *Let D be an open subset of $\mathbb{R}^d$. A function u defined on $\mathbb{R}^d$ is said to be*

(1) *harmonic in D with respect to X if $\mathbb{E}_x[|u(X_{\tau_B})|] < \infty$ and $u(x) = \mathbb{E}_x[u(X_{\tau_B})]$, $x \in B$, for every open set B whose closure is a compact subset of D;*

(2) *regular harmonic in D with respect to X if it is harmonic in D with respect to X and for each $x \in D$, $u(x) = \mathbb{E}_x[u(X_{\tau_D})]$.*

Theorem 2.13 (Harnack inequality) *There exists $C_6 = C_6(\phi) > 0$ such that, for any $r \in (0, 1/4)$, $x_0 \in \mathbb{R}^d$, and any function u which is nonnegative on $\mathbb{R}^d$ and harmonic with respect to X in $B(x_0, 16r)$, we have*

$$\sup_{y \in B(x_0, r/2)} u(y) \le C_6 \inf_{y \in B(x_0, r/2)} u(y).$$

From now we will always assume that ϕ is a complete Bernstein function satisfying the assumption (**H**) for $\alpha \in (0, 2\wedge d)$ and the additional (2.14) in the case $d \le 2$. We will no longer explicitly mention these assumptions

For any open set D in $\mathbb{R}^d$, we will use $G_D(x, y)$ to denote the Green function of X in D. Using the continuity and the radial decreasing property of G, we can easily check that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$. We will frequently use the well-known fact that $G_D(\cdot, y)$ is harmonic in $D\setminus\{y\}$, and regular harmonic in $D \setminus \overline{B(y, \varepsilon)}$ for every $\varepsilon > 0$.

Using the Lévy system for X, we know that for every bounded open subset D and every $f \ge 0$ and $x \in D$,

$$\mathbb{E}_x\left[f(X_{\tau_D}); X_{\tau_D-} \neq X_{\tau_D}\right] = \int_{\overline{D}^c}\int_D G_D(x, z)J(z-y)dzf(y)dy. \tag{2.19}$$

Now we prove the following version of the Harnack inequality for X.

Theorem 2.14 *Let $L > 0$. There exists a positive constant $C_7 = C_7(L, \phi) > 0$ such that the following is true: If $x_1, x_2 \in \mathbb{R}^d$ and $r \in (0, 1)$ are such that $|x_1 - x_2| < Lr$, then for every nonnegative function u which is harmonic with respect to X in $B(x_1, r) \cup B(x_2, r)$, we have*

$$C_7^{-1}u(x_2) \le u(x_1) \le C_7u(x_2).$$

Proof. By [21, Proposition 4.10] (see also [20, Proposition 3.8]), for every $r \in (0, 1)$, every $x \in \mathbb{R}^d$ and every $y \in B(x, \frac{r}{8})$ it holds that

$$K_{B(x,\frac{r}{8})}(x, y) := \int_{B(x,\frac{r}{8})} G_{B(x,\frac{r}{8})}(x, z)J(z - y)dz \ge c_1\, j(|x - y|)\frac{r^\alpha}{\ell(r^{-2})}\,, \tag{2.20}$$

with some positive constant $c_1 > 0$.

Now let $r \in (0, 1)$, $x_1, x_2 \in \mathbb{R}^d$ be such that $|x_1 - x_2| < Lr$ and let u be a nonnegative function which is harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to X.

If $|x_1-x_2| < \frac{1}{4}r$, then since $r < 1$, the theorem is true from Theorem 2.13. Thus we only need to consider the case when $\frac{1}{4}r \le |x_1 - x_2| \le Lr$ with $L > \frac{1}{4}$.

Let $w \in B(x_1, \frac{r}{8})$. Because $|x_2 - w| \le |x_1 - x_2| + |w - x_1| < (L + \frac{1}{8})r \le 2Lr$, by the monotonicity of j and (2.20)

$$K_{B(x_2,\frac{r}{8})}(x_2, w) \ge c_1\, j(2Lr)\frac{r^\alpha}{\ell(r^{-2})}\,. \tag{2.21}$$

For any $y \in B(x_1, \frac{r}{8})$, u is regular harmonic in $B(y, \frac{7r}{8}) \cup B(x_1, \frac{7r}{8})$. Since $|y - x_1| < \frac{r}{8}$, by Theorem 2.13

$$u(y) \ge c_2u(x_1), \quad y \in B(x_1, \frac{r}{8}), \tag{2.22}$$

for some constant $c_2 > 0$. Therefore, by (2.19) and (2.21)–(2.22),

$$\begin{aligned} u(x_2) &= \mathbb{E}_{x_2}\left[u(X_{\tau_{B(x_2,\frac{r}{8})}})\right] \ge \mathbb{E}_{x_2}\left[u(X_{\tau_{B(x_2,\frac{r}{8})}}); X_{\tau_{B(x_2,\frac{r}{8})}} \in B(x_1, \frac{r}{8})\right] \\ &\ge c_2\, u(x_1)\,\mathbb{P}_{x_2}\left(X_{\tau_{B(x_2,\frac{r}{8})}} \in B(x_1, \frac{r}{8})\right) = c_2\, u(x_1)\int_{B(x_1,\frac{r}{8})} K_{B(x_2,\frac{r}{8})}(x_2, w)\, dw \\ &\ge c_3\, u(x_1)\left|B(x_1, \frac{r}{8})\right| j(2Lr)\frac{r^\alpha}{\ell(r^{-2})} = c_4\, u(x_1)j(2Lr)\frac{r^{\alpha+d}}{\ell(r^{-2})}. \end{aligned}$$

Thus, by Theorem 2.11 and the inequality above, there exists a constant $c_5 > 0$ such that for all $r \in (0, 1)$, $u(x_2) \ge c_5u(x_1)\frac{j(2Lr)}{j(r)}$. The right-hand side is by (2.17) greater than $c_5c_6^{\log 2L/\log 2}u(x_1)$ where $c_6 =$

$C_4^{-1}(4L) \in (0,1)$. We have thus proved the right-hand side inequality in the conclusion of the theorem. The inequality on the left-hand side can be proved similarly. □

In [20] we have established the boundary Harnack principle under the assumption (2.2) (and the additional transience assumption in case $d \le 2$) for κ-fat open sets. Even though we only explicitly stated the results for $d \ge 2$ in [20], the results and arguments there are in fact valid for $d = 1$ also. Under the current assumptions, the same result is reproved in [21].

Theorem 2.15 ([20, Theorem 4.8], [21, Theorem 4.22]) Suppose that D is a bounded κ-fat open set with the characteristics (R_1, κ). There exists a constant $C_8 = C_8(d, \phi, R_1, \kappa) > 1$ such that if $r \le R_1 \wedge \frac{1}{4}$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have

$$C_8^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le C_8 \frac{u(A_r(Q))}{v(A_r(Q))}, \qquad x \in D \cap B(Q, \frac{r}{2}).$$

Before concluding this section, we give some examples satisfying our assumptions.

Example 2.16 Suppose that $0 < \beta < \alpha < 2$, $0 < \gamma < 2 - \alpha$ and define

$$\phi_1(\lambda) = \lambda^{\alpha/2}, \qquad \phi_2(\lambda) = (\lambda+1)^{\alpha/2} - 1, \qquad \phi_3(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2},$$

$$\phi_4(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{\gamma/2} \qquad \text{and} \qquad \phi_5(\lambda) = \lambda^{\alpha/2}(\log(1+\lambda))^{-\beta/2}.$$

Then ϕ_i, $i = 1, \dots, 5$, are complete Bernstein functions which can be written as

$$\phi_i(\lambda) = \lambda^{\alpha/2} \ell_i(\lambda), \qquad i = 1, \dots, 5,$$

with

$$\ell_1(\lambda) = 1, \qquad \ell_2(\lambda) = \Big((\lambda+1)^{\alpha/2} - 1\Big) \lambda^{-\alpha/2}, \qquad \ell_3(\lambda) = 1 + \lambda^{(\beta-\alpha)/2},$$

$$\ell_4(\lambda) = (\log(1+\lambda))^{\gamma/2} \qquad \text{and} \qquad \ell_5(\lambda) = (\log(1+\lambda))^{-\beta/2}.$$

As already mentioned in the introduction, the subordinate Brownian motion corresponding to ϕ_1 is a symmetric α-stable process, the subordinate Brownian motion corresponding to ϕ_2 is a relativistic α-stable process and the subordinate Brownian motion corresponding to ϕ_3 is the sum of a symmetric α-stable process and an independent symmetric β-stable process. The subordinate Brownian motions corresponding to ϕ_4 and ϕ_5 were discussed in [6].

In the case $d \ge 3$, the only condition on the complete Bernstein function ϕ is (1.2), so we can use Proposition 7.1, Corollary 7.9, Propositions 7.10–7.11, Corollary 7.12 of [27] to come up with infinitely many examples of such functions, e.g.: (i) $\lambda^{\alpha/2}(\log(1+\log(1+\lambda^{\gamma/2})^{\delta/2}))^{\beta/2}$, $\alpha, \gamma, \delta \in (0,2), \beta \in (0, 2-\alpha]$; (ii) $\lambda^{\alpha/2}(\log(1+\log(1+\lambda^{\gamma/2})^{\delta/2}))^{-\beta/2}$, $\alpha, \gamma, \delta \in (0,2), \beta \in (0, \alpha]$.

All of the listed example satisfy the stronger condition (2.2). As already mentioned, [21, Example 2.8] provides an example of a complete Bernstein function which satisfies (2.1), but not (2.2).

3 Green function estimates on bounded κ-fat open sets

In this section we will establish sharp two-sided Green function estimates for X in any bounded κ-fat open subset D of $\mathbb{R}^d$, $d \ge 1$. Our standing assumption is that ϕ is a complete Bernstein function satisfying the assumption **(H)** for $\alpha \in (0, 2 \wedge d)$ and the additional assumption (2.14) when $d \le 2$.

We will first establish the interior estimates using Theorems 2.9 and 2.14. Once we have the interior estimates, we can apply Theorem 2.14 and the boundary Harnack principle (Theorem 2.15), and use the arguments of [4, 16] to get the full estimates (1.5) for bounded κ-fat open sets D. Since the arguments of [4, 16] are by now quite standard, we will omit the details of this part.

Although the conclusions of Lemmas 3.1-3.3 below can be written in terms of ϕ, we write them in terms of ℓ since it is in this form that the results are used.

Lemma 3.1 *There exist $R_2 = R_2(\phi) > 0$, $L_1 = L_1(\phi) > 2$ and $C_9 = C_9(\phi) > 0$ such that*

$$G(x) - G(L_1 x) \geq C_9 \frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})} \qquad \text{for every } |x| < R_2.$$

Proof. By Theorem 2.9 there exists a constant $c > 1$ such that for all $x \in \mathbb{R}^d \setminus \{0\}$ with $|x| < 1$ it holds that

$$\frac{c^{-1}}{|x|^{d-\alpha}\ell(|x|^{-2})} \leq G(x) \leq \frac{c}{|x|^{d-\alpha}\ell(|x|^{-2})}\,.$$

Choose $L_1 = (4c^2)^{\frac{1}{d-\alpha}} \vee 2$ so that $c^2/L_1^{d-\alpha} \leq 1/4$. Since ℓ is slowly varying at infinity, there exists $r_1 < 1$ such that

$$\frac{\ell(|x|^{-2})}{\ell(|L_1 x|^{-2})} \leq 2$$

whenever $0 < |x| < r_1$. Let $R_2 = r_1 \wedge L_1^{-1}$. Then for $x \in \mathbb{R}^d \setminus \{0\}$ we have

$$\begin{aligned} G(x) - G(L_1 x) &\geq \frac{c^{-1}}{|x|^{d-\alpha}\ell(|x|^{-2})} - \frac{c}{|L_1 x|^{d-\alpha}\ell(|L_1 x|^{-2})} \\ &= \frac{c^{-1}}{|x|^{d-\alpha}\ell(|x|^{-2})}\left(1 - \frac{c^2}{L_1^{d-\alpha}}\frac{\ell(|x|^{-2})}{\ell(|L_1 x|^{-2})}\right) \\ &\geq \frac{1}{2c}\frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})}\,. \end{aligned}$$

□

Lemma 3.2 *For every bounded open set D, there exists a constant $C_{10} = C_{10}(\phi) > 1$ such that*

$$G_D(x,y) \leq C_{10}\frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}\,, \qquad \text{for } x, y \in D\,, \tag{3.1}$$

and

$$G_D(x,y) \geq C_{10}^{-1}\frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}, \qquad \text{for } x, y \in D \text{ with } L_1|x-y| \leq \delta_D(x) \wedge \delta_D(y), \tag{3.2}$$

where L_1 is the constant in Lemma 3.1.

Proof. Since $G_D(x,y) \leq G(x,y)$, D is bounded and ℓ locally bounded from above and below by positive constants, (3.1) is an immediate consequence of Theorem 2.9. Now we show (3.2). Without loss of generality, we assume $\delta_D(y) \leq \delta_D(x)$, and let $M := \mathrm{diam}(D)$. We consider three cases separately:
(a) $\delta_D(y) \leq R_2$: Since $|x-y| \leq \delta_D(y)/L_1$, $|X_{\tau_{B(y,\delta_D(y))}} - y| \geq \delta_D(y) \geq L_1|x-y|$. Thus by the monotonicity of G and Lemma 3.1,

$$G_D(x,y) \,\geq\, G_{B(y,\delta_D(y))}(x,y) \,=\, G(x,y) - \mathbb{E}_x\left[G(X_{\tau_{B(y,\delta_D(y))}}, y)\right]$$

$$\geq \quad G(x-y) - G(L_1(x-y)) \;\geq\; c_1 \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}, \qquad \forall |x-y| \leq \frac{\delta_D(y)}{L_1}\,.$$

(b) $\delta_D(y) > R_2$ and $|x-y| \leq R_2/L_1$: In this case, $|X_{\tau_{B(y,R_2)}} - y| \;\geq\; R_2 \;\geq\; L_1|x-y|$ and, by the monotonicity of G and Lemma 3.1, we get

$$\begin{aligned} G_D(x,y) \;&\geq\; G_{B(y,R_2)}(x,y) \;=\; G(x,y) - \mathbb{E}_x\left[G(X_{\tau_{B(y,R_2)}}, y)\right] \\ &\geq\; G(x-y) - G(L_1(x-y)) \;\geq\; c_1 \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}, \qquad \forall |x-y| \leq \frac{R_2}{L_1}\,. \end{aligned}$$

(c) $\delta_D(y) > R_2$ and $|x-y| > R_2/L_1$: Choose a point $w \in \partial B(y, R_2/L_1)$. Then from the argument in (b), we get

$$G_D(w,y) \geq G(w,y) - \mathbb{E}_w\left[G(X_{\tau_{B(y,R_2)}}, y)\right] \;\geq\; c_1 \frac{1}{(R_2/L_1)^{d-\alpha}\ell((R_2/L_1)^{-2})}.$$

Since D is bounded and $G_D(\,\cdot\,, y)$ is harmonic with respect to X in $B(x, R_2/(2L_1)) \cup B(w, R_2/(2L_1))$, by Theorem 2.14 we have

$$\begin{aligned} G_D(x,y) \;&\geq\; c_2 G_D(w,y) \;\geq\; c_3 \frac{1}{(R_2/L_1)^{d-\alpha}\ell((R_2/L_1)^{-2})} \\ &\geq\; \frac{c_3}{\ell((R_2/L_1)^{-2})}\left(\inf_{\frac{R_2}{L_1} \leq s \leq M} \ell(s^{-2})\right) \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}, \qquad \forall |x-y| > \frac{R_2}{L_1}\,. \end{aligned}$$

□

Lemma 3.3 *For every $L > 0$ and bounded open set D, there exists $C_{11} = C_{11}(L, \phi) > 0$ such that for every $|x-y| \leq L(\delta_D(x) \wedge \delta_D(y))$,*

$$G_D(x,y) \geq C_{11} \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}. \tag{3.3}$$

Proof. Without loss of generality, we assume $\delta_D(x) \leq \delta_D(y)$. Moreover, by Lemma 3.2 we can assume that $L > 1/L_1$ and we only need to show (3.3) for $\frac{1}{L_1}\delta_D(x) \leq |x-y| \leq L\delta_D(x)$.

Choose a point $w \in \partial B(x, \delta_D(x)/L_1)$. Then by Lemma 3.2, we get

$$G_D(x,w) \;\geq\; c_1 \frac{1}{(\delta_D(x)/L_1)^{d-\alpha}\ell((\delta_D(x)/L_1)^{-2})}.$$

Since $|y-w| \leq |x-y| + |x-w| \leq (L+1)\delta_D(x)$ and $G_D(x,\,\cdot\,) = G_D(\,\cdot\,, x)$ is harmonic with respect to X in $B(y, \delta_D(x)/L_1) \cup B(w, \delta_D(x)/L_1)$, by Theorem 2.14 we have

$$\begin{aligned} G_D(x,y) &\geq c_2 G_D(x,w) \geq c_3 \frac{1}{(\delta_D(x)/L_1)^{d-\alpha}\ell((\delta_D(x)/L_1)^{-2})} \\ &\geq \quad c_4 \left(\frac{\ell(|x-y|^{-2})}{\ell((\delta_D(x)/L_1)^{-2})}\right) \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}. \end{aligned} \tag{3.4}$$

By the uniform convergence theorem ([2, Theorem 1.2.1]), we can choose a small $r_1 = r_1(\ell, L) > 0$ such that

$$\inf_{\lambda \in [1,2L]} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \;\geq\; \frac{1}{2}, \qquad \forall r \leq r_1\,. \tag{3.5}$$

If $\frac{1}{L_1}\delta_D(x) \leq |x-y| \leq L\delta_D(x) \leq r_1$, by (3.4)–(3.5),

$$G_D(x,y) \;\geq\; c_4 \left(\inf_{r \leq r_1, \lambda \in [1,2L]} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})}\right) \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})} \;\geq\; \frac{c_4}{2} \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}.$$

On the other hand, if $\frac{1}{L_1}\delta_D(x) \le |x-y| \le L\delta_D(x)$ and $\delta_D(x) \ge r_1/L_1$, then $|x-y| \ge \frac{r_1}{LL_1}$. Thus from (3.4), we see that

$$G_D(x,y) \ \ge \ c_4 \left(\inf_{r\in[\frac{r_1}{LL_1},M]} \ell(r^{-2}) \right) \left(\inf_{r\in[\frac{r_1}{LL_1},M]} \ell(r^{-2})^{-1} \right) \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}$$

where $M = \mathrm{diam}(D)$. □

For the remainder of this section, we assume that D is a bounded κ-fat open set with characteristics (R_1,κ). Without loss of generality we may assume that $R_1 \le 1/4$. We recall that for each $Q \in \partial D$ and $r \in (0,R_1)$, $A_r(Q)$ denotes a point in $D \cap B(Q,r)$ satisfying $B(A_r(Q),\kappa r) \subset D \cap B(Q,r)$. Recall also that $G_D(\cdot,y)$ is regular harmonic in $D \setminus \overline{B(y,\varepsilon)}$ for every $\varepsilon > 0$ and vanishes outside D.

Using the uniform convergence theorem ([2, Theorem 1.2.1]), we further choose $R_3 \le R_1$ such that

$$\frac{1}{2} \le \min_{\frac{1}{6}\le\lambda\le 2\kappa^{-1}} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \le \max_{\frac{1}{6}\le\lambda\le 2\kappa^{-1}} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \le 2, \quad \text{if } r \le R_3\,. \tag{3.6}$$

Let us fix $z_0 \in D$ with $\kappa R_3 < \delta_D(z_0) < R_3$ and let $\varepsilon_1 := \kappa R_3/24$. For $x,y \in D$, we define $r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and

$$\mathcal{B}(x,y) := \begin{cases} \left\{A \in D:\ \delta_D(A) > \frac{\kappa}{2} r(x,y),\ |x-A| \vee |y-A| < 5r(x,y)\right\} & \text{if } r(x,y) < \varepsilon_1 \\ \{z_0\} & \text{if } r(x,y) \ge \varepsilon_1. \end{cases} \tag{3.7}$$

Note that for every $(x,y) \in D \times D$ with $r(x,y) < \varepsilon_1$

$$\frac{1}{6}\delta_D(A) \ \le\ \delta_D(x) \vee \delta_D(y) \vee |x-y| \ \le\ 2\kappa^{-1}\delta_D(A), \qquad A \in \mathcal{B}(x,y). \tag{3.8}$$

So by (3.6), if $r(x,y) < \varepsilon_1$

$$\frac{1}{2} \le \frac{\ell((\delta_D(A))^{-2})}{\ell((r(x,y))^{-2})} \le 2, \qquad A \in \mathcal{B}(x,y). \tag{3.9}$$

Let

$$C_{12} := C_{10} 2^{d-\alpha} \delta_D(z_0)^{-d+\alpha} \left(\sup_{\delta_D(z_0)/2 \le r \le M} \ell(r^{-2})^{-1} \right).$$

It follows from Lemma 3.2 that $G_D(\cdot,z_0)$ is bounded above by C_{12} on $D \setminus B(z_0,\delta_D(z_0)/2)$. Now we define

$$g(x) := G_D(x,z_0) \wedge C_{12}. \tag{3.10}$$

Note that if $\delta_D(z) \le 6\varepsilon_1$, then $|z-z_0| \ge \delta_D(z_0) - 6\varepsilon_1 \ge \delta_D(z_0)/2$ since $6\varepsilon_1 < \delta_D(z_0)/4$, and therefore $g(z) = G_D(z,z_0)$.

Proof of Theorem 1.2. Note that, by Theorem 2.14, for all $x,y \in D$ and $A_1,A_2 \in \mathcal{B}(x,y)$, $g(A_1)$ is comparable to $g(A_2)$. With this observation, (3.6), Lemma 3.1–3.3, Theorem 2.14 and the boundary Harnack principle (Theorem 2.15) in hand, one can easily adapt the arguments leading to the proofs, as well as the proofs of [4, Proposition 6] and [16, Theorem 2.4] to finish the proof of Theorem 1.2. Since these are more or less standard now, we omit the details. □

4 Explicit Green function estimates on bounded $C^{1,1}$-open sets

In this section we refine the estimates from Theorem 1.2 in the case of bounded $C^{1,1}$ open sets.

Recall that $X = (X_t : t \geq 0)$ is the d-dimensional subordinate Brownian motion defined by $X_t = W_{S_t}$ where $W = (W^1, \dots, W^d)$ is a d-dimensional Brownian motion and $S = (S_t : t \geq 0)$ an independent subordinator with the Laplace exponent ϕ which is a complete Bernstein function satisfying assumption **(H)** for $\alpha \in (0, 2 \wedge d)$ and the additional assumption (2.14) when $d \leq 2$. Let $Z = (Z_t : t \geq 0)$ be the one-dimensional subordinate Brownian motion defined as $Z_t := W^d_{S_t}$. Recall that the potential measure of the ladder height process of Z is denoted by V and its density by v. We also use V to denote the renewal function of the ladder height process of Z. In Corollary 2.8 we have established that both V and v are C^∞ functions. Recall the notation $\mathbb{R}^d_+ := \{x = (x_1, \dots, x_{d-1}, x_d) := (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ for the half-space. The next result, which follows from [28, Theorem 2], is very important in this paper.

Define $w(x) := V((x_d)^+)$.

Theorem 4.1 *The function w is harmonic in $\mathbb{R}^d_+$ with respect to X and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ (in $(0, r)$ when $d = 1$) with respect to X.*

Proof. Since $Z_t := W^d_{S_t}$ has a transition density, it satisfies the condition ACC in [28], namely the resolvent kernels are absolutely continuous. The assumption in [28] that 0 is regular for $(0, \infty)$ is also satisfied since Z is symmetric and has infinite Lévy measure. Indeed, if 0 were irregular for $(0, \infty)$, it would be, by symmetry, irregular for $(-\infty, 0)$ as well. But then Z would be a compound Poisson process which contradicts the fact that it has infinite Lévy measure. Further, again by symmetry of Z, the notions of coharmonic and harmonic functions coincide. Let $Z^{(0,\infty)}$ (respectively $X^{\mathbb{R}^d_+}$) denote the process Z killed upon exiting $(0, \infty)$ (respectively X killed upon exiting $\mathbb{R}^d_+$). By [28, Theorem 2], the renewal function V of the ladder height process of Z is invariant for $Z^{(0,\infty)}$. Thus w is invariant for $X^{\mathbb{R}^d_+}$. Being invariant for $X^{\mathbb{R}^d_+}$, w is also harmonic for $X^{\mathbb{R}^d_+}$, and consequently, harmonic in $\mathbb{R}^d_+$ with respect to X. We show now that w is regular harmonic for X in $\mathbb{R}^{d-1} \times (0, r)$ for any $r > 0$. First note that since V is continuous at zero and $V(0) = 0$, it follows that

$$\lim_{x_d \to 0} w(x) = \lim_{x_d \to 0} w(\tilde{x}, x_d) = \lim_{x_d \to 0} V(x_d) = 0 \,. \tag{4.1}$$

Thus, by harmonicity of w and (4.1)

$$w(x) = w(\tilde{x}, x_d) = \lim_{\varepsilon \to 0} \mathbb{E}_x \left[w\left(X_{\tau_{\mathbb{R}^{d-1} \times (\varepsilon, r)}} \right) \right] = \mathbb{E}_x \left[w\left(X_{\tau_{\mathbb{R}^{d-1} \times (0, r)}} \right) \right], \quad x_d > 0 \,.$$

□

Proposition 4.2 *For all positive constants r_0 and L, we have*

$$\sup_{x \in \mathbb{R}^d :\, 0 < x_d < L} \int_{B(x, r_0)^c \cap \mathbb{R}^d_+} w(y) j(|x - y|) \, dy < \infty \,.$$

Proof. By Theorem 4.1 and (2.19), for every $x \in \mathbb{R}^d_+$,

$$\begin{aligned} w(x) &\geq \mathbb{E}_x \left[w(X_{\tau_{B(x, r_0/2) \cap \mathbb{R}^d_+}}) : X_{\tau_{B(x, r_0/2) \cap \mathbb{R}^d_+}} \in B(x, r_0)^c \cap \mathbb{R}^d_+ \right] \\ &= \int_{B(x, r_0)^c \cap \mathbb{R}^d_+} \int_{B(x, r_0/2) \cap \mathbb{R}^d_+} G_{B(x, r_0/2) \cap \mathbb{R}^d_+}(x, z) j(|z - y|) w(y) \, dz \, dy \,. \end{aligned} \tag{4.2}$$

Since $|z-y| \le |x-z| + |x-y| \le r_0 + |x-y| \le 2|x-y|$ for $(z,y) \in B(x,r_0/2) \times B(x,r_0)^c$, using (2.17) and (2.18), we have $j(|z-y|) \ge c_1 j(|x-y|)$. Thus, combining this with (4.2), we obtain that

$$\sup_{x\in\mathbb{R}^d:\,0<x_d<L} \int_{B(x,r_0)^c\cap\mathbb{R}^d_+} w(y)j(|x-y|)dy \le c_1^{-1} \sup_{\widetilde{x}=0,0<x_d<L} \frac{w(x)}{\mathbb{E}_x[\tau_{B(x,r_0/2)\cap\mathbb{R}^d_+}]}.$$

We claim that the supremum on the right-hand side above is finite. Clearly, if $L > x_d \ge r_0/(64)$ and $\widetilde{x} = 0$,

$$\frac{w(x)}{\mathbb{E}_x[\tau_{B(x,r_0/2)\cap\mathbb{R}^d_+}]} \le \frac{V(L)}{\mathbb{E}_0[\tau_{B(0,r_0/(64))}]}.$$

Suppose $x_d < r_0/(64)$ and $\widetilde{x} = 0$. Let $U := B((\widetilde{0}, 16r_0), r_0)$. By the Lévy system, we have

$$\mathbb{P}_x\left(X_{\tau_{B(0,r_0/2)\cap\mathbb{R}^d_+}} \in U\right) = \int_U \int_{B(0,r_0/2)\cap\mathbb{R}^d_+} G_{B(0,r_0/2)\cap\mathbb{R}^d_+}(x,z)j(|z-y|)dzdy \le c_2\mathbb{E}_x[\tau_{B(0,r_0/2)\cap\mathbb{R}^d_+}].$$

Thus, by the above and the boundary Harnack principle (Theorem 2.15),

$$\frac{w(x)}{\mathbb{E}_x[\tau_{B(x,r_0/2)\cap\mathbb{R}^d_+}]} \le c_2 \frac{w(x)}{\mathbb{P}_x(X_{\tau_{B(0,r_0/2)\cap\mathbb{R}^d_+}} \in U)} \le c_3 \frac{w(x_1)}{\mathbb{P}_{x_1}(X_{\tau_{B(0,r_0/2)\cap\mathbb{R}^d_+}} \in U)} \le c_4 V(r_0/(16))$$

where $x_1 = (\widetilde{0}, r_0/(16))$. We have thus proved the claim. □

We now define the operator $(\mathcal{A}, \mathfrak{D}(\mathcal{A}))$ by the following formula:

$$\begin{aligned} \mathcal{A}f(x) &:= \text{P.V.} \int_{\mathbb{R}^d} (f(y)-f(x))\, j(|y-x|)\, dy := \lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^d:|x-y|>\varepsilon\}} (f(y)-f(x))\, j(|y-x|)\, dy \\ \mathfrak{D}(\mathcal{A}) &:= \left\{ f:\mathbb{R}^d\to\mathbb{R} : \lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^d:|x-y|>\varepsilon\}} (f(y)-f(x))\, j(|y-x|)\, dy \text{ exists and is finite} \right\}. \end{aligned} \tag{4.3}$$

It is well known that $C_0^2 \subset \mathfrak{D}(\mathcal{A})$, where C_0^2 is the collection of C^2 functions in $\mathbb{R}^d$ vanishing at infinity, and that by the rotational symmetry of X, $\mathcal{A}$ restricted to C_0^2 coincides with the infinitesimal generator of the process X (e.g. [26, Theorem 31.5]).

Theorem 4.3 *$\mathcal{A}w(x)$ is well defined and $\mathcal{A}w(x) = 0$ for all $x \in \mathbb{R}^d_+$.*

Proof. We first note that it follows from Proposition 4.2 and the fact that J is a Lévy density that for any $L > 0$

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d:\ 0<x_d<L} \left| \int_{\{y\in\mathbb{R}^d:|y-x|>1\}} (w(y)-w(x))j(|y-x|)dy \right| \\ &\le \sup_{x\in\mathbb{R}^d:\ 0<x_d<L} \int_{\{y\in\mathbb{R}^d:|y-x|>1\}} w(y)j(|y-x|)\, dy + V(L) \int_{\{y\in\mathbb{R}^d:|y|>1\}} j(|y|)dy < \infty\, . \end{aligned} \tag{4.4}$$

Hence, for every $\varepsilon \in (0, 1/2)$

$$\mathcal{A}_\varepsilon w(x) := \int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}} (w(y)-w(x))j(|y-x|)dy$$

is well defined. Note that since $w(x) = V((x_d)^+)$ and V is smooth in $(0,\infty)$ by Corollary 2.8, it holds that w is smooth in $\mathbb{R}^d_+$. Hence,

$$\mathcal{A}_\varepsilon w(x) = \int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}} \big(w(y)-w(x) - \mathbf{1}_{\{|y-x|<1\}}(y-x)\cdot\nabla w(x)\big)\, j(|y-x|)dy.$$

Moreover, by the smoothness of w,

$$x \mapsto \int_{\{y\in\mathbb{R}^d: |y-x|\le\varepsilon\}} (w(y) - w(x) - (y-x)\cdot\nabla w(x))\, j(|y-x|)\, dy$$

converges to 0 locally uniformly in $\mathbb{R}^d_+$ as $\varepsilon \to 0$. Combining this with (4.4), we see that $\mathcal{A}w$ is well defined in $\mathbb{R}^d_+$ and $\mathcal{A}_\varepsilon w(x)$ converges to

$$\mathcal{A}w(x) = \int_{\mathbb{R}^d} \big(w(y) - w(x) - \mathbf{1}_{\{|y-x|<1\}}(y-x)\cdot\nabla w(x)\big)\, j(|y-x|)dy$$

locally uniformly in $\mathbb{R}^d_+$ as $\varepsilon \to 0$.

Moreover, for every $x \in \mathbb{R}^d_+$, $z \in B(x, (\varepsilon\wedge x_d)/2)$, and $y \in B(z,\varepsilon)^c$ it holds that $\frac{1}{2}|y-z| \le |y-x| \le \frac{3}{2}|y-z|$. So, using (2.17),

$$\begin{aligned}
&\mathbf{1}_{\{|y-z|>\varepsilon\}}\left|(w(y) - w(z) - \mathbf{1}_{\{|y-z|<1\}}(y-z)\cdot\nabla w(z))\right| j(|y-z|)\\
\le& c\left(\sup_{\varepsilon/2<s<x_d+2} V''(s)\right)|y-x|^2\mathbf{1}_{\{\varepsilon/2<|y-x|<2\}}\, j(|y-x|/2)\\
&\quad + (w(y) + V(x_d+1))\mathbf{1}_{\{|y-x|>1/2\}}\, j(|y-x|/2).
\end{aligned}$$

It follows from Proposition 4.2 and the fact that J is a Lévy density, by using the dominated convergence theorem, that $x \to \mathcal{A}_\varepsilon w(x)$ is continuous for each ε. Therefore, by this and the local uniform convergence of $\mathcal{A}_\varepsilon w$, the function $\mathcal{A}w(x)$ is continuous in $\mathbb{R}^d_+$.

Suppose that U_1 and U_2 are relatively compact open subsets of $\mathbb{R}^d_+$ such that $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{R}^d_+$. Let $r_0 := \mathrm{dist}(U_1, U_2^c) > 0$. Then, by Proposition 4.2

$$\begin{aligned}
\int_{U_1}\int_{U_2^c} w(y)j(|x-y|)dydx \;&\le\; |U_1| \sup_{x\in U_1}\int_{U_2^c} w(y)j(|x-y|)dy\\
&\le\; |U_1| \sup_{x\in U_1}\int_{B(x,r_0)^c} w(y)j(|x-y|)dy < \infty\,. \qquad (4.5)
\end{aligned}$$

By harmonicity of w, clearly $w(X_{\tau_{U_1}}) \in L^1(\mathbb{P}_x)$ and

$$\sup_{x\in U_1}\mathbb{E}_x\left[\mathbf{1}_{U_2^c}(X_{\tau_{U_1}})w(X_{\tau_{U_1}})\right] \le \sup_{x\in U_1}\mathbb{E}_x\left[w(X_{\tau_{U_1}})\right] = \sup_{x\in U_1} w(x) < \infty.$$

The last two displays show that the conditions [7, (2.4), (2.6)] are true. Thus, by [7, Lemma 2.3, Theorem 2.11(ii)], we have that for any $f \in C^2_c(\mathbb{R}^d_+)$,

$$0 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (w(y) - w(x))(f(y) - f(x))j(|y-x|)\, dx\, dy. \qquad (4.6)$$

For $f \in C^2_c(\mathbb{R}^d_+)$ with $\mathrm{supp}(f) \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{R}^d_+$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w(y) - w(x)||f(y) - f(x)|j(|y-x|)dxdy\\
=&\int_{U_2}\int_{U_2} |w(y) - w(x)||f(y) - f(x)|j(|y-x|)dxdy + 2\int_{U_1}\int_{U_2^c} |w(y) - w(x)||f(x)|j(|y-x|)dxdy\\
\le& c_1\int_{U_2\times U_2} |y-x|^2 j(|y-x|)dxdy + 2\|f\|_\infty |U_1|\left(\sup_{x\in U_1} w(x)\right)\int_{U_2^c} j(|y-x|)dy
\end{aligned}$$

$$+ 2\|f\|_\infty \int_{U_1} \int_{U_2^c} w(y)j(|x-y|)dydx$$

is finite by (4.5) and the fact that J is a Lévy density. Thus by (4.6), Fubini's theorem and the dominated convergence theorem, we have for any $f \in C_c^2(\mathbb{R}^d_+)$,

$$\begin{aligned} 0 &= \lim_{\varepsilon \downarrow 0} \int_{\{(x,y)\in \mathbb{R}^d\times\mathbb{R}^d,\ |y-x|>\varepsilon\}} (w(y)-w(x))(f(y)-f(x))j(|y-x|)\,dx\,dy \\ &= -2\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d_+} f(x)\left(\int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}}(w(y)-w(x))j(|y-x|)dy\right)dx \;=\; -2\int_{\mathbb{R}^d_+} f(x)\mathcal{A}w(x)\,dx, \end{aligned}$$

where we have used the fact $\mathcal{A}_\varepsilon w \to \mathcal{A}w$ converges uniformly on the support of f. Hence, by the continuity of $\mathcal{A}w$, we have $\mathcal{A}w(x) = 0$ in $\mathbb{R}^d_+$. □

For $x \in \mathbb{R}^d$, let $\delta_{\partial D}(x)$ denote the Euclidean distance between x and ∂D. It is well known that any $C^{1,1}$ open set D with characteristics (R, Λ) satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition* with the radius r_1: there exists $r_1 < R$ such that for every $x \in D$ with $\delta_D(x) < r_1$ and $y \in \mathbb{R}^d \setminus \overline{D}$ with $\delta_D(y) < r_1$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_D(y)$ and that $B(x_0, r_1) \subset D$ and $B(y_0, r_1) \subset \mathbb{R}^d \setminus \overline{D}$ for $x_0 = z_x + r_1(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_1(y - z_y)/|y - z_y|$.

In the remainder of this section, we assume D is a $C^{1,1}$ open set with characteristics (R, Λ) and D satisfies the uniform interior ball condition and the uniform exterior ball condition with the radius R (by choosing R smaller if necessary).

Lemma 4.4 *Fix $Q \in \partial D$ and let*

$$h(y) := V\left(\delta_D(y)\right)\mathbf{1}_{D\cap B(Q,R)}(y).$$

There exist $C_{13} = C_{13}(\Lambda, R, \phi) > 0$ and $R_4 \le R/4$ independent of the point $Q \in \partial D$ such that $\mathcal{A}h$ is well defined in $D \cap B(Q, R_4)$ and

$$|\mathcal{A}h(x)| \le C_{13} \quad \text{for all } x \in D \cap B(Q, R_4)\,. \tag{4.7}$$

Proof. We first note that when $d = 1$, the lemma follows from Proposition 4.2 and Theorem 4.3. In fact, suppose that $d = 1$ and $x \in D \cap B(Q, R/2)$. Without loss of generality we may assume that Q is the origin and $D \cap B(Q, R) = (0, R)$ (due to uniform exterior ball condition). Since $h(y) = w(y)$ for $y \in D \cap B(Q, R) = (0, R)$, we have

$$\begin{aligned} \mathcal{A}(h-w)(x) &= \lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^1:|x-y|>\varepsilon\}}(h-w)(y)j(|y-x|)\,dy \\ &= -\lim_{\varepsilon\downarrow 0}\int_{\{y\ge R:|x-y|>\varepsilon\}} w(y)j(|y-x|)\,dy. \end{aligned}$$

Since $0 < x < R/2$ and $y \ge R$, we have $|x - y| > R/2$, and thus

$$|\mathcal{A}(h-w)(x)| \le \int_{\{y\ge R,|x-y|>R/2\}} w(y)j(|y-x|)\,dy.$$

Therefore, by using Theorem 4.3 (which gives $\mathcal{A}h(x) = \mathcal{A}(h-w)(x)$), Proposition 4.2 and the above display, we conclude that

$$|\mathcal{A}h(x)| = |\mathcal{A}(h-w)(x)| \le \sup_{0<z<R/2}\int_{\{y\in\mathbb{R}^1_+:|y-z|>R/2\}} w(y)j(|y-z|)dy < \infty.$$

Throughout the remainder of the proof, $d \geq 2$ and q is a fixed positive constant such that

$$0 < q < \frac{\alpha \wedge (2-\alpha)}{20} \quad \text{and} \quad \alpha + 2q - 1 \neq 0.$$

Since ℓ is slowly varying at ∞, by Potter's Theorem (Theorem 1.4), we can find a small $R_4 < 1 \wedge (R/4)$ such that for every $r \leq 2R_4^2$

$$\frac{\ell(r^{-2})}{(\ell((2R^{-1})^{-2} r^{-4}))^{1/2}} \leq 2 \frac{\ell((2R_4^2)^{-2})}{(\ell((2R^{-1})^{-2}(2R_4^2)^{-4}))^{1/2}} (2R_4^2)^{1/2} r^{-1/2} \leq c_1 r^{-1/2}, \tag{4.8}$$

$$\ell(r^{-1}) \leq 2\,\ell((2R_4^2)^{-1})(2R_4^2)^q r^{-q} \leq c_1 r^{-q}, \tag{4.9}$$

$$\ell(r^{-2})^{-1/2} \leq 2\,\ell((2R_4^2)^{-2})^{1/2}(2R_4^2)^q r^{-q} \leq c_1 r^{-q}. \tag{4.10}$$

In the remainder of this proof, we fix $x \in D \cap B(Q, R_4)$ and $x_0 \in \partial D$ satisfying $\delta_D(x) = |x - x_0|$. We also fix the $C^{1,1}$ function ψ and the coordinate system $CS = CS_{x_0}$ in the definition of $C^{1,1}$ open set so that $x = (0, x_d)$ with $0 < x_d < R_4$ and $B(x_0, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS : y_d > \psi(\widetilde{y})\}$. Let

$$\psi_1(\widetilde{y}) := R - \sqrt{R^2 - |\widetilde{y}|^2} \quad \text{and} \quad \psi_2(\widetilde{y}) := -R + \sqrt{R^2 - |\widetilde{y}|^2}.$$

Due to the uniform interior ball condition and the uniform exterior ball condition with the radius R, we have

$$\psi_2(\widetilde{y}) \leq \psi(\widetilde{y}) \leq \psi_1(\widetilde{y}) \quad \text{for every } y \in D \cap B(x, R_4). \tag{4.11}$$

Define $H^+ := \{y = (\widetilde{y}, y_d) \in CS : y_d > 0\}$ and let

$$A := \{y = (\widetilde{y}, y_d) \in (D \cup H^+) \cap B(x, R_4) : \psi_2(\widetilde{y}) \leq y_d \leq \psi_1(\widetilde{y})\},$$

$$E := \{y = (\widetilde{y}, y_d) \in B(x, R_4) : y_d > \psi_1(\widetilde{y})\}.$$

Note that, since $|y - Q| \leq |y - x| + |x - Q| \leq R/2$ for $y \in B(x, R_4)$, we have

$$B(x, R_4) \cap D \subset B(Q, R/2) \cap D\,. \tag{4.12}$$

Let

$$h_x(y) := V\left(\delta_{H^+}(y)\right).$$

Note that $h_x(x) = h(x)$. Moreover, since $\delta_{H^+}(y) = (y_d)^+$ in CS, by Theorem 4.3 it follows that $\mathcal{A}h_x$ is well defined in H^+ and

$$\mathcal{A}h_x(y) = 0, \quad \forall y \in H^+. \tag{4.13}$$

We show now that $\mathcal{A}(h - h_x)(x)$ is well defined. For each $\varepsilon > 0$ we have that

$$\begin{aligned}
&\left|\int_{\{y \in D \cup H^+ :\, |y-x| > \varepsilon\}} (h(y) - h_x(y)) j(|y-x|)\, dy\right| \\
\leq &\int_{B(x,R_4)^c} (h(y) + h_x(y)) j(|y-x|) dy + \int_A (h(y) + h_x(y)) j(|y-x|)\, dy + \int_E |h(y) - h_x(y)| j(|y-x|) dy \\
=: &I_1 + I_2 + I_3.
\end{aligned}$$

We claim that

$$I_1 + I_2 + I_3 \leq C_{13} \tag{4.14}$$

for some constant $C_{13} = C_{13}(\Lambda, R, \phi)$. This shows in particular that the limit

$$\lim_{\varepsilon \downarrow 0} \int_{\{y \in D \cup H^+ : |y-x| > \varepsilon\}} (h(y) - h_x(y)) j(|y-x|)\, dy$$

exists and hence $\mathcal{A}(h-h_x)(x)$ is well defined, and $|\mathcal{A}(h-h_x)(x)| \le C_{13}$. By linearity and (4.13), we get that $\mathcal{A}h(x)$ is well defined and $|\mathcal{A}h(x)| \le C_{13}$. Therefore, it remains to prove (4.14).

By the fact that $h(y)=0$ for $y \in B(Q,R)^c$,

$$I_1 \le \sup_{z\in\mathbb{R}^d:\ 0<z_d<R} \int_{B(z,R_4)^c\cap H^+} V(y_d) j(|z-y|)dy + c_3 \int_{B(0,R_4)^c} j(|y|)dy =: K_1 + K_2.$$

K_2 is clearly finite since J is the Lévy density of X and K_1 is finite by Proposition 4.2.

For $y \in A$, since V is increasing and $(R-\sqrt{R^2-|\widetilde{y}|^2}) \le R^{-1}|\widetilde{y}|^2$, we see that

$$h_x(y) + h(y) \le 2V(\psi_1(\widetilde{y}) - \psi_2(\widetilde{y})) \le 2V(2R^{-1}|\widetilde{y}|^2). \tag{4.15}$$

Using (4.15), (2.12) and Theorem 2.11, we have

$$\begin{aligned} I_2 \le& \int_0^{R_4} \int_{|\widetilde{y}|=r} \mathbf{1}_A(y)(h_x(y)+h(y)) j((r^2+|y_d-x_d|^2)^{1/2})\ m_{d-1}(dy)dr \\ \le& 2\int_0^{R_4} \int_{|\widetilde{y}|=r} \mathbf{1}_A(y) V(2R^{-1}r^2) j(r)\ m_{d-1}(dy)dr \\ \le& c_4 \int_0^{R_4} r^{-d} \frac{\ell(r^{-2})}{(\ell(2^{-2}R^2r^{-4}))^{1/2}} m_{d-1}(\{y\in A : |\widetilde{y}|=r\})dr \end{aligned} \tag{4.16}$$

where m_{d-1} is the surface measure, that is, the $(d-1)$-dimensional Lebesgue measure. Furthermore, since $|\psi_2(\widetilde{y}) - \psi_1(\widetilde{y})| \le 2R^{-1}|\widetilde{y}|^2 = 2R^{-1}r^2$ on $|\widetilde{y}| = r$, we have for $r \le R_4$, $m_{d-1}(\{y : |\widetilde{y}| = r, \psi_2(\widetilde{y}) < y_d < \psi_1(\widetilde{y})\}) \le c_5 r^d$ for some constant c_5. Using the above inequality and (4.8), from (4.16) we get

$$I_2 \le c_6 \int_0^{R_4} \frac{\ell(r^{-2})}{(\ell((2R^{-1})^{-2}r^{-4}))^{1/2}} dr \le c_7 \int_0^{R_4} r^{-1/2} dr < \infty.$$

For I_3, we consider two cases separately: If $0 < y_d = \delta_{H^+}(y) \le \delta_D(y)$, since v is decreasing,

$$h(y) - h_x(y) \le V(y_d + R^{-1}|\widetilde{y}|^2) - V(y_d) = \int_{y_d}^{y_d+R^{-1}|\widetilde{y}|^2} v(z)dz \le R^{-1}|\widetilde{y}|^2 v(y_d). \tag{4.17}$$

If $y_d = \delta_{H^+}(y) > \delta_D(y)$ and $y \in E$, using the fact that $\delta_D(y)$ is greater than or equal to the distance between y and the graph of ψ_1 and

$$y_d - R + \sqrt{|\widetilde{y}|^2 + (R-y_d)^2} = \frac{|\widetilde{y}|^2}{\sqrt{|\widetilde{y}|^2+(R-y_d)^2} + (R-y_d)} \le \frac{|\widetilde{y}|^2}{2(R-y_d)} \le \frac{|\widetilde{y}|^2}{R},$$

we have

$$h_x(y) - h(y) \le \int_{R-\sqrt{|\widetilde{y}|^2+(R-y_d)^2}}^{y_d} v(z)dz \le R^{-1}|\widetilde{y}|^2\, v(R - \sqrt{|\widetilde{y}|^2+(R-y_d)^2}). \tag{4.18}$$

By (4.17)-(4.18),

$$\begin{aligned} I_3 \le& R^{-1} \int_{E\cap\{y: y_d\le\delta_D(y)\}} |\widetilde{y}|^2 v(y_d) j(|x-y|)dy \\ &+ R^{-1} \int_{E\cap\{y: y_d>\delta_D(y)\}} |\widetilde{y}|^2 v(R-\sqrt{|\widetilde{y}|^2+(R-y_d)^2}) j(|x-y|)dy =: R^{-1}(L_1+L_2). \end{aligned}$$

Since $E \subset \{z = (\widetilde{z}, z_d) \in \mathbb{R}^d : |\widetilde{z}| < R_4 \text{ and } 0 < z_d \le 2R_4\}$, using polar coordinates for $\widetilde{y}$ and by Theorem 2.11, (2.12), (4.9) and (4.10), we have that

$$
\begin{aligned}
L_1 &\le c_8 \int_0^{2R_4} v(y_d) \left(\int_0^{R_4} r^2 j((r^2 + |y_d - x_d|^2)^{1/2}) r^{d-2} dr \right) dy_d \\
&\le c_9 \int_0^{2R_4} \frac{1}{(y_d)^{1-\alpha/2} (\ell(y_d^{-2}))^{1/2}} \left(\int_0^{R_4} \frac{r^d \ell((r^2 + |y_d - x_d|^2)^{-1})}{(r^2 + |y_d - x_d|^2)^{(d+\alpha)/2}} dr \right) dy_d \\
&\le c_{10} \int_0^{2R_4} \frac{1}{(y_d)^{1-\alpha/2+q}} \left(\int_0^{R_4} \frac{r^d}{(r^2 + |y_d - x_d|^2)^{(d+\alpha+2q)/2}} dr \right) dy_d \\
&\le c_{10} \int_0^{2R_4} \frac{1}{(y_d)^{1-\alpha/2+q}} \left(\int_0^{R_4} \frac{1}{(r + |y_d - x_d|)^{\alpha+2q}} dr \right) dy_d \\
&\le c_{11} \int_0^{2R_4} \frac{1}{(y_d)^{1-\alpha/2+q}} \left(\frac{1}{|y_d - x_d|^{\alpha+2q-1}} + \frac{1}{(R_4 + |y_d - x_d|)^{\alpha+2q-1}} \right) dy_d \le c_{12}
\end{aligned}
$$

for some constant $c_8, \dots, c_{12} > 0$. The last inequality is due to the fact that $q < (2-\alpha)/20$, which implies $(1 - \alpha/2) + \alpha + 3q - 1 < (6 + 7\alpha)/20 < 1$, so by the dominated convergence theorem,

$$
x_d \mapsto \int_0^{2R_4} \frac{1}{(y_d)^{1-\alpha/2+q} |y_d - x_d|^{\alpha+2q-1}} dy_d \tag{4.19}
$$

is a strictly positive continuous function in $x_d \in [0, R_4]$ and hence it is bounded.

On the other hand, we have, using polar coordinates for $\widetilde{y}$, and by Theorem 2.11, (2.12) and (4.9)–(4.10),

$$
\begin{aligned}
L_2 &\le c_{13} \int_0^{x_d+R_4} \left(\int_0^{R_4 \wedge \sqrt{2Ry_d - y_d^2}} v(R - \sqrt{r^2 + (R - y_d)^2}) r^d j((r^2 + |y_d - x_d|^2)^{1/2}) dr \right) dy_d \\
&\le c_{14} \int_0^{x_d+R_4} \left(\int_0^{R_4 \wedge \sqrt{2Ry_d - y_d^2}} \frac{(R - \sqrt{r^2 + (R - y_d)^2})^{\alpha/2-1} \ell((r^2 + |y_d - x_d|^2)^{-1})}{(\ell((R - \sqrt{r^2 + (R - y_d)^2})^{-2}))^{1/2} (r^2 + |y_d - x_d|^2)^{(d+\alpha)/2}} r^d dr \right) dy_d \\
&\le c_{15} \int_0^{x_d+R_4} \left(\int_0^{R_4 \wedge \sqrt{2Ry_d - y_d^2}} \frac{r^d}{(R - \sqrt{r^2 + (R - y_d)^2})^{1-\alpha/2+q} (r^2 + |y_d - x_d|^2)^{(d+\alpha+2q)/2}} dr \right) dy_d \\
&\le c_{16} \int_0^{x_d+R_4} \left(\int_0^{R_4 \wedge \sqrt{2Ry_d - y_d^2}} \frac{1}{(R - \sqrt{r^2 + (R - y_d)^2})^{1-\alpha/2+q} (r + |y_d - x_d|)^{\alpha+2q}} dr \right) dy_d .
\end{aligned}
$$

Since, for $0 < r < R_4 \wedge \sqrt{2Ry_d - y_d^2}$,

$$
\frac{1}{R - \sqrt{r^2 + (R - y_d)^2}} = \frac{R + \sqrt{r^2 + (R - y_d)^2}}{(\sqrt{2Ry_d - y_d^2} + r)(\sqrt{2Ry_d - y_d^2} - r)} \le \frac{c_{17}}{\sqrt{y_d}(\sqrt{2Ry_d - y_d^2} - r)},
$$

we have

$$
L_2 \le \int_0^{x_d+R_4} \frac{c_{18}}{(y_d)^{(1-\alpha/2+q)/2}} \int_0^{R_4 \wedge \sqrt{2Ry_d - y_d^2}} \frac{dr}{(\sqrt{2Ry_d - y_d^2} - r)^{1-\alpha/2+q} (r + |y_d - x_d|)^{\alpha+2q}} dy_d .
$$

Using the fact that $q \le \frac{\alpha}{20}$, we see that with $a := \sqrt{2Ry_d - y_d^2}$ and $b := |y_d - x_d|$,

$$
\int_0^{R_4 \wedge a} \frac{dr}{(a - r)^{1-\alpha/2+q} (r + b)^{\alpha+2q}}
$$

$$= \int_0^{(R_4\wedge a)/2} \frac{dr}{(a-r)^{1-\alpha/2+q}\,(r+b)^{\alpha+2q}} + \int_{(R_4\wedge a)/2}^{R_4\wedge a} \frac{dr}{(a-r)^{1-\alpha/2+q}\,(r+b)^{\alpha+2q}}$$
$$\le \frac{2^{1-\alpha/2+q}}{a^{1-\alpha/2+q}} \int_0^{(R_4\wedge a)/2} \frac{dr}{(r+b)^{\alpha+2q}} + \frac{1}{(b+(R_4\wedge a)/2)^{\alpha+2q}} \int_{(R_6\wedge a)/2}^{R_4\wedge a} \frac{dr}{(a-r)^{1-\alpha/2+q}}$$
$$\le \frac{c_{19}}{a^{1-\alpha/2+q} b^{(\alpha+2q-1)^+}} + \frac{c_{19}}{(R_6\wedge a)^{\alpha+2q}} a^{\alpha/2-q} \le \frac{c_{20}}{(y_d)^{(1-\alpha/2+q)/2}|x_d-y_d|^{(\alpha+2q-1)^+}} + \frac{c_{20}}{(y_d)^{(\alpha+6q)/4}}\,.$$

Thus we obtain

$$L_2 \,\le\, c_{21} \int_0^{2R_4} \frac{dy_d}{(y_d)^{(1-\alpha/2+q)}|y_d-x_d|^{(\alpha+2q-1)^+}} + c_{21}\int_0^{2R_4} \frac{dy_d}{(y_d)^{(1+4q)/2}}. \tag{4.20}$$

Since $q<1/10$, the second integral in (4.20) is bounded. And by the same argument as the one for (4.19), the first integral in (4.20) is also bounded. We have proved the claim (4.14). □

When $d\ge 2$, define $\rho_Q(x) := x_d - \psi_Q(\widetilde{x})$, where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_Q. Note that for every $Q\in\partial D$ and $x\in B(Q,R)\cap D$ we have

$$(1+\Lambda^2)^{-1/2}\,\rho_Q(x) \,\le\, \delta_D(x) \,\le\, \rho_Q(x). \tag{4.21}$$

For $a,b>0$, we define $D_Q(a,b) := \{y\in D : a>\rho_Q(y)>0,\ |\widetilde{y}|<b\}$ when $d\ge 2$. When $d=1$, we simply take $D_Q(a,b)=D_Q(a) := B(Q,a)\cap D$.

Lemma 4.5 *There are constants $R_5 = R_5(R,\Lambda,\phi)\in(0, R_4/(4\sqrt{1+(1+\Lambda)^2}))$ and $C_i = C_i(R,\Lambda,\phi)>0$, $i=14,15$, such that for every $r\le R_5$, $Q\in\partial D$ and $x\in D_Q(r,r)$,*

$$\mathbb{P}_x\left(X_{\tau_{D_Q(r,r)}} \in D\right) \ge C_{14} V(\delta_D(x)) \tag{4.22}$$

and

$$\mathbb{E}_x\left[\tau_{D_Q(r,r)}\right] \,\le\, C_{15} V(\delta_D(x)). \tag{4.23}$$

Proof. Without loss of generality, we assume $Q=0$. For $d\ge 2$, let $\psi=\psi_0:\mathbb{R}^{d-1}\to\mathbb{R}$ be the $C^{1,1}$ function and CS_0 be the coordinate system in the definition of $C^{1,1}$ open set so that $B(0,R)\cap D = \big\{(\widetilde{y},\, y_d)\in B(0,R) \text{ in } CS_0 : y_d > \psi(\widetilde{y})\big\}$. Let $\rho(y) := y_d - \psi(\widetilde{y})$ and $D(a,b) := D_0(a,b)$ for $d\ge 2$. When $d=1$, $D(a,b)$ is simply $B(0,a)\cap D$. The remainder of the proof is written for $d\ge 2$. The interpretation in the case $d=1$ is obvious.

Note that

$$|y|^2 = |\widetilde{y}|^2 + |y_d|^2 < r^2 + (|y_d-\psi(\widetilde{y})| + |\psi(\widetilde{y})|)^2 < (1+(1+\Lambda)^2)r^2 \quad \text{for every } y\in D(r,r)\,. \tag{4.24}$$

Hence, by letting $A := R_4/\sqrt{1+(1+\Lambda)^2}$, $D(r,s)\subset D(A,A)\subset B(0,R_4)\cap D\subset B(0,R)\cap D$ for every $r,s\le A$. Define

$$h(y) := V(\delta_D(y))\mathbf{1}_{B(0,R)\cap D}(y).$$

Let g be a non-negative smooth radial function with compact support such that $g(x)=0$ for $|x|>1$ and $\int_{\mathbb{R}^d} g(x)dx = 1$. For $k\ge 1$, define $g_k(x) = 2^{kd} g(2^k x)$ and

$$h^{(k)}(z) := (g_k * h)(z) := \int_{\mathbb{R}^d} g_k(y)h(z-y)dy\,,$$

and let $B_k := \big\{x\in D\cap B(0,R_4) : \delta_{D\cap B(0,R_4)}(x)\ge 2^{-k}\big\}$. Since $h^{(k)}$ is C^∞, $\mathcal{A}h^{(k)}$ is well defined everywhere. We claim that

$$-C_{13} \le \mathcal{A}h^{(k)} \le C_{13} \quad \text{on } B_k\,, \tag{4.25}$$

where C_{13} is the constant from Lemma 4.4. Indeed, for $x \in B_k$ and $z \in B(0, 2^{-k})$ it holds that $x - z \in D \cap B(0, R_4)$. Hence, by Lemma 4.4 the following limit exists:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} (h(y-z) - h(x-z))\, j(|x-y|)\, dy \\
&= \lim_{\varepsilon \to 0} \int_{|(x-z)-y'|>\varepsilon} (h(y') - h(x-z))\, j(|(x-z)-y'|)\, dy' \;=\; \mathcal{A}h(x-z)\,.
\end{aligned}$$

Moreover, by the same Lemma 4.4 it holds that $-C_{13} \le \mathcal{A}h(x-z) \le C_{13}$. Next,

$$\begin{aligned}
&\int_{|x-y|>\varepsilon} (h^{(k)}(y) - h^{(k)}(x))\, j(|x-y|)\, dy \\
&= \int_{|x-y|>\varepsilon} \left(\int_{\mathbb{R}^d} g_k(z)(h(y-z) - h(x-z))\, dz \right) j(|x-y|)\, dy \\
&= \int_{|z|<2^{-k}} g_k(z) \left(\int_{|x-y|>\varepsilon} (h(y-z) - h(x-z))\, j(|x-y|)\, dy \right) dz.
\end{aligned}$$

By letting $\varepsilon \to 0$ and using the dominated convergence theorem, it follows that

$$\mathcal{A}h^{(k)}(x) = \int_{|z|<2^{-k}} g_k(z)\mathcal{A}h(x-z)\, dz \le C_{13} \int_{|z|<2^{-k}} g_k(z)\, dz = C_{13}\,.$$

The left-hand side inequality in (4.25) is obtained in the same way.

Using the fact that $\mathcal{A}$ restricted to C_c^∞ coincides with the infinitesimal generator of the process X, we see that the following Dynkin formula is true; for $f \in C_c^\infty(\mathbb{R}^d)$ and any bounded open subset U of $\mathbb{R}^d$,

$$\mathbb{E}_x \int_0^{\tau_U} \mathcal{A}f(X_t)dt = \mathbb{E}_x[f(X_{\tau_U})] - f(x). \tag{4.26}$$

Let $U \subset D \cap B(0, R_4)$. By using (4.26) for $U \cap B_k$ and $h^{(k)}$, the estimates (4.25), the fact that $h^{(k)}$ are in $C_c^\infty(\mathbb{R}^d)$, and by letting $k \to \infty$ we get

$$h(x) \ge \mathbb{E}_x\left[h(X_{\tau_U})\right] - C_{13}\mathbb{E}_x[\tau_U] \quad \text{and} \quad h(x) \le \mathbb{E}_x\left[h(X_{\tau_U})\right] + C_{13}\mathbb{E}_x[\tau_U]. \tag{4.27}$$

Now, we have by (4.21) and (4.27), for every $\lambda \ge 1$ and $x \in D(\lambda^{-1}A, \lambda^{-1}A)$,

$$\begin{aligned}
&V(\delta_D(x)) \;=\; h(x) \\
&\ge \mathbb{E}_x\left[h\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}}\right); X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}} \in D(A, \lambda^{-1}A) \setminus D(\lambda^{-1}A, \lambda^{-1}A)\right] - C_{13}\mathbb{E}_x\left[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}\right] \\
&\ge V\big(\lambda^{-1}(1+\Lambda^2)^{-1/2}A\big)\mathbb{P}_x\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}} \in D(A, \lambda^{-1}A) \setminus D(\lambda^{-1}A, \lambda^{-1}A)\right) - C_{13}\mathbb{E}_x\left[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}\right].
\end{aligned} \tag{4.28}$$

We also have from (4.27)

$$\begin{aligned}
V(\delta_D(x)) = h(x) \;&\le\; \mathbb{E}_x\left[h\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}}\right)\right] + C_{13}\mathbb{E}_x[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}] \\
&\le\; V(R)\mathbb{P}_x\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}} \in D\right) + C_{13}\mathbb{E}_x[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}].
\end{aligned} \tag{4.29}$$

By (2.19) and the monotonicity of j, for every $\lambda \ge 4$ and $x \in D(\lambda^{-1}A, \lambda^{-1}A)$,

$$\mathbb{P}_x\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}} \in D\right) \;\ge\; \mathbb{P}_x\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}} \in D(3\lambda^{-1}A, \lambda^{-1}A) \setminus D(2\lambda^{-1}A, \lambda^{-1}A)\right)$$

$$
\begin{aligned}
&= \mathbb{E}_x\left[\int_0^{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}}\int_{D(3\lambda^{-1}A,\lambda^{-1}A)\setminus D(2\lambda^{-1}A,\lambda^{-1}A)} j(|X_s-y|)dyds\right]\\
&\geq \left(\int_{D(3\lambda^{-1}A,\lambda^{-1}A)\setminus D(2\lambda^{-1}A,\lambda^{-1}A)} dy\right) j(10\lambda^{-1}A)\,\mathbb{E}_x\left[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}\right]\\
&\geq c_1(\lambda^{-1}A)^d j(10\lambda^{-1}A)\,\mathbb{E}_x\left[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}\right].
\end{aligned}
$$

Now, applying Theorem 2.11, we get for $x\in D(\lambda^{-1}A,\lambda^{-1}A)$

$$
\mathbb{P}_x\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}}\in D\right)\geq c_2\ell\big((10\lambda^{-1}A)^{-2}\big)\lambda^\alpha\mathbb{E}_x\left[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}\right]. \tag{4.30}
$$

Thus from (4.28)–(4.30), for every $x\in D(\lambda^{-1}A,\lambda^{-1}A)$

$$
V(\delta_D(x))\geq\left(c_2V\big(\lambda^{-1}(1+\Lambda^2)^{-1/2}A\big)\ell\big((10\lambda^{-1}A)^{-2}\big)\lambda^\alpha-C_{13}\right)\mathbb{E}_x[\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}] \tag{4.31}
$$

and

$$
V(\delta_D(x))\leq c_3\left(1+\left(\ell\big((10\lambda^{-1}A)^{-2}\big)\right)^{-1}\lambda^{-\alpha}\right)\mathbb{P}_x\left(X_{\tau_{D(\lambda^{-1}A,\lambda^{-1}A)}}\in D\right). \tag{4.32}
$$

Using first (2.12) and then Potter's Theorem (Theorem 1.4), we see that there exists a large $\lambda_0>4$ such that for every $\lambda\geq\lambda_0$

$$
\begin{aligned}
&V\big(\lambda^{-1}(1+\Lambda^2)^{-1/2}A\big)\ell\big((10\lambda^{-1}A)^{-2}\big)\lambda^\alpha\\
&\geq c_4A^{\alpha/2}(1+\Lambda^2)^{-\alpha/4}\lambda^{\alpha/2}\left(\ell\Big(\big(\lambda^{-1}(1+\Lambda^2)^{-1/2}A\big)^{-2}\Big)\right)^{-1/2}\ell\big((10\lambda^{-1}A)^{-2}\big)\geq 2C_{13}/c_2
\end{aligned} \tag{4.33}
$$

and

$$
\left(\ell\big((10\lambda^{-1}A)^{-2}\big)\right)^{-1}\lambda^{-\alpha}\leq c_5. \tag{4.34}
$$

Combining (4.31)–(4.34), we have proved the lemma with $R_5:=\lambda_0^{-1}A$. □

It is clear that every $C^{1,1}$ open set is κ-fat, i.e., for any $C^{1,1}$ open set with $C^{1,1}$ characteristics (R,Λ), there exists a constant $\kappa\in(0,1/2]$, which depends only on (R,Λ), such that for each $Q\in\partial D$ and $r\in(0,R)$, $D\cap B(Q,r)$ contains a ball $B(A_r(Q),\kappa r)$ of radius κr. In the rest of this paper, whenever we deal with $C^{1,1}$ open sets, the constants Λ, R and κ will have the meaning described above.

Recall that g is defined in (3.10).

Theorem 4.6 *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R,Λ). Then there exists $C_{16}=C_{16}(R,\Lambda,\phi,diam(D))>0$ such that*

$$
C_{16}^{-1}\,(V(\delta_D(x))\wedge 1)\;\leq\;g(x)\;\leq\;C_{16}\,(V(\delta_D(x))\wedge 1)\,,\qquad x\in D. \tag{4.35}
$$

Proof. Since the case $d=1$ is simpler, we give the proof for $d\geq 2$ only. Recall that R_3 is the constant in (3.6) and $\varepsilon_1=R_3\kappa/24$. Since $g(x)=G_D(x,z_0)\wedge C_{12}$ and $g(x)=G_D(x,z_0)$ for $\delta_D(x)<6\varepsilon_1$, it suffices to show that there exist $r^*\in(0,6\varepsilon_1)$ and $c_1>1$ such that

$$
c_1^{-1}V(\delta_D(x))\;\leq\;G_D(x,z_0)\;\leq\;c_1\,V(\delta_D(x)),\qquad \delta_D(x)<r^*\,. \tag{4.36}
$$

Let $r^*:=(R_5/4)\wedge(\varepsilon_1/(4\sqrt{1+(1+\Lambda)^2}))$ and suppose that $\delta_D(x)<r^*$. Choose $x_0\in\partial D$ satisfying $\delta_D(x)=|x-x_0|$. We fix the $C^{1,1}$ function ψ and the coordinate system $CS=CS_{x_0}$ in the definition of $C^{1,1}$ open set so that $x=(\widetilde{0},x_d)$ with $0<x_d<r^*$,

$$
B(x_0,R)\cap D=\{y=(\widetilde{y},\,y_d)\in B(0,R)\text{ in }CS:y_d>\psi(\widetilde{y})\}.
$$

Let $x_1 := (\widetilde{0}, r^*/2)$ and $D_* := D(r^*, r^*) = \{y \in D : r^* > y_d - \psi(\widetilde{y}) > 0, |\widetilde{y}| < r^*\}$. Since $B(x_1, c_2 r^*) \subset D_*$ for small $c_2 > 0$, by Theorem 2.9, Theorem 2.15 and the fact that D is bounded,

$$\begin{aligned} G_D(x, z_0) &\le c_3 G_D(x_1, z_0) \frac{\mathbb{P}_x\left(X_{\tau_{D_*}} \in B(z_0, \varepsilon_1/4)\right)}{\mathbb{P}_{x_1}\left(X_{\tau_{D_*}} \in B(z_0, \varepsilon_1/4)\right)} \\ &\le c_3\, G(x_1, z_0) \frac{\mathbb{P}_x\left(X_{\tau_{D_*}} \in B(z_0, \varepsilon_1/4)\right)}{\mathbb{P}_{x_1}\left(X_{\tau_{B(x_1, c_2 r^*)}} \in B(z_0, \varepsilon_1/4)\right)} \\ &\le c_4\, \mathbb{P}_x\left(X_{\tau_{D_*}} \in B(z_0, \varepsilon_1/4)\right) \le c_5\, \mathbb{E}_x\left[\tau_{D_*}\right] \end{aligned}$$

where in the last inequality we used (2.19) and the fact that $\mathrm{dist}(D_*, B(z_0, \varepsilon_1/4)) \ge \delta_D(z_0) - \varepsilon_1/4 - \sqrt{1 + (1+\Lambda)^2} r^* \ge \varepsilon_1$ (see (4.24)). On the other hand, by Theorem 2.15, Lemma 3.3 and the fact that D is bounded,

$$G_D(x, z_0) \ge c_5 G_D(x_1, z_0) \frac{\mathbb{P}_x\left(X_{\tau_{D_*}} \in D\right)}{\mathbb{P}_{x_1}\left(X_{\tau_{D_*}} \in D\right)} \ge c_6 \mathbb{P}_x\left(X_{\tau_{D_*}} \in D\right).$$

By applying (4.22)–(4.23), we arrive at (4.35). □

Remark 4.7 *Using* (2.12) *and Theorem 2.9 one can easily see that the inequalities* (1.4), (1.3) *and* (1.6) *are equivalent to each other.*

Now we give the proof of Theorem 1.1, which is the main result of this paper.

Proof of Theorem 1.1. We will prove Theorem 1.1 by showing that (1.6) holds. It follows from Theorems 1.2 and 4.6, we have that

$$c_1^{-1} \frac{(V(\delta_D(x)) \wedge 1)(V(\delta_D(y)) \wedge 1)}{(V(\delta_D(A)) \wedge 1)^2 |x-y|^d \phi(|x-y|^{-2})} \le G_D(x, y) \le c_1 \frac{(V(\delta_D(x)) \wedge 1)(V(\delta_D(y)) \wedge 1)}{(V(\delta_D(A)) \wedge 1)^2 |x-y|^d \phi(|x-y|^{-2})}.$$

Observe that (3.7)–(3.9) imply that for every $A \in \mathcal{B}(x, y)$

$$\left(\frac{1}{6} \wedge \frac{\varepsilon_1}{R_3}\right) \delta_D(A) \le \delta_D(x) \vee \delta_D(y) \vee |x-y| \le 2\kappa^{-1} \left(\frac{M}{2R_3} \vee 1\right) \delta_D(A), \tag{4.37}$$

where $M := 6\, \mathrm{diam}(D)$. Combining this with Theorem 2.9 we see that, to complete the proof, it suffices to show that

$$\frac{(V(\delta_D(x)) \wedge 1)(V(\delta_D(y)) \wedge 1)}{(V(\delta_D(x) \vee \delta_D(y) \vee |x-y|) \wedge 1)^2} \asymp \left(1 \wedge \frac{V(\delta_D(x)) V(\delta_D(y))}{V(|x-y|)^2}\right). \tag{4.38}$$

Using the assumption that D is bounded, one can easily check that

$$\frac{(V(\delta_D(x)) \wedge 1)(V(\delta_D(y)) \wedge 1)}{(V(\delta_D(x) \vee \delta_D(y) \vee |x-y|) \wedge 1)^2} \asymp \frac{V(\delta_D(x)) V(\delta_D(y))}{V^2(\delta_D(x) \vee \delta_D(y) \vee |x-y|)}. \tag{4.39}$$

We claim that for any $a, b, c > 0$ with $|a - b| \le c$,

$$\frac{V(a)V(b)}{V^2(a \vee b \vee c)} \le 1 \wedge \frac{V(a)V(b)}{V^2(c)} \le 4 \frac{V(a)V(b)}{V^2(a \vee b \vee c)}. \tag{4.40}$$

The first inequality in (4.40) follows immediately from the fact that V is an increasing function. So we only need to check the second inequality. By symmetry, we assume $b \le a$. Also the case $b \le a \le c$ is obvious.

To deal with the other two cases, we first note that V is subadditive (see, for instance, [1, page 74]). In the case $b \le c \le a$, we have $c \le a \le b+c \le 2c$. Using the subadditivity of V we get $V(a) \le 2V(c)$, and hence

$$1 \wedge \frac{V(a)V(b)}{V^2(c)} \le 1 \wedge 4\frac{V(a)V(b)}{V^2(a)} = 1 \wedge 4\frac{V(b)}{V(a)} \le 4\frac{V(a)V(b)}{V^2(a \vee b \vee c)}.$$

In the case $c \le b \le a$, we have $b \le a \le b+c \le 2b$. Using the subadditivity of V we get $V(a) \le 2V(b)$, and hence

$$1 \wedge \frac{V(a)V(b)}{V^2(c)} = 1 \le 2\frac{V(b)}{V(a)} = 2\frac{V(a)V(b)}{V^2(a \vee b \vee c)}.$$

Thus (4.40) is valid.

Now applying (4.40) with $a = \delta_D(x)$, $b = \delta_D(y)$ and $c = |x-y|$, and then using (4.39) we arrive at (4.38). □

Now we give the proof of Theorem 1.3, which is a consequence of Theorems 1.1 and 2.15.

Proof of Theorem 1.3. Using the interior ball condition of D, the following holds: For every $Q \in \partial D$ and $r \le R$ there is a ball $B = B(z_Q^r, r)$ of radius r such that $B \subset D$ and $\partial B \cap \partial D = \{Q\}$. In addition, it follows from [29, Lemma 2.2] that, for each $Q \in \partial D$, we can choose a constant $c_1 = c_1(d,\Lambda) \in (0, 1/8]$ and a bounded $C^{1,1}$ open set U_Q with uniform characteristics (R_*, Λ_*) depending only on (R,Λ) and d such that $B(Q, c_1R) \cap D \subset U_Q \subset B(Q,R) \cap D$ and

$$\delta_D(y) = \delta_{U_Q}(y) \quad \text{for every } y \in B(Q, c_1R) \cap D. \tag{4.41}$$

Assume that $r \in (0, c_1R]$, $Q \in \partial D$ and u is nonnegative function in $\mathbb{R}^d$ harmonic in $D \cap B(Q,r) = U_Q \cap B(Q,r)$ with respect to X and vanishes continuously on $D^c \cap B(Q,r)$. Let $z_Q := z_Q^{c_2R}$. By [9, Lemma 4.2] and its proof, we see that u and $x \to G_{U_Q}(x, z_Q)$ are regular harmonic in $U_Q \cap B(Q, 2r/3)$ with respect to X. Since the $C^{1,1}$ characteristics of U_Q depend only on (R,Λ) and d, by the boundary Harnack principle (Theorem 2.15), there exist $r_1 = r_1(\phi, R, \Lambda) \in (0, 1/4]$ and $c_2 = c_2(\phi, R, \Lambda) > 0$ such that for any $r \in (0, r_1]$ we have

$$\frac{u(x)}{u(y)} \le c_2 \frac{G_{U_Q}(x, z_Q)}{G_{U_Q}(y, z_Q)} \quad \text{for every } x, y \in B(Q, r/2) \cap D.$$

Now applying Theorem 1.1 to $G_{U_Q}(x, z_Q)$ and $G_{U_Q}(y, z_Q)$, then using (4.41), we conclude that for $r \in (0, (c_1R \wedge r_1)]$

$$\frac{u(x)}{u(y)} \le c_3 \frac{(\phi((\delta_{U_Q}(y))^{-2}))^{-1/2}}{(\phi((\delta_{U_Q}(x))^{-2}))^{-1/2}} = c_3 \frac{(\phi((\delta_D(y))^{-2}))^{-1/2}}{(\phi((\delta_D(x))^{-2}))^{-1/2}} \quad \text{for every } x, y \in B(Q, r/2) \cap D$$

for some $c_3 = c_3(\phi, R, \Lambda) > 0$. The form (1.7) given in the statement of the theorem is equivalent to the one in the display above for $r \in (0, (c_1R \wedge r_1)]$. Now the case $r \in ((c_1R \wedge r_1), (R \wedge 1)/4]$ follows from the case $r \in (0, (c_1R \wedge r_1)]$ and Theorem 2.14. □

Acknowledgements. We thank the referee for very helpful comments on the first version of this paper, especially for suggesting the short proof of Theorem 1.1 in the current version.

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