ON POTENTIAL THEORY OF MARKOV PROCESSES WITH JUMP KERNELS DECAYING AT THE BOUNDARY

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ABSTRACT. Motivated by some recent potential theoretic results on subordinate killed Lévy processes in open subsets of the Euclidean space, we study processes in an open set $D \subset \mathbb{R}^d$ defined via Dirichlet forms with jump kernels of the form $J^D(x, y) = j(|x - y|)\mathcal{B}(x, y)$ and critical killing functions. Here $j(|x - y|)$ is the Lévy density of an isotropic stable process (or more generally, a pure jump isotropic unimodal Lévy process) in $\mathbb{R}^d$. The main novelty is that the term $\mathcal{B}(x, y)$ tends to 0 when x or y approach the boundary of D. Under some general assumptions on $\mathcal{B}(x, y)$, we construct the corresponding process and prove that non-negative harmonic functions of the process satisfy the Harnack inequality and Carleson's estimate. We give several examples of boundary terms satisfying those assumptions. The examples depend on four parameters, $\beta_1, \beta_2, \beta_3, \beta_4$, roughly governing the decay of the boundary term near the boundary of D.

In the second part of this paper, we specialize to the case of the half-space $D = \mathbb{R}^d_+ = \{x = (\widetilde{x}, x_d) : x_d > 0\}$, the α-stable kernel $j(|x - y|) = |x - y|^{-d-\alpha}$ and the killing function $\kappa(x) = cx_d^{-\alpha}$, $\alpha \in (0, 2)$, where c is a positive constant. Our main result in this part is a boundary Harnack principle which says that, for any $p > (\alpha - 1)_+$, there are values of the parameters $\beta_1, \beta_2, \beta_3, \beta_4$, and the constant c such that non-negative harmonic functions of the process must decay at the rate x_d^p if they vanish near a portion of the boundary. We further show that there are values of the parameters $\beta_1, \beta_2, \beta_3, \beta_4$, for which the boundary Harnack principle fails despite the fact that Carleson's estimate is valid.

AMS 2010 Mathematics Subject Classification: Primary 60J45; Secondary 60J50, 60J75.

Keywords and phrases: Jump processes, jumping kernel with boundary part, Harnack inequality, Carleson estimate, boundary Harnack principle.

1. INTRODUCTION

When studying discontinuous Markov processes, one of the common assumptions is that the jump kernel, which describes the intensity of jumps from one point in the state space to another, is a function of the distance (or is comparable to a function of the distance) between these two points. In the context of stable (or even stable-like) processes and Lévy processes in the Euclidean space such an assumption is quite natural. For example, the jump kernel of the isotropic α-stable process in $\mathbb{R}^d$ is given by $c|x - y|^{-d-\alpha}$, $d \geq 1$, $\alpha \in (0, 2)$. The part process of this process in an open subset $D \subset \mathbb{R}^d$, the so-called killed α-stable process in D, has the same jump kernel – the intensity of jumps again depends only on the distance between the points. By removing the killing part of the killed process one is led to the censored α-stable process with the state space $D \subset \mathbb{R}^d$. Still, the jump kernel remains the same.

We note that both the killed stable process and the censored stable process, although living in an open subset D of $\mathbb{R}^d$, are closely related to (and in some sense derived from) the stable process in $\mathbb{R}^d$. On the other hand, subordinate killed Brownian motions in $D \subset \mathbb{R}^d$, or more generally, subordinate killed Lévy processes, are processes which are intrinsically defined in D and are not a part of some larger processes living in $\mathbb{R}^d$. Potential theory of those processes was recently studied in [13, 14]. One of the key features of subordinate killed processes is that their jump kernels depend not only on the distance between the points, but also on the distance of each point to the boundary ∂D. This can be easily seen from the fact that the jump kernel of the subordinate killed process is equal to the integral of the transition density of the killed process (which depends also on the distance to the boundary) against the Lévy measure of the subordinator.

To be more precise, let us describe some of the results from [13, 14] in the case of stable processes and stable subordinators. Let D be a $C^{1,1}$ domain (connected open set) in $\mathbb{R}^d$. For $\delta \in (0, 2]$, let Z^D be a killed δ-stable process in D and $p^D(t, x, y)$ be its transition density. Further, let (T_t) be an independent (of Z^D) $\gamma/2$-stable subordinator, $\gamma \in (0, 2)$, and let $Y_t^D := Z_{T_t}^D$ be the subordinate process. Away from the boundary of D, the process Y^D behaves like the α-stable process with $\alpha = \delta\gamma/2$. On the other hand, the boundary behavior of the jump kernel $J^D(x, y)$ of Y^D, which is studied in [13, 14], exhibits some sort of phase transition. In case $\delta = 2$ (then Y^D is a subordinate killed Brownian motion), it holds that

$$J^D(x, y) \asymp \left(\frac{\delta_D(x)}{|x-y|} \wedge 1 \right) \left(\frac{\delta_D(y)}{|x-y|} \wedge 1 \right) |x-y|^{-d-\alpha}. \tag{1.1}$$

Here and below, $a \asymp b$ means that $c \leq b/a \leq c^{-1}$ for some $c \in (0, 1)$, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $\delta_D(x)$ denotes the distance between x and the boundary ∂D.

In case $\delta \in (0, 2)$ (then Y^D is a subordinate killed δ-stable process) the situation is more complicated and more interesting. It holds that (recall that $2\alpha = \delta\gamma$),

$$J^D(x, y) \asymp \begin{cases} \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|} \wedge 1 \right)^{\delta(1-\gamma/2)} |x-y|^{-d-\alpha}, & \gamma \in (1, 2), \\ \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|} \wedge 1 \right)^{\delta/2} \log \left(1 + \frac{(\delta_D(x)\vee\delta_D(y))\wedge|x-y|}{\delta_D(x)\wedge\delta_D(y)\wedge|x-y|} \right) |x-y|^{-d-\alpha}, & \gamma = 1, \\ \left(\frac{\delta_D(x)\wedge\delta_D(y)}{|x-y|} \wedge 1 \right)^{\delta/2} \left(\frac{\delta_D(x)\vee\delta_D(y)}{|x-y|} \wedge 1 \right)^{(\delta/2)(1-\gamma)} |x-y|^{-d-\alpha}, & \gamma \in (0, 1). \end{cases} \tag{1.2}$$

Define $\mathcal{B}(x, y) := J^D(x, y)|x-y|^{d+\alpha}$ so that $J^D(x, y) = \mathcal{B}(x, y)|x-y|^{-d-\alpha}$. Then we can think of $\mathcal{B}(x, y)$ as the boundary term of the jump kernel of Y^D which depends on $\delta_D(x), \delta_D(y)$ and $|x-y|$ (as opposed to the term $|x-y|^{-d-\alpha}$ which depends solely on $|x-y|$). Sharp two-sided estimates for $\mathcal{B}(x, y)$ can be seen from (1.1) and (1.2).

We note that the process Y^D is transient with its killing function κ^{Y^D} satisfying the lower bound $\kappa^{Y^D}(x) \geq c\delta_D(x)^{-\alpha}$, cf. [14, (2.4)]. Note that the killing function is not in the usual Kato class, i.e., $\lim_{t\to0} \sup_{x \in D} \mathbb{E}_x \int_0^t \kappa^{Y^D}(Y_s^D)ds$ is not zero.

One of the main goals of [13, 14] was to prove a (scale invariant) boundary Harnack principle (BHP) with exact decay rate for non-negative functions harmonic with respect to Y^D in case D is a $C^{1,1}$ domain. For the subordinate killed Brownian motion ($\delta = 2$) it was shown in [13, Theorem 1.2] that BHP holds with the decay rate $\delta_D(x)$. In case of the subordinate killed δ-stable process via an independent $\gamma/2$-stable subordinator ($\delta, \gamma \in (0, 2)$), the scale invariant BHP holds in case $\gamma \in (1, 2)$ with the decay rate $\delta_D(x)^{\delta/2}$, cf. [14, Theorem 7.2]. Surprisingly, it turns out that, despite the fact that Carleson's estimate is valid, the (non-scale invariant) BHP fails when $\gamma \in (0, 1]$, see [14, Section 9].

The goal of this paper is to study processes in open subsets of $\mathbb{R}^d$ associated with Dirichlet forms with jump kernels decaying at the boundary and critical killing functions, and to build

a general framework which includes the processes studied in [13, 14] as examples. To be more precise, let $X = (X_t, \mathbb{P}_x)$ be a pure-jump isotropic unimodal Lévy process whose jump kernel $j(x, y) = j(|x - y|)$ satisfies

$$j(r) \asymp r^{-d}\Phi(r)^{-1}, \quad \text{for all } r > 0, \tag{1.3}$$

where Φ is an increasing function satisfying the following weak scaling condition: There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1(R/r)^{2\delta_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq a_2(R/r)^{2\delta_2}, \quad 0 < r < R < \infty. \tag{1.4}$$

A prototype of such a process X is the isotropic α-stable process in which case $\Phi(r) = r^\alpha$. This particular case already contains all the essential features of our results.

For a given open set $D \subset \mathbb{R}^d$, we consider the process Y^D on D associated with a pure jump Dirichlet form whose jump kernel has the form $J^D(x, y) = \mathcal{B}(x, y)j(|x - y|)$. Here $\mathcal{B}(x, y)$ – the boundary term – depends on $\delta_D(x), \delta_D(y)$ and $|x - y|$, and is allowed to approach 0 at the boundary. This is in contrast with previous works (see, for instance, [6, 7]) where $\mathcal{B}(x, y)$ is assumed to be bounded between two positive constants, which can be viewed as a uniform ellipticity condition for non-local operators. In this sense our paper is the first systematic attempt to study the potential theory of degenerate non-local operators. Throughout this paper we will assume that $\mathcal{B}$ is symmetric and bounded, namely

(B1) $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in D$.

(B2) There exists a constant $C_2 > 0$ such that $\mathcal{B}(x, y) \leq C_2$ for all $x, y \in D$.

Under these two conditions, the jump kernel J^D gives rise to a regular Dirichlet form $(\mathcal{F}^D, \mathcal{E}^D)$ on $L^2(D, dx)$. Thus (see, for instance, [10, Example 1.2.4]) there exists a Hunt process Y^D associated with $(\mathcal{F}^D, \mathcal{E}^D)$. Such a process is defined up to an exceptional set $\mathcal{N}$. We next kill the process Y^D by using a killing function $\kappa : D \to [0, \infty)$ satisfying

$$\kappa(x) \leq C_1 \frac{1}{\Phi(\delta_D(x))}, \quad x \in D, \tag{1.5}$$

for some $C_1 > 0$, to obtain the Hunt process $Y^{D,\kappa}$. Once killed, the process is sent to the cemetery ∂. We will use the convention that every function is automatically extended to be zero at ∂. We further impose the following general conditions on the boundary term $\mathcal{B}(x, y)$. These conditions are satisfied by a quite a few classes of examples.

(B3) For every $a \in (0, 1)$ there exists $C_3 = C_3(a) > 0$ such that $\mathcal{B}(x, y) \geq C_3$ whenever $\delta_D(x) \wedge \delta_D(y) \geq a|x - y|$.

Condition **(B3)** ensures that, away from the boundary, the jumping kernel $J^D(x, y)$ is comparable to $j(|x - y|)$. In Subsection 3.1 we use this condition to remove the exceptional set $\mathcal{N}$ so that the process $Y^{D,\kappa}$ can start from every point in D. The next condition is needed in the analysis of the generator of the process in Subsection 3.2.

(B4) If $\delta_2 \geq 1/2$, then there exist $\theta > 2\delta_2 - 1$ and $C_4 > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \leq C_4 \left(\frac{|x - y|}{\delta_D(x) \wedge \delta_D(y)} \right)^\theta.$$

(B5) For every $\epsilon \in (0, 1)$ there exists $C_5 = C_5(\epsilon) \geq 1$ with the following property: For all $x_0 \in D$ and $r > 0$ with $B(x_0, (1 + \epsilon)r) \subset D$, we have

$$C_5^{-1}\mathcal{B}(x_1, z) \leq \mathcal{B}(x_2, z) \leq C_5\mathcal{B}(x_1, z), \quad \text{for all } x_1, x_2 \in B(x_0, r), \ z \in D \setminus B(x_0, (1 + \epsilon)r).$$

Roughly speaking, this condition says that jumping intensity from points close to each other to a faraway point is comparable. We note that **(B3)** implies that $\mathcal{B}(x,y) > 0$ for all $x, y \in D$.

Under these conditions we prove two versions of Harnack inequality for non-negative harmonic functions with respect to $Y^{D,\kappa}$. Recall that a non-negative Borel function f defined on D is said to be *harmonic* in an open set $V \subset D$ with respect to $Y^{D,\kappa}$ if for every bounded open set $U \subset \overline{U} \subset V$,

$$f(x) = \mathbb{E}_x\left[f(Y^{D,\kappa}_{\tau_U})\right] \text{ and } \mathbb{E}_x\left[f(Y^{D,\kappa}_{\tau_U})\right] < \infty, \qquad \text{for all } x \in U,$$

where $\tau_U = \tau_U^{Y^{D,\kappa}} := \inf\{t > 0 : Y^{D,\kappa}_t \notin U\}$ is the first exit time of $Y^{D,\kappa}$ from U.

Theorem 1.1 (Harnack inequality). *Suppose $D \subset \mathbb{R}^d$ is an open set and assume that **(B1)**-**(B5)** and (1.3)-(1.5) hold true.*

(a) *There exists a constant $C > 0$ such that for any $r \in (0,1]$, $B(x_0, r) \subset D$ and any non-negative function f in D which is harmonic in $B(x_0, r)$ with respect to $Y^{D,\kappa}$, we have*

$$f(x) \leq Cf(y), \qquad \text{for all } x, y \in B(x_0, r/2).$$

(b) *There exists a constant $C > 0$ such that for anyy $L > 0$, any $r \in (0,1]$, all $x_1, x_2 \in D$ with $|x_1 - x_2| < Lr$ and $B(x_1, r) \cup B(x_2, r) \subset D$ and any non-negative function f in D which is harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to $Y^{D,\kappa}$ we have*

$$f(x_2) \leq CC_3(1/(2(L+1)))^{-1}L^{d+2\delta_2}f(x_1).$$

We then proceed towards Carleson's estimate. Here we need two additional assumptions: one on the killing function κ, the other on the boundary term $\mathcal{B}$. In addition to (1.5) we assume the corresponding lower bound: There exists $C_8 > 0$ such that

$$\kappa(x) \geq C_8 \frac{1}{\Phi(\delta_D(x))}, \quad x \in D. \tag{1.6}$$

We note in passing that the killing function of a subordinate killed Brownian motion in a Lipschitz domain satisfies (1.5) and (1.6), cf. [15, Proposition 3.2]. For the boundary term we assume

(B6) There exist $\widehat{\beta} > 0$ and $C_6 > 0$ such that if $x, y, z \in D$ satisfy $\delta_D(x) \leq \delta_D(z)$ and $|y - z| \leq M|y - x|$ with $M \geq 1$, then $\mathcal{B}(x,y) \leq C_6 M^{\widehat{\beta}}\mathcal{B}(z,y)$.

An open set $D \subset \mathbb{R}^d$ is κ-fat if there are $\overline{\kappa} \in (0, 1/2]$ and $R > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, R]$, there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$. The pair $(R, \overline{\kappa})$ is called the characteristics of the κ-fat open set D.

Theorem 1.2 (Carleson estimate). *Suppose that $D \subset \mathbb{R}^d$ is a κ-fat open set with characteristics $(R, \overline{\kappa})$. Assume that **(B1)**-**(B6)** and (1.3)-(1.6) hold true. There exists a constant $C = C(R, \overline{\kappa}) > 0$ such that for every $Q \in \partial D$, $0 < r < (R \wedge 1)/2$, and every non-negative Borel function f in D which is harmonic in $D \cap B(Q, r)$ with respect to $Y^{D,\kappa}$ and vanishes continuously on $\partial D \cap B(Q, r)$, we have*

$$f(x) \leq Cf(x_0) \qquad \text{for all } x \in D \cap B(Q, r/2), \tag{1.7}$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) \geq \overline{\kappa}r/2$.

These results comprise the first part of the paper and are foundations for the second part in which we study in more detail the boundary behavior of non-negative harmonic functions in case $D = \mathbb{R}^d_+ = \{x = (\widetilde{x}, x_d) : x_d > 0\}$, $j(x, y) = |x - y|^{-d-\alpha}$ and $\kappa(x) = cx_d^{-\alpha}$ (where c is any positive constant). The main goal is to explore conditions on $\mathcal{B}$ which ensure that the BHP holds true and find the rate of decay of non-negative harmonic functions. Motivated by the three examples of the boundary term in Section 2 that satisfy conditions **(B1)**-**(B6)**,

in the second part we assume that the boundary term $\mathcal{B}(x,y)$ satisfies **(B1)**, **(B4)** and the following sharp two-sided estimates:

(B7) There exist $C_7 \geq 1$ and $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$ such that

$$C_7^{-1}\widetilde{B}(x,y) \leq \mathcal{B}(x,y) \leq C_7\widetilde{B}(x,y), \qquad x,y \in \mathbb{R}_+^d, \tag{1.8}$$

where $\widetilde{B}(x,y) = \widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x,y)$ is

$$\left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{\beta_2} \left[\log\left(1 + \frac{(x_d \vee y_d) \wedge |x-y|}{x_d \wedge y_d \wedge |x-y|}\right)\right]^{\beta_3} \left[\log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)\right]^{\beta_4}.$$

The first three terms in the definition of $\widetilde{B}(x,y)$ with the parameters β_1, β_2 and β_3 are motivated by (1.2), and the last term in the definition of $\widetilde{B}(x,y)$ with the parameter β_4 is motivated by [9] (see the paragraph above Lemma 2.2).

We show in Section 2 that the boundary term $\widetilde{B}(x,y)$ satisfies **(B1)**-**(B6)**. Thus $\mathcal{B}$ also satisfies **(B2)**, **(B3)**, **(B5)** and **(B6)**.

The following additional assumption on $\mathcal{B}(x,y)$ is natural for the state space $\mathbb{R}_+^d$.

(B8) For all $x,y \in \mathbb{R}_+^d$ $a > 0$, $\mathcal{B}(ax,ay) = \mathcal{B}(x,y)$. In case $d \geq 2$, for all $x,y \in \mathbb{R}_+^d$ and $\widetilde{z} \in \mathbb{R}^{d-1}$, $\mathcal{B}(x + (\widetilde{z},0), y + (\widetilde{z},0)) = \mathcal{B}(x,y)$.

We now describe the BHP we establish in more detail. Let $J^{\mathbb{R}_+^d}(x,y) = |x-y|^{-d-\alpha}\mathcal{B}(x,y)$, $x,y \in \mathbb{R}_+^d$, where the boundary term $\mathcal{B}(x,y)$ satisfies **(B1)**, **(B4)**, **(B7)** and **(B8)**. To every parameter $p \in ((\alpha-1)_+, \alpha+\beta_1)$ we associate a constant $C(\alpha, p, \mathcal{B}) \in (0,\infty)$ depending on α, p and $\mathcal{B}$ defined, in case $d \geq 2$, as

$$C(\alpha, p, \mathcal{B}) = \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\alpha)/2}} \left(\int_0^1 \frac{(s^p-1)(1-s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}((1-s)\widetilde{u}, 1), s\mathbf{e}_d) \, ds\right) d\widetilde{u} \tag{1.9}$$

where $\mathbf{e}_d = (\widetilde{0}, 1)$. In case $d = 1$, $C(\alpha, p, \mathcal{B})$ is defined as

$$C(\alpha, p, \mathcal{B}) = \int_0^1 \frac{(s^p-1)(1-s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}(1, s) \, ds.$$

Note that the function $p \mapsto C(\alpha, p, \mathcal{B})$ is non-decreasing, $\lim_{p \downarrow (\alpha-1)_+} C(\alpha, p, \mathcal{B}) = 0$ and that $\lim_{p \uparrow \alpha+\beta_1} C(\alpha, p, \mathcal{B}) = \infty$ (see Lemma 5.4 and Remark 5.5).

Let

$$\kappa(x) = C(p, \alpha, \mathcal{B})x_d^{-\alpha}, \qquad x \in \mathbb{R}^d, \tag{1.10}$$

be the killing function. Note that κ depends on p, but we omit this dependence from the notation for simplicity. We denote by $Y^{\mathbb{R}_+^d, \kappa}$ the corresponding process killed by κ.

We are now ready to state the BHP. We will only give the statement of the result for $d \geq 2$. The statement in the $d = 1$ case is similar and simpler. For any $a, b > 0$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, we define $D_{\widetilde{w}}(a,b) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x} - \widetilde{w}| < a, 0 < x_d < b\}$.

Theorem 1.3. *Assume that **(B1)**, **(B4)**, **(B7)**-**(B8)** and (1.10) hold true. Suppose that $p \in ((\alpha-1)_+, \alpha+\beta_1)$ and either (a) $\beta_1 = \beta_2 = \beta \geq 0$ and $\beta_3 = \beta_4 = 0$, or (b) $p < \alpha$. Then there exists $C > 0$ such that for every $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, any non-negative function f in $\mathbb{R}_+^d$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to $Y^{\mathbb{R}_+^d, \kappa}$ and vanishes continuously on $B(\widetilde{w}, 2r) \cap \partial\mathbb{R}_+^d$, we have*

$$\frac{f(x)}{x_d^p} \leq C\frac{f(y)}{y_d^p}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2). \tag{1.11}$$

Note that the power p in the decay rate comes from the constant $C(\alpha, p, \mathcal{B})$ in the killing function. In [3], a BHP is established for censored α-stable processes, $\alpha \in (1, 2)$, in smooth domains. The censored process has no killing, $\widetilde{B}(x, y) \equiv 1$, and the explicit decay rate p in the BHP of [3] is $\alpha - 1$.

In case $\beta_1 = \beta_2 = 1$, $\beta_3 = \beta_4 = 0$ and $p = 1$, this result covers the BHP (for $D = \mathbb{R}_+^d$) from [13]. In case $\frac{\alpha}{2} < p < \alpha$, $\beta_1 = 2p - \alpha$ and $\beta_2 = \beta_3 = \beta_4 = 0$ it covers the BHP from [14]. The main novelty of Theorem 1.3 comes from part (a) where we can allow arbitrarily large β, hence arbitrarily large p, implying very fast decay rate of harmonic functions. For fixed β, this decay rate is a consequence of the constant $C(\alpha, p, \mathcal{B})$ in the killing term κ. By increasing β we enlarge the possible range for parameter $p \in ((\alpha - 1)_+, \alpha + \beta)$. Note that when $\mathcal{B}(x, y) \equiv 1$ (corresponding to the case $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$), the possible range for the parameter p is $((\alpha - 1)_+, \alpha)$. This says that the power p in the decay rate of non-negative harmonic functions of $-(-\Delta)^{\alpha/2}$ with critical killing can not be higher than α (see also [8]). This is in sharp contrast with the case of Laplacian with critical killing, where the power p can be arbitrarily large. Indeed, for $f(x) = x_d^p$, it clearly holds that $\Delta f(x) - p(p-1)x_d^{-2}f(x) = 0$ for all $p \in (1, \infty)$. The second novelty of Theorem 1.3 is that in case $p < \alpha$ we can allow $\beta_2 > 0$ as well as logarithmic terms in $\mathcal{B}(x, y)$.

On the other hand, similarly as in [14], for certain values of the parameters β_1, β_2, β_3 and β_4, we can show that BHP fails. We say that the non-scale-invariant BHP holds near the boundary of $\mathbb{R}_+^d$ if there is a constant $\widehat{R} \in (0, 1)$ such that for any $r \in (0, \widehat{R}]$, there exists a constant $c = c(r) \geq 1$ such that for all $Q \in \partial \mathbb{R}_+^d$ and non-negative functions f, g in $\mathbb{R}_+^d$ which are harmonic in $\mathbb{R}_+^d \cap B(Q, r)$ with respect to $Y^{\mathbb{R}_+^d, \kappa}$ and vanish continuously on $\partial \mathbb{R}_+^d \cap B(Q, r)$, we have

$$\frac{f(x)}{f(y)} \leq c \frac{g(x)}{g(y)} \qquad \text{for all } x, y \in (\mathbb{R}_+^d \cap B(Q, r/2)).$$

Theorem 1.4. *Assume that **(B1)**, **(B4)**, **(B7)**-**(B8)** and* (1.10) *hold true. Suppose that* $\alpha + \beta_2 < p < \alpha + \beta_1$. *Then the non-scale-invariant boundary Harnack principle is not valid for* $Y^{\mathbb{R}_+^d, \kappa}$.

Theorems 1.2 and 1.4 imply that, when $\alpha + \beta_2 < p < \alpha + \beta_1$, under the assumptions **(B1)**, **(B4)**, **(B7)**-**(B8)** and (1.10), the Carleson estimate holds but the BHP fails. This is quite interesting since it is known that, for diffusions, the Carleson estimate and BHP are equivalent. See [1] and the references therein.

Now we describe the structure of the paper. We start with a section on examples of the boundary term and check that they satisfy all of the conditions **(B1)**-**(B8)**. The examples are motivated by and are extensions of the boundary terms of the processes studied in [13, 14]. This section is not essential for understanding of the main development later in the paper and the reader may wish to only glance through it first.

Section 3 is on the interior results and uses only assumptions **(B1)**-**(B5)** and (1.3)-(1.5). It consists of three subsections. In the first one we remove the exceptional set $\mathcal{N}$. The main step towards this is the identification of the process $Y^{D, \kappa}$ killed upon exiting a relatively compact $C^{1,1}$-open subset U of D with a certain process which can start from every point of U. In Subsection 3.2 we analyze the generator $L^{\mathcal{B}}$ of the process $Y^{D, \kappa}$, cf. (3.6). The difficulty comes from the boundary term which destroys the symmetry at the singularity. Assumption **(B4)** plays a central role in getting around this difficulty. An important result of this subsection is Proposition 3.7 which gives a Dynkin formula for smooth functions of compact support. Note that our process may not be a Feller process. In Subsection 3.3 we prove Theorem 1.1 by following the approach in [2, 16].

In Section 4 we add assumption **(B6)** and the lower bound (1.6) on the killing function κ, and prove Theorem 1.2. The proof follows the method already used before in different contexts. This section concludes the first part of the paper.

In the second part of this paper, we specialize to the case of the half-space $D = \mathbb{R}^d_+ = \{x = (\widetilde{x}, x_d) : x_d > 0\}$, the α-stable kernel $j(|x - y|) = |x - y|^{-d-\alpha}$ and the killing function $\kappa(x) = cx_d^{-\alpha}$, $\alpha \in (0, 2)$. We assume that **(B1)**, **(B2)**, **(B4)**, **(B7)** and **(B8)** hold true. The main purpose of Section 5 is to establish estimates related to exit probabilities from small boxes near the boundary. To accomplish this, we first show that the action of the operator $L^{\mathcal{B}}$ on the function $x \mapsto x_d^p$ is equal to 0. This indicates that if the BHP holds, the correct decay rate has to be x_d^p. Then we establish the following two-sided estimates:

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} \, dt \asymp x_d^p, \quad x \in U, \tag{1.12}$$

where $U := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d_+ : |\widetilde{x}| < \frac{1}{2}, x_d < \frac{1}{2}\}$ and $Y = (Y^1, \ldots, Y^d)$ stands for $Y^{\mathbb{R}^d_+, \kappa}$. One direction is relatively easy, while the other requires constructing suitable test functions and estimating the action of the operator $L^{\mathcal{B}}$ on them, cf. Lemma 5.8. This turns out to be a formidable task, and therefore we postpone the proof of this lemma to Section 8. Another important ingredient in the proof of (1.12) as well as some other results from Section 5 are some extensions of Proposition 3.7 to functions which are neither smooth nor of compact support. In order not to interrupt the flow of the presentation, we relegate the rigorous proofs of the extensions to Section 9. Estimates (1.12) are used in a rather straightforward way to establish the already mentioned exit probability estimates from small boxes, cf. Lemma 5.10. These exit probability estimates together with (1.12) are the main tools in proving the BHP in Section 6. Additionally, Section 5 contains the following two-sided estimates when $\alpha < p < \alpha + \beta_1$ and $\alpha + \beta_2 < p$:

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_2} \, dt \asymp x_d^{\alpha+\beta_2}, \quad x \in U. \tag{1.13}$$

These will be used in proving the failure of the BHP for the above range of parameters in Section 7.

Sections 6 and 7 contain proofs of Theorem 1.1 and Theorem 1.4, respectively. These are modeled after the corresponding proofs in [14] and use the box method (first adapted to the case jump processes in [4]), the Harnack inequality and Carleson's estimate.

Throughout this paper, the positive constants β_1, β_2, β_3, β_4, θ, δ_1, δ_2, δ_*, $\widehat{\beta}$ will remain the same. We will use the following convention: Capital letters $C, C_i, i = 1, 2, \ldots$ will denote constants in the statements of results and assumptions. The labeling of these constants will remain the same. Lower case letters $c, c_i, i = 1, 2, \ldots$ are used to denote the constants in the proofs and the labeling of these constants starts anew in each proof. The notation $c_i = c_i(a, b, c, \ldots)$, $i = 0, 1, 2, \ldots$ indicates constants depending on $a, b, c, \ldots$. We will use ":=" to denote a definition, which is read as "is defined to be". For any $x \in \mathbb{R}^d$, $r > 0$ and $0 < r_1 < r_2$, we use $B(x, r)$ to denote the open ball of radius r centered at x and use $A(x, r_1, r_2)$ to denote the annulus $\{y \in \mathbb{R}^d : r_1 \leq |y - x| < r_2\}$. For a Borel subset V in $\mathbb{R}^d$, $|V|$ denotes the Lebesgue measure of V in $\mathbb{R}^d$, $\delta_U := \text{dist}(U, \partial D)$ and $d_U := \text{diam}(U)$. For a function space $\mathbb{H}(U)$ on an open set U in $\mathbb{R}^d$, we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f \text{ has compact support}\}$ and $\mathbb{H}_b(U) := \{f \in \mathbb{H}(U) : f \text{ is bounded}\}$. We use the shorthand notation $\log(1 + a)^\gamma$ for the more precise $(\log(1 + a))^\gamma$.

For the convenience of the reader we give below the table of important constants and operators used throughout the paper.

Notation	Description
δ_1, δ_2	Indices of weak scaling condition of Φ in (1.4)
$\beta_1, \beta_2, \beta_3, \beta_4$	Parameters in the definition of $\widetilde{B}(x,y)$ in **(B7)**
$C(\alpha, p, \mathcal{B})$	The constant in the killing function κ defined in (1.9)
$L^{\mathcal{B}}$	The extended infinitesimal generator of the process $Y^{D,\kappa}$ defined in (3.6)
$L_\alpha^{\mathcal{B}}$	The extended infinitesimal generator of the process $Y^{\mathbb{R}_+^d, 0}$ defined in (5.7).

TABLE 1.   List of of the important notations

2. Examples of the boundary term

In this section we give examples of the boundary term $\mathcal{B}(x,y)$ that satisfy assumptions **(B1)-(B8)**. Our motivation comes from the boundary terms appearing in (1.2).

We start with three elementary lemmas. We recall that a function $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$, is said to be almost increasing if there exists $c > 0$ such that $f(x) \le cf(y)$ whenever $x, y \in I$ and $x < y$. An almost decreasing function is defined analogously.

Let $\beta_1, \beta_2, \beta_3, \beta_4 \ge 0$ so that $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$. For $s, t, u > 0$, define

$$\mathcal{A}(s,t,u) := \left(\frac{s \wedge t}{u} \wedge 1 \right)^{\beta_1} \left(\frac{s \vee t}{u} \wedge 1 \right)^{\beta_2} \log \left(1 + \frac{(s \vee t) \wedge u}{s \wedge t \wedge u} \right)^{\beta_3} \log \left(1 + \frac{u}{(s \vee t) \wedge u} \right)^{\beta_4}.$$

Lemma 2.1.   (a) *The function $u \mapsto \mathcal{A}(s,t,u)$ is almost decreasing on $(0,\infty)$.*
(b) *The function $s \mapsto \mathcal{A}(s,t,u)$ is almost increasing on $(0,\infty)$.*

Proof. In the proof we will use the following fact (which can be proved by elementary calculus) several times:

$$\text{For } \beta > 0 \text{ and } \gamma \ge 0, \ s \mapsto s^\beta \log \left(1 + s^{-1} \right)^\gamma \text{ is almost increasing on } (0,1]. \qquad (2.1)$$

(a) Note that $\mathcal{A}$ is symmetric in s and t, hence we may assume that $s \ge t$. Note that

$$\mathcal{A}(s,t,u) = \begin{cases} (\log 2)^{\beta_3 + \beta_4}, & \text{if } s \ge t \ge u; \\ \left(\frac{t}{u} \right)^{\beta_1} \log \left(1 + \frac{u}{t} \right)^{\beta_3} (\log 2)^{\beta_4}, & \text{if } s \ge u \ge t; \\ \left(\frac{t}{u} \right)^{\beta_1} \left(\frac{s}{u} \right)^{\beta_2} \log \left(1 + \frac{u}{t} \right)^{\beta_3} \log \left(1 + \frac{u}{s} \right)^{\beta_4}, & \text{if } u \ge s \ge t. \end{cases}$$

Thus, by (2.1), we see that $u \mapsto \mathcal{A}(s,t,u)$ is almost decreasing on $(0,\infty)$.
(b) Here we distinguish two cases:
Case (i): $t \ge u$. Then

$$\mathcal{A}(s,t,u) = \begin{cases} \left(\frac{s}{u} \right)^{\beta_1} \log \left(1 + \frac{u}{s} \right)^{\beta_3} (\log 2)^{\beta_4}, & \text{if } s \le u \le t; \\ (\log 2)^{\beta_3 + \beta_4}, & \text{if } u \le s \le t \text{ or } u \le t \le s. \end{cases}$$

It follows from (2.1) that $s \mapsto \mathcal{A}(s,t,u)$ is almost increasing.
Case (ii): $t \le u$. Then

$$\mathcal{A}(s,t,u) = \begin{cases} \left(\frac{s}{t} \right)^{\beta_1} \left(\frac{t}{u} \right)^{\beta_1 + \beta_2} \log \left(1 + \frac{t}{s} \right)^{\beta_3} \log \left(1 + \frac{u}{t} \right)^{\beta_4}, & \text{if } s \le t \le u; \\ \left(\frac{t}{u} \right)^{\beta_1} \left(\frac{s}{u} \right)^{\beta_2} \log \left(1 + \frac{s}{t} \right)^{\beta_3} \log \left(1 + \frac{u}{s} \right)^{\beta_4}, & \text{if } t \le s \le u; \\ \left(\frac{t}{u} \right)^{\beta_1} \log \left(1 + \frac{u}{t} \right)^{\beta_3} (\log 2)^{\beta_4}, & \text{if } t \le u \le s. \end{cases}$$

By (2.1), the function $s \mapsto \mathcal{A}(s,t,u)$ is almost increasing on $(0,t]$ and almost increasing on $[t,u]$. It is clearly constant on $[u,\infty)$. Thus, $s \mapsto \mathcal{A}(s,t,u)$ is almost increasing on $(0,\infty)$. $\square$

Let $D \subset \mathbb{R}^d$ be an open set, and recall that $\delta_D(x)$ denotes the distance between $x \in D$ and the boundary ∂D. For simplicity, let

$$L(x, y) := \log \left(1 + \frac{(\delta_D(x) \vee \delta_D(y)) \wedge |x - y|}{\delta_D(x) \wedge \delta_D(y) \wedge |x - y|} \right)$$

and

$$K(x, y) = \log \left(1 + \frac{|x - y|}{(\delta_D(x) \vee \delta_D(y)) \wedge |x - y|} \right).$$

Let $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ so that $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$. We define the function $\widetilde{B} : D \times D \to (0, \infty)$ by

$$\widetilde{B}(x, y) := \left(\frac{\delta_D(x) \wedge \delta_D(y)}{|x - y|} \wedge 1 \right)^{\beta_1} \left(\frac{\delta_D(x) \vee \delta_D(y)}{|x - y|} \wedge 1 \right)^{\beta_2} L(x, y)^{\beta_3} K(x, y)^{\beta_4}. \tag{2.2}$$

Observe that $\lim_{y \to x} \widetilde{B}(x, y) = (\log 2)^{\beta_3 + \beta_4}$, so for $y = x$ we interpret the right-hand side above as $\widetilde{B}(x, x) = (\log 2)^{\beta_3 + \beta_4}$. Note that $\widetilde{B}(x, y) = \mathcal{A}(\delta_D(x), \delta_D(y), |x - y|)$.

Note that when $\beta_1 = \beta_2 = \beta$ and $\beta_3 = \beta_4 = 0$, we have

$$\widetilde{B}(x, y) = \left(\frac{\delta_D(x)}{|x - y|} \wedge 1 \right)^{\beta} \left(\frac{\delta_D(y)}{|x - y|} \wedge 1 \right)^{\beta}, \tag{2.3}$$

which is, in case $\beta = 1$, comparable to the boundary term in (1.1). When $\beta_1 = \delta(1 - \gamma/2)$, $\beta_2 = 0$ and $\beta_3 = \beta_4 = 0$, $\widetilde{B}(x, y)$ is comparable to the boundary term in the first line of (1.2). When $\beta_1 = \delta/2$, $\beta_2 = 0$, $\beta_3 = 1$ and $\beta_4 = 0$, $\widetilde{B}(x, y)$ is comparable to the boundary term in the second line of (1.2). If $\beta_1 = \delta/2$, $\beta_2 = (\delta/2)(1 - \gamma)$ $(0 < \gamma < 1)$ and $\beta_3 = \beta_4 = 0$, then $\widetilde{B}(x, y)$ is comparable to the boundary term in the third line of (1.2).

Finally, we outline an example where $\beta_4 = 1$. For more details we refer the reader to [9, Example 7.3] (where different notation is used). Let $D = \{x \in \mathbb{R}^d : x_d > 0\}$ be the upper half space in $\mathbb{R}^d$, $\delta \in (0, 2)$ and $q \in [\delta - 1, \delta) \cap (0, \delta)$. The underlying process Z^D corresponds to a Feynman-Kac semigroup of the part process in D of the reflected δ-stable process in $\overline{D}$ via a multiplicative functional involving parameter q. The process Z^D is subordinated by an independent $\gamma/2$ subordinator, $\delta\gamma = 2\alpha$, to obtain the process Y^D. In the case $q = \frac{1}{2}\delta(1 - \frac{\gamma}{2})(= \frac{1}{2}(\delta - \alpha))$, the jump kernel $J(x, y)$ of Y^D is comparable to $|x - y|^{-\alpha - d}\widetilde{B}(x, y)$ where in $\widetilde{B}(x, y)$ we have $\beta_1 = \beta_2 = q$, $\beta_3 = 0$ and $\beta_4 = 1$.

Lemma 2.2. *The boundary term $\widetilde{B}(x, y)$ satisfies Assumptions (B1)-(B6).*

Proof. Assumptions **(B1)** and **(B2)** are clear.
(B3) Let $a \in (0, 1)$ and assume that $\delta_D(x) \wedge \delta_D(y) > a|x - y|$. Then the quantities in the first two parentheses in $\widetilde{B}(x, y)$ are larger than a, and those in $L(x, y)$ and $K(x, y)$ are equal to $\log 2$. Therefore, $\widetilde{B}(x, y) \geq (\log 2)^{\beta_3 + \beta_4} a^{\beta_1 + \beta_2} =: C_3(a)$.
(B4) Suppose $\delta_D(x) \wedge \delta_D(y) \geq |x - y|$. Then clearly $\widetilde{B}(x, y) = (\log 2)^{\beta_3 + \beta_4}$, so $(\log 2)^{\beta_3 + \beta_4} - \widetilde{B}(x, y) = 0$, and the assumption holds with any $\theta > 0$, in particular, we can take $\theta = 1$.
(B5) Let $\varepsilon \in (0, 1)$, $x_0 \in D$, and $r > 0$ so that $B(x_0, (1 + \varepsilon)r) \subset D$, and let $x_1, x_2 \in B(x_0, r)$, $z \in D \setminus B(x_0, (1 + \varepsilon)r)$. It is easy to see that

$$\frac{1}{3}\delta_D(x_2) \leq \left(1 + \frac{2}{1 + \varepsilon} \right)^{-1} \delta_D(x_2) \leq \delta_D(x_1) \leq \left(1 + \frac{2}{1 + \varepsilon} \right) \delta_D(x_2) \leq 3\delta_D(x_2),$$

and similarly $3^{-1}|x_2 - z| \leq |x_1 - z| \leq 3|x_2 - z|$. Now it is straightforward to obtain that

$$\left(\frac{\delta_D(x_1) \wedge \delta_D(z)}{|x_1 - z|} \wedge 1 \right) \leq 9 \left(\frac{\delta_D(x_2) \wedge \delta_D(z)}{|x_2 - z|} \wedge 1 \right)$$

and the analogous estimate with the minimum replaced by the maximum, $L(x_1, z) \leq 9L(x_2, z)$ and $K(x_1, z) \leq 9K(x_2, z)$. Hence $\widetilde{B}(x_1, z) \leq 9^{\beta_1 + \beta_2 + \beta_3 + \beta_4} \widetilde{B}(x_2, z)$.

(B6) Let $M \geq 1$ and let $x, y, z \in D$ satisfy $\delta_D(x) \leq \delta_D(z)$ and $|y - z| \leq M|y - x|$. By Lemma 2.1, there is a constant $c = c(\beta_1, \beta_2, \beta_3, \beta_4) > 0$ such that

$$\widetilde{B}(x, y) = \mathcal{A}(\delta_D(x), \delta_D(y), |x - y|) \leq c\mathcal{A}(\delta_D(z), \delta_D(y), M^{-1}|z - y|)$$

$$= c\Big(\frac{\delta_D(z) \wedge \delta_D(y)}{M^{-1}|z - y|} \wedge 1\Big)^{\beta_1} \Big(\frac{\delta_D(z) \vee \delta_D(y)}{M^{-1}|z - y|} \wedge 1\Big)^{\beta_2} \log\Big(1 + \frac{(\delta_D(z) \vee \delta_D(y)) \wedge (M^{-1}|z - y|)}{\delta_D(z) \wedge \delta_D(y) \wedge (M^{-1}|z - y|)}\Big)^{\beta_3}$$

$$\times \log\Big(1 + \frac{M^{-1}|z - y|)}{(\delta_D(z) \vee \delta_D(y)) \wedge M^{-1}|z - y|}\Big)^{\beta_4}$$

$$\leq cM^{\beta_1 + \beta_2}\Big(\frac{\delta_D(z) \wedge \delta_D(y)}{|z - y|} \wedge 1\Big)^{\beta_1} \Big(\frac{\delta_D(z) \vee \delta_D(y)}{|z - y|} \wedge 1\Big)^{\beta_2} \log\Big(1 + \frac{(\delta_D(z) \vee \delta_D(y)) \wedge (|z - y|)}{\delta_D(z) \wedge \delta_D(y) \wedge (|z - y|)}\Big)^{\beta_3}$$

$$\times \log\Big(1 + \frac{|z - y|)}{(\delta_D(z) \vee \delta_D(y)) \wedge |z - y|}\Big)^{\beta_4}$$

$$= cM^{\beta_1 + \beta_2}\widetilde{B}(z, y).$$

The second inequality above follows from the fact that, when $s \geq u$, $t \to \frac{s \wedge t}{u \wedge t}$ and $t \mapsto \frac{t}{s \wedge t}$ are increasing on $(0, \infty)$. $\qquad\square$

Let

$$\widehat{B}(x, y) := \Big(\frac{(\delta_D(x) \wedge \delta_D(y))^{\beta_1}(\delta_D(x) \vee \delta_D(y))^{\beta_2}}{|x - y|^{\beta_1 + \beta_2}} \wedge 1\Big) L(x, y)^{\beta_3} K(x, y)^{\beta_4}. \qquad (2.4)$$

The next result shows that $\widetilde{B}$ and $\widehat{B}$ are comparable.

Lemma 2.3. *For all* $x, y \in D$, $\widehat{B}(x, y) \geq \widetilde{B}(x, y) \geq 2^{-(\beta_1 \vee \beta_2)}\widehat{B}(x, y)$.

Proof. The first inequality is trivial, so we only prove the second. Without loss of generality, we assume that $\delta_D(x) \leq \delta_D(y)$. It is clear that the two sides are equal when $\delta_D(x) \wedge \delta_D(y) \geq |x - y|$ or when $\delta_D(x) \vee \delta_D(y) \leq |x - y|$. So we assume that $\delta_D(x) < |x - y| < \delta_D(y)$. Then $\delta_D(y) \leq \delta_D(x) + |y - x| \leq 2|x - y|$. Thus

$$\frac{\delta_D(x)^{\beta_1}\delta_D(y)^{\beta_2}}{|x - y|^{\beta_1 + \beta_2}} \wedge 1 \leq 2^{\beta_2}\Big(\frac{\delta_D(x)^{\beta_1}}{|x - y|^{\beta_1}} \wedge 1\Big) = 2^{\beta_2}\Big(\frac{\delta_D(x)^{\beta_1}}{|x - y|^{\beta_1}} \wedge 1\Big)\Big(\frac{\delta_D(y)^{\beta_2}}{|x - y|^{\beta_2}} \wedge 1\Big).$$

$\qquad\square$

Since $\widehat{B}$ is comparable to $\widetilde{B}$, we immediately obtain that $\widehat{B}(x, y)$ also satisfies **(B1)-(B3)**, **(B5)** and **(B6)**. If $\delta_D(x) \wedge \delta_D(y) \geq |x - y|$, we have that $\widehat{B}(x, y) = (\log 2)^{\beta_3 + \beta_4}$, so **(B4)** also holds with any $\theta > 0$. In particular, we can take $\theta = 1$.

We now introduce yet another boundary term. Let

$$\bar{B}(x, y) := \frac{(\delta_D(x) \wedge \delta_D(y))^{\beta_1}(\delta_D(x) \vee \delta_D(y))^{\beta_2}}{|x - y|^{\beta_1 + \beta_2} + (\delta_D(x) \wedge \delta_D(y))^{\beta_1}(\delta_D(x) \vee \delta_D(y))^{\beta_2}} L(x, y)^{\beta_3} K(x, y)^{\beta_4}. \qquad (2.5)$$

The elementary inequalities $\frac{1}{2}\big(\frac{a}{b} \wedge 1\big) \leq \frac{a}{a+b} \leq \frac{a}{b} \wedge 1$, $a, b > 0$, imply

Lemma 2.4. *For all* $x, y \in D$, $2^{-1}\widehat{B}(x, y) \leq \bar{B}(x, y) \leq \widehat{B}(x, y)$.

Thus $\bar{B}$ is comparable to $\widetilde{B}$ and $\widehat{B}$. Since Assumptions **(B1)-(B3)**, **(B5)** and **(B6)** depend only on comparability, we conclude that they are also satisfied by the boundary term $\bar{B}(x, y)$.

Let $x, y \in D$ such that $\delta_D(x) \wedge \delta_D(y) \geq |x - y|$. Then $L(x, y) = K(x, y) = \log 2$, so

$$(\log 2)^{\beta_3 + \beta_4} - \bar{B}(x, y) = \frac{|x - y|^{\beta_1 + \beta_2}}{|x - y|^{\beta_1 + \beta_2} + (\delta_D(x) \wedge \delta_D(y))^{\beta_1}(\delta_D(x) \vee \delta_D(y))^{\beta_2}} \leq \frac{|x - y|^{\beta_1 + \beta_2}}{(\delta_D(x) \wedge \delta_D(y))^{\beta_1 + \beta_2}}.$$

Hence **(B4)** is satisfied with $\theta = \beta_1 + \beta_2$.

Note that when $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$, $\widetilde{B}(x, y)$, $\widehat{B}(x, y)$ and $\bar{B}(x, y)$ reduce to 1, so the trivial case $\mathcal{B}(x, y) \equiv 1$ satisfies all our assumptions.

Now we consider a subordinate killed Lévy process in an arbitrary open subset $D \subset \mathbb{R}^d$ and show that, under some mild assumptions, its boundary term satisfies **(B1)**-**(B6)**.

Let Z be a subordinate Brownian motion $\mathbb{R}^d$, $d \geq 2$, via a subordinator whose Laplace exponent φ is a complete Bernstein function satisfying the following global weak scaling condition: There exist $a_1, a_2 > 0$ and $0 < \delta_3 \leq \delta_4 < 1$ such that

$$a_1 \left(\frac{R}{r}\right)^{\delta_3} \leq \frac{\varphi(R)}{\varphi(r)} \leq a_2 \left(\frac{R}{r}\right)^{\delta_4}, \qquad 0 < r \leq R < \infty. \tag{2.6}$$

Then it has a continuous transition density $p^Z(t, |x - y|)$. Let Z^D be the subprocess of Z killed upon leaving D. Then the transition density $p^{Z,D}(t, x, y)$ of Z^D has the form

$$p^{Z,D}(s, x, y) = p^{Z,D}(s, |x - y|) - \mathbb{E}_x \left[p^Z\left(s - \tau_D^Z, |Z_{\tau_D^Z} - y|\right) : \tau_D^Z < s \right],$$

where $\tau_D^Z := \inf\{t > 0 : Z_t \notin D\}$.

Let $T = (T_t)_{t \geq 0}$ be another subordinator, independent of Z, with Laplace exponent ψ and Lévy measure ν. We assume that ψ is a complete Bernstein function (so that ν has complete monotone density $\nu(x)$) satisfying the following global weak scaling condition: There exist $b_1, b_2 > 0$ and $0 < \gamma_1 \leq \gamma_2 < 1$ such that

$$b_1 \left(\frac{R}{r}\right)^{\gamma_1} \leq \frac{\psi(R)}{\psi(r)} \leq b_2 \left(\frac{R}{r}\right)^{\gamma_2}, \qquad 0 < r \leq R < \infty. \tag{2.7}$$

So ψ also has no drift.

Let Y_t and X_t be the processes defined by $Y_t := Z_{T_t}^D$ and $X_t := Z_{T_t}$, respectively. The jump kernel $J^D(x, y)$ of Y and the jump kernel $J(|x - y|)$ of X are given respectively by

$$J^D(x, y) = \int_0^\infty p^{Z,D}(s, x, y)\nu(s)ds \quad \text{and} \quad J(|x - y|) = \int_0^\infty p^Z(s, |x - y|)\nu(s)ds.$$

It is clear that $J^D(x, y) \leq J(|x - y|)$. By [12, Theorem 3.4], there exists $c \geq 1$ such that

$$c^{-1} \frac{(\psi \circ \varphi)(|x - y|^{-2})}{|x - y|^d} \leq J(|x - y|) \leq c \frac{(\psi \circ \varphi)(|x - y|^{-2})}{|x - y|^d}, \qquad x \neq y. \tag{2.8}$$

Let $\mathcal{B}(x, y) = J^D(x, y)/j(|x - y|)$ for $x \neq y$ and $\mathcal{B}(x, x) = 1$. Then $\mathcal{B}(x, y) \leq 1$.

We first check that $\mathcal{B}$ satisfies **(B4)**. Assume that $\delta_D(y) \geq \delta_D(x) \geq |x - y|$. Then

$$j(|x - y|) - J^D(x, y) = \int_0^\infty \left(p^Z(s, |x - y|) - p^{Z,D}(s, x, y)\right)\nu(s)ds$$

$$= \int_0^\infty \mathbb{E}_x \left[p^Z\left(s - \tau_D^Z, |Z_{\tau_D^Z} - y|\right) : \tau_D^Z < s \right]\nu(s)ds = \mathbb{E}_x \int_{\tau_D^Z}^\infty p^Z\left(s - \tau_D^Z, |Z_{\tau_D^Z} - y|\right)\nu(s)ds$$

$$= \mathbb{E}_x \int_0^\infty p^Z\left(v, |Z_{\tau_D^Z} - y|\right)\nu(v + \tau_D^Z)ds \leq \mathbb{E}_x \int_0^\infty p^Z\left(v, |Z_{\tau_D^Z} - y|\right)\nu(v)ds = \mathbb{E}_x \left[J\left(|Z_{\tau_D^Z} - y|\right) \right].$$

Since $|Z_{\tau_D^Z} - y| \geq \delta_D(y)$ and j is monotone, we have $j(|x-y|) - J^D(x,y) \leq j(\delta_D(y))$. Hence by (2.6), (2.7) and (2.8),

$$\mathcal{B}(x,x) - \mathcal{B}(x,y) = 1 - \mathcal{B}(x,y) \leq \frac{j(\delta_D(y))}{j(|x-y|)}$$

$$\leq c\left(\frac{|x-y|}{\delta_D(y)}\right)^d \frac{(\psi \circ \varphi)(\delta_D(y)^{-2})}{(\psi \circ \varphi)(|x-y|^{-2})} \leq c\left(\frac{|x-y|}{\delta_D(y)}\right)^{d+2\gamma_1\delta_3}. \tag{2.9}$$

Thus **(B4)** holds with $\theta = d + 2\gamma_1\delta_3 > 1$. If D is a bounded κ-fat open set, then it follows from [14, Propositions 4.5 and 5.3, and the proof of Theorem 5.5] that $\mathcal{B}(x,y)$ satisfies assumptions **(B1)-(B3)**, **(B5)** and **(B6)** as well.

If Z is instead a Brownian motion in $\mathbb{R}^d$ with $d \geq 3$, then by following the same argument one can show that the corresponding boundary term $\mathcal{B}(x,y)$ satisfies **(B4)** with $\theta = d + 2\gamma_1$. If D is either a bounded $C^{1,1}$ domain, or a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function, or D is a $C^{1,1}$ domain with compact complement, it follows from [13] that the corresponding boundary term $\mathcal{B}(x,y)$ satisfies assumptions **(B1)-(B3)**, **(B5)** and **(B6)** as well.

In the remaining part of this section we assume that $D = \mathbb{R}_+^d$, $Z = Z^{(\delta)}$ is a δ-stable process in $\mathbb{R}^d$ and T_t is an independent $(\gamma/2)$-stable subordinator where $\delta \in (0,2]$ and $\gamma \in (0,2)$. Let $\alpha = \delta\gamma/2$ and denote by $J^{(\alpha),\mathbb{R}_+^d}(x,y)$ the jump kernel of $Y_t := Z_{T_t}^{(\delta),\mathbb{R}_+^d}$ It follows from (1.1) and (1.2) that the boundary term $\mathcal{B}(x,y) = J^D(x,y)/j(|x-y|)$ satisfies the assumption **(B7)** with $\beta_4 = 0$. Let $p^{(\delta),\mathbb{R}_+^d}(t,x,y)$ be the transition density of $Z^{(\delta),\mathbb{R}_+^d}$. By using the scaling property in the third equality and the change of variables $s = a^\delta t$ in the fourth equality below, we get

$$J^{(\alpha),\mathbb{R}_+^d}(ax,ay) = c(\gamma) \int_0^\infty p^{(\delta),\mathbb{R}_+^d}(a^\delta(a^{-\delta}s), ax, ay)s^{-1-\frac{\gamma}{2}}ds$$

$$= c(\gamma) \int_0^\infty a^{-d} p^{(\delta),\mathbb{R}_+^d}(a^{-\delta}s, x, y)s^{-1-\frac{\gamma}{2}}ds$$

$$= a^{-(d+\alpha)}c(\gamma) \int_0^\infty p^{(\delta),\mathbb{R}_+^d}(t,x,y)t^{-1-\frac{\gamma}{2}}dt = a^{-(d+\alpha)}J^{(\alpha),\mathbb{R}_+^d}(x,y).$$

This proves that $\mathcal{B}(x,y)$ is homogeneous. The second part of **(B8)** follows from translation invariance property of the transition density: $p^{(\delta),\mathbb{R}_+^d}(t, x + (\widetilde{z},0), y + (\widetilde{z},0)) = p^{(\delta),\mathbb{R}_+^d}(t,x,y)$ for all $x,y \in \mathbb{R}_+^d$ and all $\widetilde{z} \in \mathbb{R}^{d-1}$.

3. Interior results

Let $D \subset \mathbb{R}^d$ be an open set, $D \neq \mathbb{R}^d$. From now on we always assume that **(B1)-(B3)** and (1.3)–(1.5) hold. Recall that $\delta_D(x) := \text{dist}(x,\partial D)$ and that we have defined the kernel $J^D(x,y)$ as $J^D(x,y) = \mathcal{B}(x,y)j(|x-y|)$. Define

$$\mathcal{E}^D(u,v) := \frac{1}{2} \int_D \int_D (u(x) - u(y))(v(x) - v(y))J^D(x,y)\,dy\,dx.$$

By Fatou's lemma, $(C_c^\infty(D), \mathcal{E}^D)$ is closable in $L^2(D,dx)$. Let $\mathcal{F}$ be the closure of $C_c^\infty(D)$ under $\mathcal{E}_1^D := \mathcal{E}^D + (\cdot,\cdot)_{L^2(D,dx)}$. Then $(\mathcal{F}^D, \mathcal{E}^D)$ is a regular Dirichlet form on $L^2(D,dx)$.

Recall that $\kappa : D \to [0,\infty)$ is a Borel function satisfying (1.5), namely $\kappa(x) \leq C_1\Phi(\delta_D(x))^{-1}$, $x \in D$, for some $C_1 > 0$. This implies that κ is locally bounded in D. Set

$$\mathcal{E}^{D,\kappa}(u,v) := \mathcal{E}^D(u,v) + \int_D u(x)v(x)\kappa(x)\,dx.$$

Since κ is locally bounded, the measure $\kappa(x)dx$ is a positive Radon measure charging no set of zero capacity. Let $\mathcal{F}^{D,\kappa} := \widetilde{\mathcal{F}^D} \cap L^2(D, \kappa(x)dx)$, where $\widetilde{\mathcal{F}^D}$ is the family of all quasi-continuous functions on $\mathcal{F}^D$. By [10, Theorems 6.1.1 and 6.1.2], $(\mathcal{F}^{D,\kappa}, \mathcal{E}^{D,\kappa})$ is a regular Dirichlet form on $L^2(D, dx)$ having $C_c^\infty(D)$ as a special standard core. Let $(Y_t^{D,\kappa}, \mathbb{P}_x)$, $t \geq 0$, $x \in D \setminus \mathcal{N}$, be the associated Hunt process with lifetime ζ, where $\mathcal{N}$ is an exceptional set. We add a cemetery point ∂ to the state space D and define $Y_t^{D,\kappa} = \partial$ for $t \geq \zeta$. We will write $D_\partial = D \cup \{\partial\}$.

For an open set $U \subset \overline{U} \subset D$, let $Y^{D,\kappa,U}$ be the process $Y^{D,\kappa}$ killed upon exiting U, that is, the part of the process $Y^{D,\kappa}$ in U. The Dirichlet form of $Y^{D,\kappa,U}$ is equal to $(\mathcal{F}_U^{D,\kappa}, \mathcal{E}^{D,\kappa})$, where $\mathcal{F}_U^{D,\kappa} = \{u \in \mathcal{F}^{D,\kappa} : u = 0 \text{ q.e. on } D \setminus U\}$. For $u, v \in \mathcal{F}_U^{D,\kappa}$,

$$\mathcal{E}^{D,\kappa}(u,v) = \frac{1}{2} \int_U \int_U (u(x) - u(y))(v(x) - v(y)) J^D(x,y) \, dy \, dx + \int_U u(x)u(y)\kappa_U(x) \, dx, \quad (3.1)$$

where

$$\kappa_U(x) = \int_{D \setminus U} J^D(x,y) \, dy + \kappa(x), \quad x \in U. \quad (3.2)$$

Moreover, by [5, Theorem 3.3.9], since $C_c^\infty(D)$ is a special standard core of $(\mathcal{F}^{D,\kappa}, \mathcal{E}^{D,\kappa})$, the set $\{u \in C_c^\infty(D) : \mathrm{supp}(u) \subset U\} = C_c^\infty(U)$ is a core of $(\mathcal{F}_U^{D,\kappa}, \mathcal{E}^{D,\kappa})$.

3.1. Regularization of the process.

In this subsection we show that one can remove the exceptional set $\mathcal{N}$ and start the process $Y^{D,\kappa}$ from every point $x \in D$. For this purpose, we will use the process Z on $\mathbb{R}^d$, with jump kernel J_γ defined below, as a tool. The process Z can start from every point in $\mathbb{R}^d$.

Let U be a relatively compact $C^{1,1}$ open subset of D. Note that for $x, y \in U$, $\delta_D(x) \wedge \delta_D(y) \geq \mathrm{dist}(U, \partial D)$, so with $a := \mathrm{dist}(U, \partial D)/\mathrm{diam}(U)$ we have that $\delta_D(x) \wedge \delta_D(y) \geq a|x-y|$. Therefore, it follows from **(B2)** and **(B3)** that there exists a constant $c_1 = c_1(U) \geq 1$ such that $c^{-1} \leq \mathcal{B}(x,y) \leq c$ for all $x, y \in U$. Together with (1.3) this implies that there exist $c_2 > 0$ and $c_3 > 0$ such that

$$\frac{c_2}{|x-y|^d \Phi(|x-y|)} \leq J^D(x,y) \leq \frac{c_3}{|x-y|^d \Phi(|x-y|)}, \quad x, y \in U. \quad (3.3)$$

For $\gamma > 0$ define a kernel $J_\gamma(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ by $J_\gamma(x,y) = J^D(x,y)$ for $x, y \in U$, and $J_\gamma(x,y) = \gamma j(|x-y|)$ otherwise. By (1.3) and (3.3), there exist $c_4 > 0$ and $c_5 > 0$ such that

$$\frac{c_4}{|x-y|^d \Phi(|x-y|)} \leq J_\gamma(x,y) \leq \frac{c_5}{|x-y|^d \Phi(|x-y|)}, \quad x, y \in \mathbb{R}^d.$$

It is now straightforward to check that all conditions of [7, Theorem 1.2] (as well as the geometric condition of the paper) are satisfied. For $u \in L^2(\mathbb{R}^d, dx)$, define

$$\mathcal{C}(u,u) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 J_\gamma(x,y) \, dx \, dy \text{ and } \mathcal{D}(\mathcal{C}) := \{u \in L^2(\mathbb{R}^d) : \mathcal{C}(u,u) < \infty\}.$$

Note that $C_c^\infty(\mathbb{R}^d)$ is a special standard core of $\mathcal{D}(\mathcal{C})$.

Let

$$\widetilde{q}(t,x,y) := \Phi(t)^{-d} \wedge \frac{t}{|x-y|^d \Phi(|x-y|)}, \quad t > 0, \ x, y \in \mathbb{R}^d.$$

It follows from [7] that there exists a conservative Feller and strongly Feller process Z associated with $(\mathcal{D}(\mathcal{C}), \mathcal{C})$ that can start from every point in $\mathbb{R}^d$. Moreover, the process Z has a continuous transition density function $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ (with respect to the Lebesgue measure) that satisfies the following estimate: There exists $c_6 \geq 1$ such that

$$c_6^{-1} \widetilde{q}(t,x,y) \leq p(t,x,y) \leq c_6 \widetilde{q}(t,x,y), \quad t > 0, \ x, y \in \mathbb{R}^d.$$

Denote the part of the process Z killed upon exiting U by Z^U. The Dirichlet form of Z^U is $(\mathcal{D}_U(\mathcal{C}), \mathcal{C})$, where for $u, v \in \mathcal{D}_U(\mathcal{C})$,

$$
\begin{aligned}
\mathcal{C}(u, v) &= \frac{1}{2} \int_U \int_U (u(x) - u(y))(v(x) - v(y)) J_\gamma(x, y) \, dy \, dx + \int_U u(x)v(x)\kappa_U^Z(x) \, dx \\
&= \frac{1}{2} \int_U \int_U (u(x) - u(y))(v(x) - v(y)) J^D(x, y) \, dy \, dx + \int_U u(x)v(x)\kappa_U^Z(x) \, dx
\end{aligned}
$$

with

$$
\kappa_U^Z(x) = \int_{\mathbb{R}^d \setminus U} J_\gamma(x, y) \, dy = \gamma \int_{\mathbb{R}^d \setminus U} j(|x - y|) \, dy, \quad x \in U,
$$

and $\mathcal{D}_U(\mathcal{C}) = \{u \in \mathcal{D}(\mathcal{C}) : u = 0 \text{ q.e. on } \mathbb{R}^d \setminus U\}$. Again, by [5, Theorem 3.3.9], $C_c^\infty(U)$ is a core of $(\mathcal{D}_U(\mathcal{C}), \mathcal{C})$.

Recall $\delta_U = \text{dist}(U, \partial D)$ and $d_U = \text{diam}(U)$. For all $x, y \in U$ we have that $\delta_D(x) \wedge \delta_D(y) \geq (\delta_U/d_U)|x - y|$, hence by **(B3)** we have $\mathcal{B}(x, y) \geq c_7$ (where $c_7 = C_3(2^{-1} \wedge (\delta_U/d_U))$) implying together with **(B2)** that

$$
c_7 j(|x - y|) \leq J^D(x, y) = \mathcal{B}(x, y) j(|x - y|) \leq C_2 j(|x - y|), \quad x, y \in U. \tag{3.4}
$$

Next, let V be the $\delta_U/2$-neighborhood of U, that is $V := \{x \in D : \text{dist}(x, U) < \delta_U/2\}$. Then

$$
\kappa_U(x) = \int_{D \setminus V} J^D(x, y) \, dy + \int_{V \setminus U} J^D(x, y) \, dy + \kappa(x).
$$

Similarly as above we conclude that $c_8 j(|x - y|) \leq J^D(x, y) \leq C_2 j(|x - y|)$ for all $x, y \in V$ with $c_8 := C_3((\delta_U/2)/(d_U + \delta_U/2)) > 0$. Moreover, $\sup_{x \in U} \int_{D \setminus V} J^D(x, y) \, dy =: c_9 < \infty$ (this can be shown by splitting the integral into two parts – over $(D \setminus V) \cap B(0, R_0)$ and $(D \setminus V) \cap B(0, R_0)^c$ for appropriate R_0) and $\|\kappa_{|U}\| =: c_{10} < \infty$. Therefore

$$
c_8 \int_{V \setminus U} j(|x - y|) \, dy \leq \kappa_U(x) \leq c_9 + C_2 \int_{V \setminus U} j(|x - y|) \, dy + c_{10}, \quad x \in U.
$$

Since

$$
\inf_{x \in U} \int_{V \setminus U} j(|x - y|) \, dy \geq |V \setminus U| j(\text{diam}(V)) =: c_{11} > 0,
$$

we conclude that

$$
c_8 \int_{V \setminus U} j(|x - y|) \, dy \leq \kappa_U(x) \leq c_{12} \int_{V \setminus U} j(|x - y|) \, dy, \quad x \in U.
$$

Further, since

$$
\kappa_U^Z(x) = \gamma \left(\int_{\mathbb{R}^d \setminus V} j(|x - y|) \, dy + \int_{V \setminus U} j(|x - y|) \, dy \right), \quad x \in U
$$

and $\sup_{x \in U} \int_{\mathbb{R}^d \setminus V} j(|x - y|) \, dy =: c_{13} < \infty$, we see that there is a constant $c_{14} > 0$ such

$$
\int_{V \setminus U} j(|x - y|) \, dy \leq \gamma^{-1} \kappa_U^Z(x) \leq c_{14} \int_{V \setminus U} j(|x - y|) \, dy, \quad x \in U.
$$

It follows that $c_{12}^{-1} \kappa_U(x) \leq \gamma^{-1} \kappa_U^Z(x) \leq c_{14} c_8^{-1} \kappa_U(x)$ for all $x \in U$ with positive constants c_8, c_{12}, c_{14} not depending on γ. Now we choose $\gamma > 0$ so small that $\gamma c_{14} c_8^{-1} \leq 1$. With this choice we get that $\kappa_U^Z(x) \leq \kappa_U(x)$ for all $x \in U$. In particular, with $c_{15} := \gamma c_{12}^{-1}$ we see that

$$
c_{15} \kappa_U(x) \leq \kappa_U^Z(x) \leq \kappa_U(x), \quad x \in U.
$$

It follows that for $u \in C_c^\infty(U)$,

$$
\begin{aligned}
\mathcal{E}_1^{D,\kappa}(u,u) &= \mathcal{E}^{D,\kappa}(u,u) + \int_U u(x)^2\, dx \\
&= \frac{1}{2} \int_U \int_U (u(x)-u(y))^2 J^D(x,y)\, dy\, dx + \int_U u(x)^2 \kappa_U(x)\, dx + \int_U u(x)^2 dx \\
&\asymp \frac{1}{2} \int_U \int_U (u(x)-u(y))^2 J_\gamma(x,y)\, dy\, dx + \int_U u(x)^2 \kappa_U^Z(x)\, dx + \int_U u(x)^2 dx \\
&= \mathcal{C}(u,u) + \int_U u(x)^2 dx = \mathcal{C}_1(u,u)\,.
\end{aligned}
$$

Since $C_c^\infty(U)$ is a core of both $(\mathcal{F}_U^{D,\kappa}, \mathcal{E}^{D,\kappa})$ and $(\mathcal{C}, \mathcal{D}_U(\mathcal{C}))$, we conclude that $\mathcal{F}_U^{D,\kappa} = \mathcal{D}_U(\mathcal{C})$.

We now define $\widetilde{\kappa} : U \to \mathbb{R}$ by $\widetilde{\kappa}(x) := \kappa_U(x) - \kappa_U^Z(x)$, $x \in U$. By the choice of γ we have that $\widetilde{\kappa} \geq 0$. On the other hand, since κ is bounded in U and U is $C^{1,1}$, it follows from (3.3) that there is a constant $c_{16} > 0$ such that

$$
\widetilde{\kappa}(x) \leq \kappa_U(x) = \int_{D \setminus U} J^D(x,y)\, dy + \kappa(x) \leq \frac{c_{16}}{\Phi(\delta_D(x))}\,, \quad x \in U. \tag{3.5}
$$

Let $\mu(dx) = \widetilde{\kappa}(x)\, dx$ be a measure on U. For $t > 0$ and $a \geq 0$ define

$$
N_a^{U,\mu}(t) := \sup_{x \in \mathbb{R}^d} \int_0^t \int_{z \in U : \delta_U(z) > a\Phi^{-1}(t)} \widetilde{q}(s,x,z)\mu(dz)\, ds\,.
$$

By the definition of $\widetilde{q}$ and (3.5) one can check that $\sup_{t<1} N_a^{U,\mu}(t) < \infty$ and $\lim_{t \to 0} N_0^{V,\mu}(t) = 0$ for every relatively compact open set $V \subset U$, that is, $\mu \in \mathbf{K}_1(U)$ in the notation of [8, Definition 2.12].

Let $A_t := \int_0^t \widetilde{\kappa}(Z_s^U)\, ds$. Then $(A_t)_{t \geq 0}$ is a positive continuous additive functional of Z^U in the strict sense (i.e. without an exceptional set) with Revuz measure $\widetilde{\kappa}(x)dx$. For any non-negative Borel function f on U let

$$
T_t^{U,\widetilde{\kappa}} f(x) := \mathbb{E}_x[\exp(-A_t)f(Z_t^U)], \qquad t > 0, x \in U\,,
$$

be the Feynman-Kac semigroup of Z^U associated with $\widetilde{\kappa}(x)dx$. By [8, Propostion 2.14], the Hunt process $Z^{U,\widetilde{\kappa}}$ on U corresponding to the transition semigroup $(T_t^{U,\widetilde{\kappa}})_{t \geq 0}$ has a transition density $q^U(t,x,y)$ (with respect to the Lebesgue measure) such that $q^U(t,x,y) \leq c_{17}\widetilde{q}(t,x,y)$ for $t < 1$. Further, $(t,y) \mapsto q^U(t,x,y)$ is continuous for each $x \in U$.

According to [10, Theorem 6.1.2], the Dirichlet form $\mathcal{C}^{U,\widetilde{\kappa}}$ corresponding to $T_t^{U,\widetilde{\kappa}}$ is regular and is given by

$$
\mathcal{C}^{U,\widetilde{\kappa}}(u,v) = \frac{1}{2} \int_U \int_U (u(x)-u(y))(v(x)-v(y))J^D(x,y)dydx + \int_U u(x)v(x)\kappa_U(x)\, dx
$$

with the domain $\mathcal{D}_U^{\widetilde{\kappa}} = \mathcal{D}_U(\mathcal{C}) \cap L^2(U, \widetilde{\kappa}(x)dx)$. Since $(\mathcal{D}_U^{\widetilde{\kappa}}, \mathcal{C}^{U,\widetilde{\kappa}})$ is regular, the set $\mathcal{D}_U^{\widetilde{\kappa}} \cap C_c(U) = \mathcal{D}_U(\mathcal{C}) \cap C_c(U)$ is its core. By comparing with (3.1) we see that

$$
\mathcal{E}^{D,\kappa}(u,v) = \mathcal{C}^{U,\widetilde{\kappa}}(u,v)\,, \qquad u,v \in C_c^\infty(U)\,.
$$

Now we have to argue that the Dirichlet spaces $(\mathcal{F}_U^{D,\kappa}, \mathcal{E}^{D,\kappa})$ and $(\mathcal{D}_U^{\widetilde{\kappa}}, \mathcal{C}^{U,\widetilde{\kappa}})$ are equal. We know that $C_c^\infty(U)$ is a core for $\mathcal{E}^{D,\kappa}$. One can easily check that this should be also true for $\mathcal{C}^{U,\widetilde{\kappa}}$. Further, $C_c^\infty(U) \subset C_c(U) \cap \{u \in L^2(U,dx) : \mathcal{C}^U(u,u) < \infty\}$ (which is a core). Clearly, $C_c^\infty(U)$ is dense in $C_c(U)$ with uniform norm. It is easy to see that $C_c^\infty(U)$ is dense in $C_c(U) \cap \{u \in L^2(U,dx) : \mathcal{C}^U(u,u) < \infty\}$ with $\mathcal{C}_1^{U,\kappa_U}$ norm. Thus the process $Z^{U,\widetilde{\kappa}}$ coincides with $Y^{D,\kappa,U}$. We summarize the above discussion in the following lemma.

Lemma 3.1. *The process $Y^{D,\kappa,U}$ can be refined to start from every point in U. Moreover, it is strongly Feller.*

Proposition 3.2. *The process $Y^{D,\kappa}$ can be refined to start from every point in D.*

Proof. Let $(U_n)_{n\geq 1}$ be a sequence of bounded smooth open sets such that $U_n \subset \overline{U_n} \subset U_{n+1}$ and $\cup_{n\geq 1}U_n = D$. For each $n \geq 1$, let $Y^{(n)} = Y^{D,\kappa,U_n}$ and let $\zeta^{(n)}$ be the lifetime of $Y^{(n)}$. The Dirichlet form of $Y^{(n)}$ is $(\mathcal{F}^{D,\kappa}_{U_n}, \mathcal{E}^{D,\kappa})$, where $\mathcal{F}^{D,\kappa}_{U_n} = \{u \in \mathcal{F}^{D,\kappa} : u = 0 \text{ q.e. on } D \setminus U_n\}$. It follows from Lemma 3.1 that each process $Y^{(n)}$ can start from every point in U_n. These processes are consistent in the sense that $Y^{(n+1)}_t = Y^{(n)}_t$ for $t \in [0,\zeta^{(n)})$. We define a new process $\widetilde{Y}$ by $\widetilde{Y}_t = Y^{(1)}_t$ for $t \in [0,\zeta^{(1)})$, and for each $n > 1$, $\widetilde{Y}_t = Y^{(n)}_t$ for $t \in [\zeta^{(n-1)},\zeta^{(n)})$. The process $\widetilde{Y}$ can start from every point in D. It is easy to check that the Dirichlet form of $\widetilde{Y}$ is $(\mathcal{F}^{D,\kappa}, \mathcal{E}^{D,\kappa})$. Thus we can assume the process $Y^{D,\kappa}_t$ can start from every point in D. □

3.2. **Analysis of the generator.** In this subsection we assume that, in addition to **(B1)**-**(B3)**, **(B4)** also holds. Let

$$C^2_c(D;\mathbb{R}^d) = \{f : D \to \mathbb{R} : \text{ there exists } u \in C^2_c(\mathbb{R}^d) \text{ such that } u = f \text{ on } D\}$$

be the space of functions on D that are restrictions of $C^2_c(\mathbb{R}^d)$ functions. Clearly, if $f \in C^2_c(D;\mathbb{R}^d)$ then $f \in C^2_b(D) \cap L^2(D)$.

We introduce the operator

$$
\begin{aligned}
L^{\mathcal{B}}f(x) \quad &:= \quad \text{p.v.} \int_D (f(y) - f(x))J^D(x,y)\,dy - \kappa(x)f(x) \quad &(3.6)\\
&= \quad \lim_{\varepsilon\to 0} \int_{D,|y-x|>\varepsilon} (f(y) - f(x))J^D(x,y)\,dy - \kappa(x)f(x), \quad x \in D,
\end{aligned}
$$

defined for all functions $f : D \to \mathbb{R}$ for which the principal value integral makes sense. We will show that this is the case when $f \in C^2_c(D;\mathbb{R}^d)$. Let us start with the following auxiliary result.

Lemma 3.3. *There exists a constant $C_9 > 0$ such that for any bounded Lipschitz function with Lipschitz constant L, we have for all $x \in D$ and $r \in (0, \delta_D(x)]$,*

$$\int_D |f(y) - f(x)|\,j(|y-x|)|\mathcal{B}(x,x) - \mathcal{B}(x,y)|\,dy \leq C_9(\|f\|_\infty + rL)\Phi(r)^{-1}. \quad (3.7)$$

Proof. Since the proof for the case $\delta_2 < 1/2$ is simpler, we give the proof for the case $\delta_2 \geq 1/2$.
Note that

$$\int_D |f(y) - f(x)|\,j(|y-z|)|\mathcal{B}(x,x) - \mathcal{B}(x,y)|\,dy$$

$$\leq L \int_{D,|y-x|<r/2} |y-x|j(|y-x|)|\mathcal{B}(x,x) - \mathcal{B}(x,y)|\,dy$$

$$+ 4C_2\|f\|_\infty \int_{D,|y-x|\geq r/2} j(|y-x|)\,dy =: I_1 + I_2,$$

where we used **(B2)** in the inequality. It follows from $\delta_D(x) \geq r$ that, if $|y - x| < r/2$, then $\delta_D(y) > r/2$ and thus $\delta_D(y) \wedge \delta_D(x) > r/2 > |y - x|$. Thus by **(B4)**, (1.3) and (1.4),

$$
\begin{aligned}
I_1 \quad &\leq \quad c_1 C_4 L 2^\theta r^{-\theta} \int_{|y-x|<r/2} |y-x|^{1+\theta}|y-x|^{-d}\Phi(|y-x|)^{-1}\,dy\\
&\leq \quad c_2 C_4 L 2^\theta r^{-\theta} \frac{1}{\Phi(r/2)} \int_0^{r/2} \frac{s^\theta \Phi(r/2)}{\Phi(s)}\,ds
\end{aligned}
$$

$$\leq \quad a_2 c_2 C_4 L 2^\theta r^{-\theta} \frac{(r/2)^{2\delta_2}}{\Phi(r/2)} \int_0^{r/2} s^{\theta - 2\delta_2} \, ds \leq c_3 L r \Phi(r)^{-1} \, .$$

Similarly, using (1.3) and (1.4), $I_2 \leq c_4 \|f\|_\infty \int_{r/2}^\infty t^{d-1} j(t) \, dt \leq c_5 \|f\|_\infty \Phi(r)^{-1}$. Combining the estimates for I_1 and I_2 we get the statement of the lemma. $\qquad\qquad\qquad\square$

Proposition 3.4.   (a) *Let $f \in C_c^2(D; \mathbb{R}^d)$. Then $L^{\mathcal{B}} f$ is well defined and for all $x \in D$ and $r > 0$,*

$$\begin{aligned}
L^{\mathcal{B}} f(x) \;=\;& \mathcal{B}(x, x) \int_{\mathbb{R}^d} (u(y) - u(x) - \nabla u(x) \mathbf{1}_{\{|y-x|<r\}} \cdot (y - x)) j(|x - y|) dy \\
&+ \mathcal{B}(x, x) \int_{\mathbb{R}^d \setminus D} (u(x) - u(y)) j(|x - y|) dy \\
&+ \int_D (u(y) - u(x)) j(|x - y|)(\mathcal{B}(y, x) - \mathcal{B}(x, x)) dy - \kappa(x) u(x), \qquad (3.8)
\end{aligned}$$

    where $u \in C_c^2(\mathbb{R}^d)$ is any function such that $u = f$ on D.
    (b) *There exists a constant $C_{10} > 0$ such that for any $f \in C_c^2(D; \mathbb{R}^d)$, any $x \in D$ and any $r \in (0, \delta_D(x)]$ we have*

$$|L^{\mathcal{B}} f(x)| \leq C_{10} \left(r^2 \|\partial^2 u\|_\infty + r \|\nabla u\|_\infty + \|u\|_\infty \right) \Phi(r)^{-1} \, , \qquad (3.9)$$

    where $u \in C_c^2(\mathbb{R}^d)$ is any function such that $u = f$ on D.
    (c) *Let $f \in C_c^2(D)$. Then for all $0 < \epsilon_0 \leq \mathrm{dist}(\partial D, \mathrm{supp}(f))/2$,*

$$\|L^{\mathcal{B}} f\|_\infty \leq C_{10} \left(\epsilon_0^2 \|\partial^2 f\|_\infty + \epsilon_0 \|\nabla f\|_\infty + \|f\|_\infty \right) \Phi(\epsilon_0)^{-1} \, .$$

Proof. (a) Let $u \in C_c^2(\mathbb{R}^d)$ be such that $u = f$ on D. Fix $x \in D$ and let $\varepsilon < r \wedge (\delta_D(x)/2)$. Then

$$\int_{D, \, |x-y|>\varepsilon} (f(y) - f(x)) j(|x - y|) \mathcal{B}(x, y) dy$$

$$= \mathcal{B}(x, x) \int_{D, \, |x-y|>\varepsilon} (u(y) - u(x)) j(|x - y|) dy$$

$$+ \int_{D, \, |x-y|>\varepsilon} (u(y) - u(x)) j(|x - y|)(\mathcal{B}(x, y) - \mathcal{B}(x, x)) dy$$

$$= \mathcal{B}(x, x) \int_{|x-y|>\varepsilon} (u(y) - u(x)) j(|x - y|) dy + \mathcal{B}(x, x) \int_{\mathbb{R}^d \setminus D, \, |x-y|>\varepsilon} (u(x) - u(y)) j(|x - y|) dy$$

$$+ \int_{D, \, |x-y|>\varepsilon} (u(y) - u(x)) j(|x - y|)(\mathcal{B}(x, y) - \mathcal{B}(x, x)) dy$$

$$= \mathcal{B}(x, x) \int_{|x-y|>\varepsilon} (u(y) - u(x) - \nabla u(x) \mathbf{1}_{\{|y-x|<r\}} \cdot (y - x)) j(|x - y|) dy$$

$$+ \mathcal{B}(x, x) \int_{\mathbb{R}^d \setminus D, \, |x-y|>\varepsilon} (u(x) - u(y)) j(|x - y|) dy$$

$$+ \int_{D, \, |x-y|>\varepsilon} (u(y) - u(x)) j(|x - y|)(\mathcal{B}(y, x) - \mathcal{B}(x, x)) dy.$$

In the the first integral above, we have used **(B1)**. By letting $\varepsilon \to 0$ and using Lemma 3.3 (with r there being $\delta_D(x)$) for the third integral, we obtain (3.8).

(b) Let $u \in C_c^2(\mathbb{R}^d)$ be any function such that $u = f$ on D. Fix $x \in D$ and let $r \in (0, \delta_D(x)]$. Then by part (a),

$$L^{\mathcal{B}} f(x) = \mathcal{B}(x,x) \int_{\mathbb{R}^d} \left(u(y) - u(x) - \nabla u(x) \mathbf{1}_{|y-x|<r} \cdot (y-x) \right) j(|y-x|) \, dy$$

$$+ \mathcal{B}(x,x) \int_{\mathbb{R}^d \setminus D} (u(x) - u(y)) j(|y-z|) \, dy$$

$$+ \int_D (f(y) - f(x)) j(|y-x|)(\mathcal{B}(y,x) - \mathcal{B}(x,x)) \, dy - \kappa(x) f(x)$$

$$=: I + II + III + IV \,.$$

For I, we use

$$\left| u(y) - u(x) - \nabla u(x) \mathbf{1}_{|y-x|<r} \cdot (y-x) \right| \le \|\partial^2 u\|_\infty |y-x|^2 \mathbf{1}_{|y-x|\le r} + 2\|u\|_\infty \mathbf{1}_{|y-z|\ge r}$$

to get

$$I \le \mathcal{B}(x,x) \int_{\mathbb{R}^d} \left(\|\partial^2 u\|_\infty |y-x|^2 \mathbf{1}_{|y-x|\le r} + 2\|u\|_\infty \mathbf{1}_{|y-x|\ge r} \right) j(|y-x|) \, dy$$

$$\le \mathcal{B}(x,x) c(d) \left(\|\partial^2 u\|_\infty \int_0^r t^{d-1} t^2 t^{-d} \Phi(t)^{-1} \, dt + \int_r^\infty 2\|u\|_\infty t^{d-1} t^{-d} \Phi(t)^{-1} \, dt \right)$$

$$\le \mathcal{B}(x,x) c(d)(\|\partial^2 u\|_\infty r^2 + 2\|u\|_\infty) \Phi(r)^{-1} \,.$$

For II we use that $\delta_D(x) \ge r$ to get

$$II \le 2\mathcal{B}(x,x) \|u\|_\infty \int_{B(x,\delta_D(x))} j(|y-x|) \, dy$$

$$\le c(d) \mathcal{B}(x,x) \|u\|_\infty \Phi(\delta_D(x))^{-1} \le c(d) \mathcal{B}(x,x) c(d) \|u\|_\infty \Phi(r)^{-1} \,.$$

III is estimated in Lemma 3.3, while for IV we use (1.5) to get

$$IV \le C_1 \|f\|_\infty \Phi(\delta_D(x))^{-1} \le C_1 \|f\|_\infty \Phi(r)^{-1} \,.$$

(c) Take $x \in D$ such that $\delta_D(x) < \epsilon_0$. Then $B(x, \epsilon_0) \cap D \subset D \setminus \mathrm{supp}(f)$. Since $f(x) = 0$ and also $f(y) = 0$ for $y \in D \cap B(x, \epsilon_0)$, we have that

$$|L^{\mathcal{B}} f(x)| \le \int_{|y-x| \ge \epsilon_0} |f(y)| J^D(y,x) \, dy \le \|f\|_\infty \int_{|y-x| \ge \epsilon_0} j(|y-x|) \, dy \le c(d) \|f\|_\infty \Phi(\epsilon_0)^{-1} \,.$$

On the other hand, if $\delta_D(x) \ge \epsilon_0$, then we can take $r = \epsilon_0$ and a function $u \in C_c^2(D; \mathbb{R}^d)$ such that $u = f$ on D with $\|u\|_\infty = \|f\|_\infty$, $\|\nabla u\|_\infty = \|\nabla u\|_\infty$ and $\|\partial^2 u\|_\infty = \|\partial^2 u\|_\infty$ in (3.9) to get

$$|L^{\mathcal{B}} f(x)| \le C_{10} \left(\epsilon_0^2 \|\partial^2 f\|_\infty + \epsilon_0 \|\nabla f\|_\infty + \|f\|_\infty \right) \Phi(\epsilon_0)^{-1} \,.$$

$\square$

We mention in passing that the right-hand sides of (3.7) and (3.9) tend to ∞ as $r \to 0$. We will apply them with fixed $r > 0$.

Corollary 3.5. *Let $(A, D(A))$ be the L^2-generator of the semigroup $(T_t)_{t\ge0}$ corresponding to $\mathcal{E}^{D,\kappa}$. Then $C_c^2(D; \mathbb{R}^d) \subset D(A)$ and $A|_{C_c^2(D;\mathbb{R}^d)} = L^{\mathcal{B}}|_{C_c^2(D;\mathbb{R}^d)}$.*

Proof. Since κ is locally bounded, it suffices to show that, for $u \in C_c^2(D; \mathbb{R}^d)$ and $v \in C_c^2(D)$,

$$\int_D \int_D (u(y) - u(x))(v(y) - v(x)) J^D(x,y)(x,y) \, dy \, dx = 2 \int_D \left(L^{\mathcal{B}} u(x) - \kappa(x) u(x) \right) v(x) \, dx. \tag{3.10}$$

Note that, by the Lebesgue dominated convergence theorem and the symmetry of $\mathcal{B}$,

$$\int_D \int_D (u(y) - u(x))(v(y) - v(x))j(|x-y|)\mathcal{B}(x,y)dydx$$

$$= \lim_{\varepsilon \downarrow 0} \int_D \int_{y \in D: |x-y| > \varepsilon} (u(y) - u(x))(v(y) - v(x))j(|x-y|)\mathcal{B}(x,y)dydx$$

$$= 2 \lim_{\varepsilon \downarrow 0} \int_D \int_{y \in D: |x-y| > \varepsilon} (u(y) - u(x))j(|x-y|)\mathcal{B}(x,y)dy\, v(x)dx$$

$$= 2 \lim_{\varepsilon \downarrow 0} \int_{\mathrm{supp}(v)} \int_{y \in D: |x-y| > \varepsilon} (u(y) - u(x))j(|x-y|)\mathcal{B}(x,y)dy\, v(x)dx. \qquad (3.11)$$

Let $\varepsilon < \varepsilon_0 := \mathrm{dist}(\partial D, \mathrm{supp}(v))/2$. Then by the estimates of I, II and III in the proof of Proposition 3.4 (b), we have that

$$\sup_{x \in \mathrm{supp}(v), \varepsilon < \varepsilon_0} \left| \int_{y \in D: |x-y| > \varepsilon} (u(y) - u(x))j(|x-y|)\mathcal{B}(x,y)dy \right|$$
$$\leq c_1 \left(\varepsilon_0^2 \|\partial^2 u\|_\infty + \varepsilon_0 \|\nabla u\|_\infty + \|u\|_\infty \right) \Phi(\varepsilon_0)^{-1} < \infty.$$

Thus we can use the Lebesgue dominated convergence theorem to conclude that (3.10) holds.
□

Corollary 3.5 says that $L^\mathcal{B}$ is the extended generator of the semigroup $(T_t)_{t \geq 0}$ corresponding to $\mathcal{E}^{D,\kappa}$.

Consider now the process $Y^{D,\kappa,U}$ – the process $Y^{D,\kappa}$ killed upon exiting $U \subset \overline{U} \subset D$. Denote $L_U^\mathcal{B} u := L^{\mathcal{B},U} u - \kappa^U(\cdot)u$, where

$$L^{\mathcal{B},U} u(z) := \mathrm{p.v.} \int_U (u(y) - u(z))J^D(y,z)\, dy - \kappa(z)u(z), \qquad u \in U,$$

and

$$\kappa^U(z) := \int_{D \setminus U} J^D(z,y)\, dy, \qquad z \in U.$$

We can write

$$L_U^\mathcal{B} u(z) = \mathrm{p.v.} \int_U (u(y) - u(z))J^D(y,z)\, dy - \kappa_U(z)u(z), \qquad u \in U,$$

where $\kappa_U = \kappa + \kappa^U$. If U is a $C^{1,1}$-open set, then $\kappa^U(z) \leq c_1 \frac{1}{\Phi(\delta_U(z))}$. Since κ is bounded in U we have that $\kappa_U(z) \leq c_2 \frac{1}{\Phi(\delta_U(z))}$. Thus $L_U^\mathcal{B}$ has the same form in U as $L^\mathcal{B}$ in D. Hence, the analog of Proposition 3.4 is valid. In particular, if $u \in C_c^2(U)$, then $\|L_U^\mathcal{B} u\|_\infty < \infty$.

Further, if $u \in C_c^2(U) \subset C_c^2(D)$, then for $z \in U$,

$$L^\mathcal{B} u(z) = \lim_{\epsilon \to 0} \int_{D, |y-z| < \epsilon} (u(z) - u(y))J^D(y,z)dy - \kappa(z)u(z)$$

$$= \lim_{\epsilon \to 0} \int_{U, |y-z| < \epsilon} (u(z) - u(y))J^D(y,z)dy + \lim_{\epsilon \to 0} \int_{D \setminus U, |y-z| < \epsilon} (u(z) - u(y))J^D(y,z)dy - \kappa(z)u(z)$$

$$= L^{\mathcal{B},U} u(z) - \kappa^U(z)u(z) - \kappa(z)u(z) = L_U^\mathcal{B} u(z).$$

Recall that our basic process is $Y^{D,\kappa} = (Y_t^{D,\kappa}, \mathbb{P}_x, x \in D)$. The part of this process in $U \subset \overline{U} \subset D$ is $Y^{D,\kappa,U} = (Y_t^{D,\kappa,U}, \mathbb{P}_x, x \in U)$.

Lemma 3.6. *Suppose that U is a $C^{1,1}$-open set and $U \subset \overline{U} \subset D$. For any $u \in C_c^2(U)$ and any $x \in U$,*

$$M_t^u := u(Y_t^{D,\kappa,U}) - u(Y_0^{D,\kappa,U}) - \int_0^t L_U^{\mathcal{B}} u(Y_s^{D,\kappa,U})\,ds \qquad (3.12)$$

is a $\mathbb{P}_x$-martingale with respect to the filtration of $Y^{D,\kappa,U}$.

Proof. We follow the proof of [11, Lemma 2.2]. If $(A, \mathcal{D}(A))$ denotes the L^2-generator of the semigroup T_t of $Y^{D,\kappa,U}$, then using an argument similar to that used in Proposition 3.4 and Corollary 3.5 we get that $C_c^2(U) \subset \mathcal{D}(A)$ and $A_{|C_c^2(U)} = (L_U^{\mathcal{B}})_{|C_c^2(U)}$. Then in the same way as in the proof of [11, Lemma 2.2] we get that

$$T_t u(x) - u(x) = \int_0^t T_s L_U^{\mathcal{B}} u(x)\,ds \qquad \text{a.e. } x \in U\,.$$

Moreover, by the display before this lemma we know that $L_U^{\mathcal{B}} u$ is bounded in U. Also, by Lemma 3.1, $Y^{D,\kappa,U}$ is strongly Feller. Now we can follow word-by-word the second part of the proof of [11, Lemma 2.2] to get the desired conclusion. $\square$

Recall that, for an open set $U \subset D$, $\tau_U = \tau_U^{Y^{D,\kappa}} = \inf\{t > 0 : Y_t^{D,\kappa} \notin U\}$.

Note that $L_U^{\mathcal{B}} u(Y_s^{D,\kappa,U}) = L^{\mathcal{B}} u(Y_s^{D,\kappa})\mathbf{1}_{s<\tau_U \wedge \zeta} = L^{\mathcal{B}} u(Y_s^{D,\kappa})\mathbf{1}_{s<\tau_U}$ and that $u(Y_t^{D,\kappa,U}) = u(Y_t^{D,\kappa})\mathbf{1}_{t<\tau_U}$. Thus we can rewrite (3.12) as

$$M_t^u = u(Y_t^{D,\kappa})\mathbf{1}_{t<\tau_U} - u(Y_0^{D,\kappa}) - \int_0^{t\wedge\tau_U} L^{\mathcal{B}} u(Y_s^{D,\kappa})\,ds\,. \qquad (3.13)$$

Proposition 3.7. *For $u \in C_c^2(D)$ and $x \in D$,*

$$M_t^u := u(Y_t^{D,\kappa}) - u(Y_0^{D,\kappa}) - \int_0^t L^{\mathcal{B}} u(Y_s^{D,\kappa})\,ds$$

is a $\mathbb{P}_x$-martingale.

Proof. Let (U_n) be an increasing sequence of relatively compact $C^{1,1}$ open subsets of D such that $x \in U_1$, $\mathrm{supp}(u) \subset U_1$ and $U_n \uparrow D$. Let $M^{(n)}$ denote the martingale in (3.13) with $U = U_n$, that is,

$$M_t^{(n)} = u(Y_t^{D,\kappa})\mathbf{1}_{t<\tau_{U_n}} - u(Y_0^{D,\kappa}) - \int_0^{t\wedge\tau_{U_n}} L^{\mathcal{B}} u(Y_s^{D,\kappa})\,ds\,.$$

By using that both u and $L^{\mathcal{B}} u$ are bounded (by Proposition 3.4 (c)), we conclude that M_t^u is a $\mathbb{P}_x$-martingale. $\square$

Let $u \in C_c^2(D)$. Consider $y \in D$ and $r > 0$ such that $B(y, 2r) \subset D$. Let $\tau = \tau_{B(y,r)}$. Since u and $L^{\mathcal{B}} u$ are bounded, applying the optional stopping theorem to M_t^u we get

$$\mathbb{E}_y[u(Y_{t\wedge\tau}^{D,\kappa})] - u(y) = \mathbb{E}_y\left[\int_0^{t\wedge\tau} L^{\mathcal{B}} u(Y_s^{D,\kappa})\,ds\right]\,.$$

Note that for $s < \tau$ it holds that $\delta_D(Y_s^{D,\kappa}) > r$ $\mathbb{P}_y$-a.s. Now Proposition 3.4 implies that $|L^{\mathcal{B}} u(Y_s^{D,\kappa})| \le C\Phi(r)^{-1}$ for $s < \tau$. Therefore,

$$\mathbb{E}_y[u(Y_{t\wedge\tau_{B(y,r)}}^{D,\kappa})] - u(y) \le C\Phi(r)^{-1}t, \quad \text{for all } r \in (0, 1] \text{ and } B(y, 2r) \subset D. \qquad (3.14)$$

3.3. **Harnack's inequality.** In this subsection we assume that **(B1)**-**(B5)** and (1.3)-(1.5) hold true.

For any $x \in D$ and Borel subset A of D_∂, we define $N(x, A) = \int_{A \cap D} J^D(x, y) dy + \kappa(x) \mathbf{1}_A(\partial)$. Then it is known that (N, t) is a Lévy system for $Y^{D,\kappa}$ (cf. [10, Theorem 5.3.1] and the argument in [6, p.40]), that is, for any non-negative Borel function f on $D \times D_\partial$ vanishing on the diagonal and any stopping time T,

$$\mathbb{E}_x \sum_{s \leq T} f(Y_{s-}^{D,\kappa}, Y_s^{D,\kappa}) = \mathbb{E}_x \left(\int_0^T \int_{D_\partial} f(Y_s^{D,\kappa}, y) N(Y_s^{D,\kappa}, dy) ds \right), \quad x \in D.$$

The following lemma is a simple consequence of Assumptions **(B3)** and **(B5)**.

Lemma 3.8.   (a) *For all $\varepsilon \in (0, 1)$, $x_0 \in D$ and $r > 0$ such that $B(x_0, (1 + \varepsilon)r) \subset D$, we have*

$$C_3(\varepsilon/2) j(|x - y|) \leq J^D(x, y), \qquad x, y \in B(x_0, r). \tag{3.15}$$

(b) *For any $\varepsilon \in (0, 1]$, there exists a constant $= C_{11}(\varepsilon) \geq 1$ such that for all $x_0 \in D$ and all $r > 0$ satisfying $B(x_0, (1 + \varepsilon)r) \subset D$, it holds that*

$$J^D(z, x_1) \leq C_{11} J^D(z, x_2), \quad x_1, x_2 \in B(x_0, r), \quad z \in D \setminus B(x_0, (1 + \varepsilon)r). \tag{3.16}$$

Proof. (a) For $x, y \in B(x_0, r)$, we have $\delta_D(x) \wedge \delta_D(y) \geq \varepsilon r$ and $|x - y| \leq 2r$. Hence, $\delta_D(x) \wedge \delta_D(y) \geq (\varepsilon_0/2)|x - y|$. Thus by **(B3)**, $J^D(x, y) = \mathcal{B}(x, y) j(|x - y|) \geq C_3 j(|x - y|)$, where $C_3 = C_3(\varepsilon/2)$.
(b) By using **(B5)** we have $J^D(z, x_1) = \mathcal{B}(z, x_1) j(z, x_1) \leq C_5 \mathcal{B}(z, x_2) j(z, x_1)$. Since $|x_1 - z| \leq |x_2 - z| + |x_1 - x_2| \leq |x_2 - z| + 2r \leq |x_2 - z| + (2/\varepsilon)|x_2 - z| = (1 + 2\varepsilon)|x_2 - z|$, it follows from (1.4) that $j(|z - x_1|) \leq c_1 j(|z - x_2|)$. This proves the estimate. $\qquad\square$

Lemma 3.9. *There exists a constant $C_{12} > 0$ such that for all $x \in D$ and $r > 0$ with $B(x, 2r) \subset D$, it holds that*

$$\mathbb{P}_x(\tau_{B(x,r)} < t \wedge \zeta) \leq C_{12} \Phi(r)^{-1} t.$$

Proof. Let $x \in D$ and $r > 0$ be such that $B(x, 2r) \subset D$. Let $f : \mathbb{R}^d \to [-1, 0]$ be a C^2 function such that $f(z) = -1$ for $|z| \leq 1/2$, $f(y) = 0$ for $|z| \geq 1$ and that $\|\nabla f\|_\infty + \|\partial^2 f\|_\infty =: c_1 < \infty$. Define

$$f_r(y) := f\left(\frac{y - x}{r}\right).$$

Then $f_r \in C_c^2(D)$, $f_r(y) = -1$ for $y \in B(x, r/2)$ and $f_r(y) = 0$ for $y \in D \setminus B(x, r)$. Moreover, $\|\nabla f_r\|_\infty \leq c_1/r$, $\|\partial^2 f_r\|_\infty \leq c_1/r^2$. Using these and applying Proposition 3.4 (c), we get $\|L^{\mathcal{B}} f_r\|_\infty \leq c_2 \Phi(r)^{-1}$.

Since $f_r \in C_c^2(D)$, by Proposition 3.7 that

$$f_r(Y_t^{D,\kappa}) - f_r(Y_0^{D,\kappa}) - \int_0^t L^{\mathcal{B}} f_r(Y_s^{D,\kappa}) \, ds$$

is a $\mathbb{P}_y$-martingale for every $y \in D$. Thus for any $y \in B(x, r/2)$ and any stopping time S with $\mathbb{E}_y[S] < \infty$,

$$\mathbb{P}_y(|Y_S^{D,\kappa} - x| \geq r, S < \zeta) = \mathbb{E}_y[1 + f_r(Y_S^{D,\kappa}), |Y_S^{D,\kappa} - x| \geq r, S < \zeta)$$

$$\leq \mathbb{E}_y[1 + f_r(Y_S^{D,\kappa})] = -f_r(y) + \mathbb{E}_y[f_r(Y_S^{D,\kappa})] = \mathbb{E}_y\left[\int_0^S L^{\mathcal{B}} f_r(Y_s^{D,\kappa}) \, ds\right]$$

$$\leq \|L^{\mathcal{B}} f_r\|_\infty \mathbb{E}_y[S] \leq c_2 \Phi(r)^{-1} \mathbb{E}_y[S]. \tag{3.17}$$

The first inequality in the second line follows because $1 + f_r \geq 0$. Note that here $f_r(Y_S^{D,\kappa})$ makes sense regardless whether $S < \zeta$ or not (by definition $f_r(\partial) = 0$).

Now take $S = \tau_{B(x,r)} \wedge t$ and notice that $\{|Y_{\tau_{B(x,r)}\wedge t}^{D,\kappa} - x| \geq r\} = \{\tau_{B(x,r)} < t\}$. Indeed, $|Y_{\tau_{B(x,r)}\wedge t}^{D,\kappa} - x| \geq r$ implies that $\tau_{B(x,r)} \leq t$. Consequently we have that

$$\left\{|Y_{\tau_{B(x,r)}\wedge t}^{D,\kappa} - x| \geq r, \tau_{B(x,r)} \wedge t < \zeta\right\} = \left\{\tau_{B(x,r)} < t \wedge \zeta\right\}.$$

Now it follows form (3.17) that for $y \in B(x,r)$ we have

$$\mathbb{P}_y\big(\tau_{B(x,r)} < t \wedge \zeta\big) = \mathbb{P}_y\big(|Y_{\tau_{B(x,r)}\wedge t}^{D,\kappa} - x| \geq r, \tau_{B(x,r)}\wedge t < \zeta\big) \leq c_2\Phi(r)^{-1}\mathbb{E}_y[\tau_{B(x,r)}\wedge t] \leq c_2\Phi(r)^{-1}t.$$

$\square$

Lemma 3.10. *For all $x \in D$ and all $r > 0$ such that $B(x, 2r) \subset D$, it holds that $\mathbb{P}_x(\tau_{B(x,r)} = \zeta < t) \leq C_{13}\Phi(r)^{-1}t$, where $C_{13} = C_1$.*

Proof. By the Lévy system formula,

$$\mathbb{P}_x\big(\tau_{B(x,r)} = \zeta < t\big) = \mathbb{E}_x \sum_{s<t} \mathbf{1}_{B(x,r)\times\{\partial\}}(Y_{s-}^{D,\kappa}, Y_s^{D,\kappa}) = \mathbb{E}_x \int_0^t \mathbf{1}_{B(x,r)}(Y_s^{D,\kappa})\kappa(Y_s^{D,\kappa})ds.$$

Since $\kappa(y) \leq C_1/\Phi(r)$ for $y \in B(x,r)$ by (1.5), we immediately get $\mathbb{P}_x(\tau_{B(x,r)} = \zeta < t) \leq C_1\Phi(r)^{-1}t$.

$\square$

Recall that $A(x, r_1, r_2)$ denotes the annulus $\{y \in \mathbb{R}^d : r_1 \leq |y - x| < r_2\}$.

Proposition 3.11. *(a) There exists a constant $C_{14} > 0$ depending on $a_1, a_2, \delta_1, \delta_2$ such that for all $x_0 \in D$ and $r > 0$ with $B(x_0, r) \subset D$, it holds that*

$$\mathbb{E}_x\tau_{B(x_0,r)} \geq C_{14}\Phi(r), \quad x \in B(x_0, r/3).$$

(b) For every $\varepsilon > 0$, there exists $C_{15} > 0$ depending on ε, a_1, a_2, δ_1, δ_2, C_3, C_5 such that for all $x_0 \in D$ and all $r > 0$ satisfying $B(x_0, (1+\varepsilon)r) \subset D$, it holds that

$$\mathbb{E}_x\tau_{B(x_0,r)} \leq C_{15}\Phi(r), \quad x \in B(x_0, r).$$

Proof. (a) Let $x \in D$ and $r > 0$ be such that $B(x, r) \subset D$. It follows from Lemmas 3.9–3.10 and (1.4) that

$$\mathbb{P}_x\big(\tau_{B(x,r/3)} < t\big) \leq c_1\Phi(r)^{-1}t.$$

Therefore,

$$\mathbb{E}_x\tau_{B(x,r/3)} \geq t\mathbb{P}_x\big(\tau_{B(x,r/3)} \geq t\big) \geq t(1 - c_1\Phi(r)^{-1}t)$$

for all $t > 0$. Choose $t = \Phi(r)/(2c_1)$, so that $1 - c_1\Phi(r)^{-1}t = 1/2$. Then

$$\mathbb{E}_x\tau_{B(x,r/3)} \geq \frac{1}{2}\Phi(r)/(2c_1) = c_2\Phi(r).$$

Now let $B(x_0, r) \subset D$ and $x \in B(x_0, r/3)$. Then $B(x, r/2) \subset B(x_0, r) \subset D$. By what was proven above,

$$\mathbb{E}_x\tau_{B(x_0,r)} \geq \mathbb{E}_x\tau_{B(x,r/6)} \geq c_2\Phi(r/2) \geq c_3\Phi(r).$$

(b) Let $\varepsilon_0 := \varepsilon/3$, $x_0 \in D$ and $r > 0$ so that $B(x_0, (1+3\varepsilon_0)r) \subset D$. For $x \in B(x_0, r)$, by using Lemma 3.8 (b) in the third line,

$$\begin{aligned}
1 &\geq \mathbb{P}_x\big(Y_{\tau_{B(x_0,r)}}^{D,\kappa} \in A(x_0, (1+\varepsilon_0)r, (1+2\varepsilon_0)r)\big) \\
&= \mathbb{E}_x \int_0^{\tau_{B(x_0,r)}} \int_{A(x_0,(1+\varepsilon_0)r,(1+2\varepsilon_0)r)} J^D(u, Y_s^{D,\kappa}) \, du \, ds
\end{aligned}$$

$$\geq \; C_{11}^{-1}c_1\mathbb{E}_x\tau_{B(x_0,r)}\int_{A(x_0,(1+\varepsilon_0)r,(1+2\varepsilon_0)r)} J^D(u,x_0)\,du. \tag{3.18}$$

For $u \in A(x_0,r,(1+2\varepsilon_0)r)$, we have by Lemma 3.8 (a) that $J^D(u,x_0) \geq C_{11}j(|u-x_0|)$. Therefore, by (1.3) and (1.4),

$$\int_{A(x_0,(1+\varepsilon_0)r,(1+2\varepsilon_0)r)} J^D(u,x_0)\,du \;\geq\; C_{11}\int_{A(x_0,(1+\varepsilon_0)r,(1+2\varepsilon_0)r)} j(|u-x_0|)\,du$$

$$\geq\; c_3\int_{(1+\varepsilon_0)r}^{(1+2\varepsilon_0)r} \frac{1}{t\Phi(t)}\,dt \geq c_4\frac{1}{\Phi(r)}\,.$$

By inserting this in (3.18) we obtain $1 \geq c_5\mathbb{E}_x\tau_{B(x_0,r)}/\Phi(r)$, which is the required inequality.
$\square$

Lemma 3.12. *There exists $C_{16} > 0$ such that for all $x \in D$ and $r > 0$ with $B(x,5r) \subset D$, and any Borel $A \subset B(x,r)$,*

$$\mathbb{P}_y(T_A < \tau_{B(x,3r)}) \geq C_{16}\frac{|A|}{|B(x,r)|}\,, \qquad y \in B(x,2r).$$

Proof. Without loss of generality assume that $\mathbb{P}_y(T_A < \tau_{B(x,3r)}) < 1/4$. Set $\tau = \tau_{B(x,3r)}$. For $y \in B(x,2r)$ we have that $B(y,3r) \subset D$. Hence by Lemmas 3.9 and 3.10, for every $y \in B(x,2r)$, $\mathbb{P}_y(\tau < t) \leq \mathbb{P}_y(\tau_{B(y,r)} < t) \leq c_1\Phi(r)^{-1}t$. Choose $t_0 = \Phi(r)/(4c_1)$, so that $\mathbb{P}_y(\tau < t_0) \leq 1/4$. Further, if $z \in B(x,3r)$ and $u \in A \subset B(x,r)$, then $|u-z| \leq 4r$. Since j is decreasing, $j(|u-z|) \geq j(4r)$. Moreover, $\delta_D(u) \wedge \delta_D(z) \geq r \geq (1/4)|u-z|$, implying by **(B3)** that $\mathcal{B}(u,z) \geq C_3(1/4) = C_3$. Thus,

$$\mathbb{P}_y(T_A < \tau) \geq \mathbb{E}_y\sum_{s \leq T_A \wedge \tau \wedge t_0}\mathbf{1}_{\{Y_{s-}^{D,\kappa}\neq Y_s, Y_s^{D,\kappa}\in A\}}$$

$$= \mathbb{E}_y\int_0^{T_A\wedge\tau\wedge t_0}\int_A j(|u-Y_s^{D,\kappa}|)\mathcal{B}(u,Y_s^{D,\kappa})\,du\,ds \geq C_3 j(4r)|A|\mathbb{E}_y[T_A \wedge \tau \wedge t_0]\,,$$

where in the second line we used properties of the Lévy system. Next,

$$\mathbb{E}_y[T_A \wedge \tau \wedge t_0] \geq t_0\mathbb{P}_y(T_A \geq \tau \geq t_0) \geq t_0[1 - \mathbb{P}_y(T_A < \tau) - \mathbb{P}_y(\tau < t_0)] \geq \frac{t_0}{2} = \frac{\Phi(r)}{8c_1}\,.$$

The last two displays give that

$$\mathbb{P}_y(T_A < \tau) \geq C_3 j(4r)|A|\frac{\Phi(r)}{8c_1} \geq c_2\frac{1}{r^d\Phi(r)}|A|\Phi(r) \geq c_3\frac{|A|}{|B(x,r)|}\,.$$

$\square$

Lemma 3.13. *There exist $C_{17} > 0$ and $C_{18} > 0$ with the property that if $r > 0$, $x \in D$ are such that $B(x,2r) \subset D$, and H is a bounded non-negative function with support in $D \setminus B(x,2r)$, then for every $z \in B(x,r)$,*

$$C_{17}\mathbb{E}_z[\tau_{B(x,r)}]\int H(y)J^D(x,y)\,dy \leq \mathbb{E}_z H(Y_{\tau_{B(x,r)}}^{D,\kappa}) \leq C_{18}\mathbb{E}_z[\tau_{B(x,r)}]\int H(y)J^D(x,y)\,dy\,.$$

Proof. Let $y \in B(x,r)$ and $u \in D \setminus B(x,2r)$. By **(B5)** (or Lemma 3.8 (b)), $J^D(u,y) \asymp J^D(x,y)$. Thus using the Lévy system we get

$$\mathbb{E}_z\left[H(Y_{\tau_{B(x,r)}}^{D,\kappa})\right] = \mathbb{E}_z\int_0^{\tau_{B(x,r)}}\int H(u)J^D(u,Y_s^{D,\kappa})\,du\,ds \asymp \mathbb{E}_z\int_0^{\tau_{B(x,r)}}\int H(u)J^D(u,x)\,du\,ds.$$

$\square$

Proof of Theorem 1.1. (a) Proposition 3.11 (a)-(b), Lemmas 3.12 and 3.13 imply that conditions (A1), (A2) and (A3) of [16] are satisfied for the process $Y^{D,\kappa}$. Thus we can repeat the proofs of [16, Theorems 2.2 and 2.4] to finish the proof of part (a). Note that conservativeness does not play any role. We omit the details.

(b) By (a) we can and will assume that $L > 2$ and $2r < |x_1 - x_2| < Lr$. For simplicity, let $B_i = B(x_i, r)$, $i = 1, 2$. Then by using harmonicity in the first inequality, Theorem 1.1.(a) in the second inequality, and the Lévy system formula in the second line, we have

$$
\begin{aligned}
f(x_1) &\geq \mathbb{E}_{x_1}\big[f(Y^{D,\kappa}_{\tau_{B_1}}); Y^{D,\kappa}_{\tau_{B_1}} \in B(x_2, r/2)\big] \geq C_{11}^{-1} f(x_2)\mathbb{P}_{x_1}\big(Y^{D,\kappa}_{\tau_{B_1}} \in B(x_2, r/2)\big) \\
&= C_{11}^{-1} f(x_2)\mathbb{E}_{x_1} \int_0^{\tau_{B_1}} \int_{B(x_2,r/2)} J^D(Y^{D,\kappa}_s, z)\, dz\, ds\,.
\end{aligned}
\tag{3.19}
$$

For $y \in B_1$ and $z \in B(x_2, r/2)$ we have by Lemma 3.8.(b) (with $\varepsilon = 1$) that $J^D(y, z) \geq C_{11}^{-1} J^D(x_1, z)$. Further, $\delta_D(x_1) \wedge \delta_D(z) \geq r/2 \geq (2L+2)^{-1}|x_1 - z|$, hence by Lemma 3.8 (a), $J^D(x_1, z) \geq C_3 j(|x_1 - z|)$, where $C_3 = C_3((2L+2)^{-1})$. By inserting this in (3.19), and by using Proposition 3.11 (a) , we obtain

$$
\begin{aligned}
f(x_1) &\geq C_3 f(x_2)\mathbb{E}_x \tau_{B_1} \int_{B(x_2,r/2)} j(|x_1 - z|)\, dz \\
&\geq c_1 C_3 f(x_2)\Phi(r)\frac{1}{((L+1)r)^d \Phi((L+1)r)} |B(x_2, r/2)| \\
&\geq C_3 f(x_2) L^{-d} \frac{\Phi(r)}{\Phi((L+1)r)} \geq c_4 f(x_2) L^{-d} L^{-2\delta_2}\,.
\end{aligned}
$$

The last inequality follows from (1.4). $\qquad\square$

By repeating the arguments of [16, Theorem 4.9] and [2, Theorem 4.1], we immediately get the following result.

Theorem 3.14. *Let f be a non-negative bounded function in D which is harmonic in $B(x_0, r)$ with respect to $Y^{D,\kappa}$. Then there exist $C_{19} > 0$ and $\beta > 0$ such that for any $r \in (0, 1]$ and $B(x_0, r) \subset D$,*

$$
|f(x) - f(y)| \leq C_{19}\|f\|_\infty |x - y|^\beta, \qquad x, y \in B(x_0, r/2).
$$

4. Carleson's estimate

In this section we prove Carleson's estimate. In addition to **(B1)-(B5)** and (1.5), we also assume that **(B6)** and (1.6) hold.

Lemma 4.1. *There exists a constant $\delta_* > 0$ such that for all $x \in D$, $\mathbb{P}_x\big(\tau_{B(x,\delta_D(x)/2)} = \zeta\big) \geq \delta_*$ where ζ is the lifetime of $Y^{D,\kappa}$.*

Proof. Let $x \in D$. By the Lévy system formula, (1.6) and Proposition 3.11 (a),

$$
\begin{aligned}
\mathbb{P}_x\big(\tau_{B(x,\delta_D(x)/2)} = \zeta\big) &\geq \mathbb{P}_x\big(\tau_{B(x,\delta_D(x)/3)} = \zeta\big) = \mathbb{E}_x \int_0^\infty \mathbf{1}_{B(x,\delta_D(x)/3)}(Y^{D,\kappa}_s)\kappa(Y^{D,\kappa}_s)ds \\
&\geq \mathbb{E}_x \int_0^{\tau_{B(x,\delta_D(x)/3)}} \kappa(Y^{D,\kappa}_s)ds \geq \frac{c_3}{\Phi(\delta_D(x)/3)}\mathbb{E}_x \tau_{B(x,\delta_D(x)/3)} \geq c_4.
\end{aligned}
$$

$\qquad\square$

Proof of Theorem 1.2. In this proof, the constants $\delta_*, \nu, \gamma, \chi, \eta$ and c_i's are always independent of r. Let f be a non-negative function on D which is harmonic in $D \cap B(Q, r)$ and vanishes continuously on $\partial D \cap B(Q, r)$. By Theorem 1.1 (b) and a standard chain argument, it suffices to prove (1.7) for $x \in D \cap B(Q, \overline{\kappa}r/(24))$.

Let $\nu := d + 2\delta_2$. Recall that $\widehat{\beta}$ is the constant from Assumption **(B6)**. Choose $0 < \gamma < 2\delta_1/(\nu + \widehat{\beta})$. Fix an $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) \geq \overline{\kappa}r/2$. For any $x \in D \cap B(Q, \overline{\kappa}r/(12))$, define

$$B_0(x) = B(x, \delta_D(x)/2), \qquad B_1(x) = B(x, r^{1-\gamma}\delta_D(x)^\gamma)$$

and

$$B_2 = B(x_0, \overline{\kappa}\delta_D(x_0)/3), \qquad B_3 = B(x_0, 2\overline{\kappa}\delta_D(x_0)/3).$$

Since $x \in B(Q, \overline{\kappa}r/(12))$, we have $\delta_D(x) < r/(12)$. In particular, we have that $B_0(x) \subset B(x, \delta_D(x)) \subset B_1(x)$. By Lemma 4.1, there exists $\delta_* = \delta_*(R, \Lambda) > 0$ such that

$$\mathbb{P}_x(\tau_{B_0(x)}^{Y^{D,\kappa}} = \zeta) \geq \delta_*, \quad x \in D \cap B(Q, \overline{\kappa}r/(12)). \tag{4.1}$$

By Theorem 1.1.(b) there exists $\chi > 0$ such that

$$f(x) < (\delta_D(x)/r)^{-\chi}f(x_0), \quad x \in D \cap B(Q, \overline{\kappa}r/(12)). \tag{4.2}$$

Since f is harmonic in $D \cap B(Q, r)$, for every $x \in D \cap B(Q, \overline{\kappa}r/(12))$,

$$f(x) = \mathbb{E}_x\left[f\left(Y^{D,\kappa}(\tau_{B_0(x)})\right); Y^{D,\kappa}(\tau_{B_0(x)}) \in B_1(x)\right]$$
$$+ \mathbb{E}_x\left[f\left(Y^{D,\kappa}(\tau_{B_0(x)})\right); Y^{D,\kappa}(\tau_{B_0(x)}) \notin B_1(x)\right]. \tag{4.3}$$

We first show that there exists $\eta > 0$ such that for all $x \in D \cap B(Q, \overline{\kappa}r/(12))$ with $\delta_D(x) < \eta r$,

$$\mathbb{E}_x\left[f\left(Y^{D,\kappa}(\tau_{B_0(x)})\right); Y^{D,\kappa}(\tau_{B_0(x)}) \notin B_1(x)\right] \leq f(x_0). \tag{4.4}$$

Step 1: There exists $c_2 > 0$ such that

$$f(x_0) \geq c_2\Phi(r)\int_{D\setminus B_3} J^D(x_0, y)f(y)\, dy. \tag{4.5}$$

Indeed, if $z \in B_2$ and $y \in D\setminus B_3$, then by Lemma 3.8 (b), $J^D(z, y) \geq c_4 J^D(x_0, y)$. By using this estimate in the second inequality below, Proposition 3.11 (a) in the third and $\delta_D(x_0) \geq \kappa r/2$ in the fourth, we get that

$$f(x_0) \geq \mathbb{E}_{x_0}\left[f(Y^{D,\kappa}(\tau_{B_2})); Y^{D,\kappa}(\tau_{B_2}) \notin B_3\right] = \mathbb{E}_{x_0}\int_0^{\tau_{B_2}}\left(\int_{D\setminus B_3} J^D(Y_t^{D,\kappa}, y)f(y)\, dy\right) dt$$

$$\geq c_5\mathbb{E}_{x_0}[\tau_{B_2}]\int_{D\setminus B_3} J^D(x_0, y)f(y)\, dy \geq c_6\Phi(\delta_D(x_0))\int_{D\setminus B_3} J^D(x_0, y)f(y)\, dy$$

$$\geq c_7\Phi(r)\int_{D\setminus B_3} J^D(x_0, y)f(y)\, dy.$$

Step 2: For $z \in B_0(x)$ and $y \in D\setminus B_1(x)$, by Lemma 3.8 (b) it holds that $J^D(z, y) \leq c_8 J^D(x, y)$. By using this in the second line below and Proposition 3.11 (b) in the third, we get

$$\mathbb{E}_x\left[f\left(Y^{D,\kappa}(\tau_{B_0(x)})\right); Y^{D,\kappa}(\tau_{B_0(x)}) \notin B_1(x)\right] = \mathbb{E}_x\int_0^{\tau_{B_0(x)}}\int_{D\setminus B_1(x)} J^D(Y_t^{D,\kappa}, y)f(y)\, dy\, dt$$

$$\leq c_9\mathbb{E}_x\tau_{B_0(x)}\int_{D\setminus B_1(x)} J^D(x, y)f(y)\, dy$$

$$\leq c_{10}\Phi(\delta_D(x))\left(\int_{(D\setminus B_1(x))\cap B_3^c} J^D(x, y)f(y)\, dy + \int_{(D\setminus B_1(x))\cap B_3} J^D(x, y)f(y)\, dy\right)$$

$$=: c_{10}\Phi(\delta_D(x))(I_1 + I_2). \tag{4.6}$$

Step 3: Suppose now that $|y - x| \geq r^{1-\gamma}\delta_D(x)^\gamma$ and $x \in B(Q, \overline{\kappa}r/(12))$. Then

$$|y - x_0| \leq |y - x| + 2r \leq |y - x| + 2r^\gamma\delta_D(x)^{-\gamma}|y - x| \leq 3r^\gamma\delta_D(x)^{-\gamma}|y - x|,$$

and $\delta_D(x) \leq \overline{\kappa}r/12 < \overline{\kappa}r/2 \leq \delta_D(x_0)$. Set $M = 3r^\gamma/\delta_D(x)^\gamma \geq 1$ so that $|y - x_0| \leq M|y - x|$. Thus, by using **(B6)** in the first inequality and (1.4) in the second inequality,

$$J^D(x,y) \leq C_6 M^{\widehat{\beta}} \mathcal{B}(x_0,y) j(M^{-1}|y - x_0|) \leq c_{11} \left(\frac{\delta_D(x)}{r}\right)^{-\gamma(\widehat{\beta}+\nu)} J^D(x_0,y). \qquad (4.7)$$

Now, using this and (4.5) , we get

$$I_1 \leq c_{12} \left(\frac{\delta_D(x)}{r}\right)^{-\gamma(\nu+\widehat{\beta})} \int_{(D\setminus B_1(x))\cap B_3^c} J^D(x_0,y) f(y)\, dy$$

$$\leq c_{13}\Phi(r)^{-1} \left(\frac{\delta_D(x)}{r}\right)^{-\gamma(\nu+\widehat{\beta})} f(x_0). \qquad (4.8)$$

Step 4: If $y \in B_3(x)$, then $\delta_D(y) \leq c_{12}r$ and $|y - x| \geq |x_0 - Q| - |x - Q| - |y - x_0| > \delta_D(x_0)/2$. By Theorem 1.1, there exists $C > 0$ such that $f(y) \leq Cf(x_0)$ for all $y \in B_3(x)$. Thus,

$$I_2 \leq c_{14}f(x_0) \int_{(D\setminus B_1(x))\cap B_3} J^D(x,y)\, dy \leq c_{14}f(x_0) \int_{|y-x|>\delta_D(x_0)/2} J^D(x,y)\, dy$$

$$\leq c_{15}f(x_0) \int_{|y-x|>\delta_D(x_0)/2} j(|x-y|)\, dy \leq c_{16}\Phi(r)^{-1}f(x_0). \qquad (4.9)$$

Step 5: Combining (4.6), (4.8), (4.9), and using (1.4) and the definition of Φ in the last line, we obtain

$$\mathbb{E}_x[f(Y^{D,\kappa}(\tau_{B_0(x)})); Y^{D,\kappa}(\tau_{B_0(x)}) \notin B_1(x)]$$

$$\leq c_{17}f(x_0) \left(\frac{\Phi(\delta_D(x))}{\Phi(r)} \left(\frac{\delta_D(x)}{r}\right)^{-\gamma(\nu+\widehat{\beta})} + \frac{\Phi(\delta_D(x))}{\Phi(r)}\right)$$

$$\leq c_{18}f(x_0) \left(\left(\frac{\delta_D(x)}{r}\right)^{-\gamma(\nu+\widehat{\beta})+2\delta_1} + \left(\frac{\delta_D(x)}{r}\right)^{2\delta_1}\right). \qquad (4.10)$$

Since $2\delta_1 - \gamma(\nu + \widehat{\beta}) > 0$ (by the choice of γ), we can choose $\eta > 0$ so that

$$c_{18} \left(\eta^{2\delta_1-\gamma(\nu+\widehat{\beta})} + \eta^{2\delta_1}\right) \leq 1.$$

Then for $x \in D \cap B(Q, \overline{\kappa}r/(12))$ with $\delta_D(x) < \eta r$, we have by (4.10),

$$\mathbb{E}_x\left[f(Y^{D,\kappa}(\tau_{B_0(x)})); Y^{D,\kappa}(\tau_{B_0(x)}) \notin B_1(x)\right] \leq c_{19}\, f(x_0) \left(\eta^{2\delta_1-\gamma(\nu+\widehat{\beta})} + \eta^{2\delta_1}\right) \leq f(x_0).$$

This completes the proof of (4.4). We now prove Carleson's estimate (1.7) for $x \in D \cap B(Q, \overline{\kappa}r/(24))$ by a method of contradiction. Without loss of generality, we may assume that $f(x_0) = 1$. Suppose that there exists $x_1 \in D \cap B(Q, \overline{\kappa}r/(24))$ such that $f(x_1) \geq K > \eta^{-\chi} \vee (1 + \delta_*^{-1})$, where K is a constant to be specified later. By (4.2) and the assumption $f(x_1) \geq K > \eta^{-\chi}$, we have $(\delta_D(x_1)/r)^{-\chi} > f(x_1) \geq K > \eta^{-\chi}$, and hence $\delta_D(x_1) < \eta r$. By (4.3) and (4.4),

$$K \leq f(x_1) \leq \mathbb{E}_{x_1}\left[f(Y^D(\tau_{B_0(x_1)})); Y^{D,\kappa}(\tau_{B_0(x_1)}) \in B_1(x_1)\right] + 1,$$

and hence

$$\mathbb{E}_{x_1}\left[f(Y^{D,\kappa}(\tau_{B_0(x_1)})); Y^{D,\kappa}(\tau_{B_0(x_1)}) \in B_1(x_1)\right] \geq f(x_1) - 1 > \frac{1}{1+\delta_*}\, f(x_1).$$

In the last inequality of the display above we used the assumption that $f(x_1) \geq K > 1 + \delta_*^{-1}$. If $K \geq (24/\overline{\kappa})^{\chi/\gamma}$, then $(\delta_D(x_1)/r)^\gamma < \overline{\kappa}/(24)$. Thus $\overline{B_1(x_1)} \subset B(Q, \overline{\kappa}r/(12))$. We now get from (4.1) that

$$\mathbb{E}_{x_1}[f(Y^{D,\kappa}(\tau_{B_0(x_1)})), Y^{D,\kappa}(\tau_{B_0(x_1)}) \in B_1(x_1)]$$
$$= \mathbb{E}_{x_1}[f(Y^{D,\kappa}(\tau_{B_0(x_1)})), Y^{D,\kappa}(\tau_{B_0(x_1)}) \in B_1(x_1) \cap D]$$
$$\leq \mathbb{P}_x\left(Y^{D,\kappa}(\tau_{B_0(x_1)}) \in D\right)\left(\sup_{B_1(x_1)} f\right) \leq (1 - \delta_*)\left(\sup_{B_1(x_1)} f\right).$$

Therefore, $\sup_{B_1(x_1)} f > f(x_1)/(1 - \delta_*^2)$, i.e., there exists $x_2 \in D \cap B(Q, \overline{\kappa}r/(12))$ such that

$$|x_1 - x_2| \leq r^{1-\gamma}\delta_D(x_1)^\gamma \quad \text{and} \quad f(x_2) > \frac{1}{1 - \delta_*^2} f(x_1) \geq \frac{1}{1 - \delta_*^2} K.$$

Similarly, if $x_k \in D \cap B(Q, \overline{\kappa}r/(12))$ with $f(x_k) \geq K/(1 - \delta_*^2)^{k-1}$ for $k \geq 2$, then there exists $x_{k+1} \in D$ such that

$$|x_k - x_{k+1}| \leq r^{1-\gamma}\delta_D(x_k)^\gamma \quad \text{and} \quad f(x_{k+1}) > \frac{1}{1 - \delta_*^2} f(x_k) > \frac{1}{(1 - \delta_*^2)^k} K. \tag{4.11}$$

From (4.2) and (4.11) it follows that $(\delta_D(x_k)/r)^\gamma \leq (1 - \delta_*^2)^{\gamma(k-1)/\chi} K^{-\gamma/\chi}$, for every $k \geq 1$. Therefore by this and (4.11),

$$|x_k - Q| \leq |x_1 - Q| + \sum_{j=1}^{k-1} |x_{j+1} - x_j| \leq \frac{\overline{\kappa}r}{24} + r\sum_{j=1}^{\infty}(\delta_D(x_k)/r)^\gamma$$
$$\leq \frac{\overline{\kappa}r}{24} + rK^{-\gamma/\chi}\frac{1}{1 - (1 - \delta_*^2)^{\gamma/\chi}}.$$

Choose

$$K = \eta^{-\chi} \vee (1 + \delta_*^{-1}) \vee [(24/\overline{\kappa})^{\chi/\gamma}(1 - (1 - \delta_*^2)^{\gamma/\chi})^{-\chi/\gamma}]$$

so that $K^{-\gamma/\chi}(1 - (1 - \delta_*^2)^{\gamma/\chi})^{-1} \leq \overline{\kappa}/(24)$. Hence $x_k \in D \cap B(Q, \overline{\kappa}r/(12))$ for every $k \geq 1$. Since $\lim_{k\to\infty} f(x_k) = \infty$ by (4.11), this contradicts the fact that f is bounded on $B(Q, r/2)$. This contradiction shows that $f(x) < K$ for every $x \in D \cap B(Q, \overline{\kappa}r/(24))$. This completes the proof of the theorem. $\square$

5. ESTIMATES IN THE HALF-SPACE

Starting from this section we assume that $D = \mathbb{R}_+^d$ and $j(|x - y|) = |x - y|^{-d-\alpha}$, $d \geq 1$. In the rest of this section, we will only deal with the case $d \geq 2$, the case $d = 1$ being simpler.

For $x \in \mathbb{R}_+^d$, we write $x = (\widetilde{x}, x_d)$, $\widetilde{x} \in \mathbb{R}^{d-1}$, $x_d > 0$, so that $\delta_D(x) = x_d$. We have that $J^{\mathbb{R}_+^d}(x, y) = \mathcal{B}(x, y)|x - y|^{-d-\alpha}$ and note that with this assumption, $\Phi(\lambda) = \lambda^\alpha$ so that $\delta_1 = \delta_2 = \alpha/2$. Further, for $p \in ((\alpha - 1)_+, \alpha + \beta_1)$, let $\kappa(x) := C(\alpha, p, \mathcal{B})x_d^{-\alpha}$, where $C(\alpha, p, \mathcal{B})$ is the constant defined in (1.9). For simplicity, we denote the process $Y^{\mathbb{R}_+^d, \kappa}$ by Y. Thus, Y is the process associated with the Dirichlet form

$$\mathcal{E}(u, v) = \frac{1}{2}\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d}\frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}}\mathcal{B}(x, y)dydx + C(\alpha, p, \mathcal{B})\int_{\mathbb{R}_+^d}u(x)v(x)x_d^{-\alpha}dx.$$

We first record the scaling property of Y here. For any $r > 0$, define a process $Y^{(r)}$ by $Y_t^{(r)} := rY_{r^{-\alpha}t}$.

Lemma 5.1. *Assume that **(B1)**, **(B2)** and **(B8)** hold. Then $(Y^{(r)}, \mathbb{P}_{rx})$ has the same law as $(Y, \mathbb{P}_x)$.*

Proof. For any function f, define $f^{(r)}(x) = f(rx)$. Let $P_t^{(r)}$ be the semigroup of $Y^{(r)}$. Then

$$P_t^{(r)} f(x) = \mathbb{E}_x^{(r)} f(Y_t^{(r)}) = \mathbb{E}_{x/r} f(r Y_{r^{-\alpha} t}) = P_{r^{-\alpha} t} f^{(r)}(x/r).$$

Thus for any nice functions f and g,

$$\frac{1}{t} \int_{\mathbb{R}_+^d} (P_t^{(r)} f(x) - f(x)) g(x) dx = r^{d-\alpha} \frac{1}{r^{-\alpha} t} \int_{\mathbb{R}_+^d} (P_{r^{-\alpha} t} f^{(r)}(x/r) - f^{(r)}(x/r)) g^{(r)}(x/r) dx.$$

Since by **(B8)** $\mathcal{B}(x/r, y/r) = \mathcal{B}(x, y)$, by letting $t \downarrow 0$ we get

$$\mathcal{E}^{(r)}(f, g) = r^{d-\alpha} \mathcal{E}(f^{(r)}, g^{(r)})$$

$$= r^{d-\alpha} \left(\frac{1}{2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{(f^{(r)}(x) - f^{(r)}(y))(g^{(r)}(x) - g^{(r)}(y))}{|x - y|^{d+\alpha}} \mathcal{B}(x, y) dy dx \right.$$

$$\left. + C(\alpha, p, \mathcal{B}) \int_{\mathbb{R}_+^d} f^{(r)}(x) g^{(r)}(x) x_d^{-\alpha} dx \right)$$

$$= r^{d-\alpha} \left(\frac{1}{2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{(f(rx) - f(ry))(g(rx) - g(ry))}{|x - y|^{d+\alpha}} \mathcal{B}(x, y) dy dx \right.$$

$$\left. + C(\alpha, p, \mathcal{B}) \int_{\mathbb{R}_+^d} f(rx) g(rx) x_d^{-\alpha} dx \right)$$

$$= \frac{1}{2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \mathcal{B}(x, y) dy dx + C(\alpha, p, \mathcal{B}) \int_{\mathbb{R}_+^d} f(x) g(x) x_d^{-\alpha} dx = \mathcal{E}(f, g).$$

Thus $(Y^{(r)}, \mathbb{P}_{rx})$ has the same law as $(Y, \mathbb{P}_x)$. $\qquad\square$

An easy consequence of Lemma 5.1 is the following: Let V be an open subset of $\mathbb{R}_+^d$, and for $r > 0$ denote $rV := \{rx : x \in V\}$. If $\tau_V = \inf\{t > 0 : Y_t \notin V\}$, then

$$\mathbb{E}_{rx} \tau_{rV} = r^{\alpha} \mathbb{E}_x \tau_V, \qquad x \in V. \tag{5.1}$$

Throughout the remainder of this section and below we always assume that $\mathcal{B}$ satisfies **(B1)**, **(B2)** *and* **(B7)-(B8)**. Note that **(B8)** implies that $x \mapsto \mathcal{B}(x, x)$ is a constant on D. Without loss of generality, will assume that $\mathcal{B}(x, x) = 1$. Additional assumptions on $\mathcal{B}$ will be imposed later when needed.

The following observation will be used several times in the sequel. Recall the function $\widetilde{B} = \widetilde{B}_{\beta_1, \beta_2, \beta_3, \beta_4}$ appearing in **(B7)**. We note that if $\beta_4 > 0$, then, for any $\varepsilon \in (0, \beta_2)$, there exists $c_\varepsilon > 0$ such that

$$(\log 2)^{-\beta_4} \widetilde{B}_{\beta_1, \beta_2, \beta_3, 0}(x, y) \le \widetilde{B}_{\beta_1, \beta_2, \beta_3, \beta_4}(x, y) \le c_\varepsilon \widetilde{B}_{\beta_1, \beta_2 - \varepsilon, \beta_3, 0}(x, y). \tag{5.2}$$

We first establish some auxiliary estimates that will be useful throughout the rest of the paper. Recall $\log(1 + a)^\gamma = (\log(1 + a))^\gamma$.

Lemma 5.2.   (a) *There exists a constant $C_{20} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \ge x_d$, it holds that*

$$\mathcal{B}(x, y) \le C_{20} x_d^{\beta_1} (|\log x_d|^{\beta_3} \vee 1)(1 + \mathbf{1}_{|y| \ge 1} (\log |y|)^{\beta_3}) |x - y|^{-\beta_1}. \tag{5.3}$$

(b) *For all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \ge x_d$, $x_d \le e^{-1}$ and $|y| \le M$, it holds that*

$$\mathcal{B}(x, y) \le C_{20}(1 + (\log M)^{\beta_3}) x_d^{\beta_1} |\log x_d|^{\beta_3} |x - y|^{-\beta_1}.$$

Proof. By using (5.2), without loss of generality, throughout the proof we assume that $\beta_4 = 0$.

(a) Since $|x - y| \geq x_d$ we get that

$$\frac{(x_d \vee y_d) \wedge |x - y|}{x_d \wedge y_d \wedge |x - y|} \leq \frac{x_d \vee y_d}{x_d \wedge y_d}.$$

Thus it follows from our assumptions on $\mathcal{B}$ that

$$\mathcal{B}(x, y) \leq c_1 \left(\frac{x_d}{|x - y|}\right)^{\beta_1} \log\left(1 + \frac{x_d \vee y_d}{x_d \wedge y_d}\right)^{\beta_3}. \tag{5.4}$$

If $x_d/2 \leq y_d \leq 2x_d$, then the log term above is estimated from above by a constant, implying (5.3).

Note that for $a, b > 0$ with $a \geq 2b$, it holds that $\log(1 + a/b) \leq 2\log(a/b)$. Using this and (5.4) we see that if $y_d \geq 2x_d$ or $y_d \leq x_d/2$, then

$$\mathcal{B}(x, y) \leq c_2 \left(\frac{x_d \wedge y_d}{|x - y|}\right)^{\beta_1} \left|\log\left(\frac{y_d}{x_d}\right)\right|^{\beta_3} \leq c_3 \left(\frac{x_d}{|x - y|}\right)^{\beta_1} \left(|\log x_d|^{\beta_3} + |\log y_d|^{\beta_3}\right).$$

We now give the the proof for the case $y_d \geq 2x_d$ and the case $y_d \leq x_d/2$ separately.

Case 1. $y_d \geq 2x_d$: If $y_d < 1$, then $|\log y_d| \leq |\log x_d|$, so $\mathcal{B}(x, y) \leq 2c_3 x_d^{\beta_1} |\log x_d|^{\beta_3} |x - y|^{-\beta_1}$.

If $y_d \geq 1$, we use the estimate $\log y_d \leq \log|y|$ to get

$$\mathcal{B}(x, y) \leq c_4 x_d^{\beta_1} \left(|\log x_d|^{\beta_3} + (\log|y|)^{\beta_3}\right) |x - y|^{-\beta_1}.$$

Therefore

$$\begin{aligned}
\mathcal{B}(x, y) &\leq c_5 x_d^{\beta_1} \left(|\log x_d|^{\beta_3} + \mathbf{1}_{|y|\geq 1}(\log|y|)^{\beta_3}\right) |x - y|^{-\beta_1} \\
&\leq c_5 x_d^{\beta_1} (|\log x_d|^{\beta_3} \vee 1)\left(1 + \mathbf{1}_{|y|\geq 1}(\log|y|)^{\beta_3}\right)|x - y|^{-\beta_1},
\end{aligned}$$

which is precisely (5.3).

Case 2. $y_d \leq x_d/2$: It is straightforward to see that $f(t) = t^{\beta_1}((\log(1/t))^{\beta_3} \vee 1)$ is almost increasing on $(0, \infty)$. Therefore, there exists $c_6 > 0$ such that $y_d^{\beta_1}|\log y_d|^{\beta_3} \leq c_6 x_d^{\beta_1}(|\log x_d|^{\beta_3} \vee 1)$. Hence, $\mathcal{B}(x, y) \leq c_7 x_d^{\beta_1}(|\log x_d|^{\beta_3} \vee 1)|x - y|^{-\beta_1}$, implying (5.3).

(b) This follows immediately from part (a). $\qquad \square$

Lemma 5.3. *Let $q < \alpha + \beta_1$.*

(a) *For all $x \in \mathbb{R}_+^d$, it holds that*

$$\int_{\mathbb{R}_+^d, |y - x| \geq 2|x|} \frac{|y|^q}{|x - y|^{d+\alpha}} \mathcal{B}(x, y) \, dy < \infty. \tag{5.5}$$

(b) *For every $b > 0$, there exists a constant $C_{21} = C_{21}(b) > 0$ such that, for all $x \in \mathbb{R}_+^d$, it holds that*

$$\int_{\mathbb{R}_+^d, |y| \geq (1+b)|x|} \frac{|y|^q}{|x - y|^{d+\alpha}} \mathcal{B}(x, y) \, dy$$

$$\leq C_{21}(b) x_d^{\beta_1}(|\log x_d|^{\beta_3} \vee 1) \int_{|y| \geq |x|} |y|^{-\beta_1 + q - d - \alpha}\left(1 + \mathbf{1}_{|y|\geq 1}(\log|y|)^{\beta_3}\right) dy < \infty. \tag{5.6}$$

Proof. First note that since $q < \alpha + \beta_1$, it holds that

$$\int_{|y| \geq |x|} |y|^{-\beta_1 + q - d - \alpha}\left(1 + \mathbf{1}_{|y|\geq 1}(\log|y|)^{\beta_3}\right) dy < \infty.$$

Further, if $|y - x| \geq |2|x|$ or $|y| \geq (1+b)|x|$, then $|y - x| \asymp |y|$. Now the conclusion for both (a) and (b) follows immediately from Lemma 5.2. $\qquad \square$

For $f : \mathbb{R}_+^d \to \mathbb{R}$, we set for $x \in \mathbb{R}_+^d$,

$$L_\alpha^\mathcal{B} f(x) := \text{p.v.} \int_{\mathbb{R}_+^d} \frac{f(y) - f(x)}{|y - x|^{d+\alpha}} \mathcal{B}(x, y) \, dy = \text{p.v.} \int_{\mathbb{R}_+^d} (f(y) - f(x)) J^{\mathbb{R}_+^d}(x, y) \, dy \,, \qquad (5.7)$$

whenever the principal value integral on the right hand side makes sense.

For $q > 0$, let $g_q(y) := y_d^q = \delta_{\mathbb{R}_+^d}(y)^q$. Recall that $C(\alpha, p, \mathcal{B})$ is given in (1.9). The following is the analog of [3, (5.4)].

Lemma 5.4. *Let* $p \in ((\alpha - 1)_+, \alpha + \beta_1)$*. Then* $C(\alpha, p, \mathcal{B})$ *is finite and*

$$L_\alpha^\mathcal{B} g_p(x) = C(\alpha, p, \mathcal{B}) x_d^{p-\alpha}, \quad x \in \mathbb{R}_+^d.$$

Proof. Recall $\mathbf{e}_d = (\widetilde{0}, 1)$. By **(B8)**, we can for simplicity take $x = (\widetilde{0}, x_d)$.

By the change of variables $y = x_d z$, and by using **(B8)**, we have

$$L_\alpha^\mathcal{B} g_p(x) = x_d^{p-\alpha} \text{p.v.} \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{z_d^p - 1}{|(\widetilde{z}, z_d) - \mathbf{e}_d|^{d+\alpha}} \mathcal{B}(\mathbf{e}_d, (\widetilde{z}, z_d)) \, dz_d d\widetilde{z} =: x_d^{p-\alpha} I_1 \,.$$

Note that Lemma 5.3(a) takes care of integrability away of the point x. Using the change of variables $\widetilde{z} = |z_d - 1| \widetilde{u}$, we get $I_1 = \int_{\mathbb{R}^{d-1}} (|\widetilde{u}|^2 + 1)^{-(d+\alpha)/2} I_2(\widetilde{u}) \, d\widetilde{u}$, where

$$I_2(\widetilde{u}) = I_2 := \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^\infty \right) \frac{z_d^p - 1}{|z_d - 1|^{1+\alpha}} \mathcal{B}(\mathbf{e}_d, (|z_d - 1| \widetilde{u}, z_d)) dz_d.$$

Fix $\widetilde{u}$ and, by using the change of variables $s = 1/z_d$, **(B1)** and **(B8)**, we get that the second integral is equal to

$$\int_0^{\frac{1}{1+\epsilon}} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}((1-s)\widetilde{u}, 1), s\mathbf{e}_d) ds$$

$$= \int_0^{1-\epsilon} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}((1-s)\widetilde{u}, 1), s\mathbf{e}_d) ds + \int_{1-\epsilon}^{\frac{1}{1+\epsilon}} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\widetilde{u}, 1), s\mathbf{e}_d) ds \,.$$

Therefore,

$$I_2 = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{(s^p - 1) + (s^{\alpha-1-p} - s^{\alpha-1})}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\widetilde{u}, 1), s\mathbf{e}_d) \, ds$$

$$+ \lim_{\epsilon \to 0} \int_{1-\epsilon}^{\frac{1}{1+\epsilon}} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\widetilde{u}, 1), s\mathbf{e}_d) \, ds =: I_{21} + I_{22} \,.$$

Next

$$I_{21} = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{(1 - s^p)(1 - s^{p-(\alpha-1)})}{(1-s)^{1+\alpha}} s^{\alpha-1-p} \mathcal{B}(((1-s)\widetilde{u}, 1), s\mathbf{e}_d) \, ds.$$

Since $\mathcal{B}$ is bounded (uniformly in $\widetilde{u}$) by **(B7)** and $\alpha < 2$, the fraction is integrable near 1. Further, if $s \in (0, 1/2)$, then by using **(B7)** and (5.2) we obtain the following estimate

$$\mathcal{B}(((1-s)\widetilde{u}, 1), s\mathbf{e}_d) \leq c_1 s^{\beta_1} \log(1/s)^{\beta_3} \,,$$

where the constant $c_1 > 0$ does not depend on $\widetilde{u}$. The fact that $p < \alpha + \beta_1$ implies that the function $s \mapsto s^{\alpha-1-p} s^{\beta_1} \log(1/s)^{\beta_3}$ is integrable near 0. Thus there exists a constant $c_2 > 0$ independent of $\widetilde{u} \in \mathbb{R}^{d-1}$ such that

$$I_{21} = \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}(((1-s)\widetilde{u}, 1), s\mathbf{e}_d) \, ds < c_2 < \infty \,. \qquad (5.8)$$

Here we used that the integrand is non-negative since $p > \alpha - 1$. Now we show that $I_{22} = 0$. Indeed, since $\mathcal{B}$ is bounded, we conclude from [3, p.121] that

$$\left| \int_{1-\epsilon}^{\frac{1}{1+\epsilon}} \frac{s^{\alpha-1-p} - s^{\alpha-1}}{(1-s)^{1+\alpha}} \mathcal{B}\big((1-s)\widetilde{u}, 1), s\mathbf{e}_d\big) ds \right| \leq c\epsilon^2 .$$

Therefore,

$$I_1 = \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2 + 1)^{(d+\alpha)/2}} \left(\int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}\big(((1-s)\widetilde{u}, 1), s\mathbf{e}_d\big) ds \right) d\widetilde{u} .$$

By combining the uniform (in $\widetilde{u}$) estimate for I_{21} and the fact that $\int_{\mathbb{R}^{d-1}} (|\widetilde{u}|^2 + 1)^{(-d-\alpha)/2} d\widetilde{u} < \infty$, we see that $I_1 = C(\alpha, p, \mathcal{B}) \in (0, \infty)$. The proof is complete. $\square$

Remark 5.5. (a) Note that $p \mapsto (s^p - 1)(1 - s^{\alpha-p-1})$ is increasing for $s \in (0, 1)$. This implies that $p \mapsto C(\alpha, p, \mathcal{B})$ is also increasing. Further, it is clear $\lim_{p \downarrow (\alpha-1)_+} C(\alpha, p, \mathcal{B}) = 0$ and, by inspecting the integrand in (5.8) near 0 and using Fatou's lemma, we see $\lim_{p \uparrow \alpha + \beta_1} C(\alpha, p, \mathcal{B}) = \infty$.
(b) Note also that if $\alpha \in [1, 2)$, then the above proof shows that $L_\alpha^{\mathcal{B}} g_p = 0$ for $p = \alpha - 1$.

Next we set

$$L^{\mathcal{B}} f(x) := L_\alpha^{\mathcal{B}} f(x) - C(\alpha, p, \mathcal{B}) x_d^{-\alpha} f(x), \quad x \in \mathbb{R}_+^d , \tag{5.9}$$

where $L_\alpha^{\mathcal{B}}$ is defined in (5.7). Thus, $L^{\mathcal{B}} g_p(x) = 0$, $x \in \mathbb{R}_+^d$. Recall that $L^{\mathcal{B}}$ was in a more general setting already introduced in (3.6).

For $a, b, r > 0$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, we define

$$D_{\widetilde{w}}(a, b) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x} - \widetilde{w}| < a, 0 < x_d < b\}.$$

Without loss of generality, we will deal with the case $\widetilde{w} = \widetilde{0}$ only. We will write $D(a, b)$ for $D_{\widetilde{0}}(a, b)$ and $U(r) = D_{\widetilde{0}}(\frac{r}{2}, \frac{r}{2})$. Further we use U for $U(1)$.

Let $h_q(x) := g_q(x) \mathbf{1}_{D(1,1)}(x) = x_d^q \mathbf{1}_{D(1,1)}(x)$.

Lemma 5.6.   (a) Let $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. There exists $C_{22} > 0$ such that for every $z \in U$,

$$0 \geq L^{\mathcal{B}} h_p(z) \geq -C_{22} z_d^{\beta_1} |\log z_d|^{\beta_3} .$$

   (b) Let $(\alpha - 1)_+ < p < q < \alpha + \beta_1$. There exist $r_0 \in (0, 1/2]$ and $C_{23} > 0$ and $C_{24} > 0$ such that for every $z \in D(\frac{1}{2}, r_0)$,

$$C_{23} z_d^{q-\alpha} \leq L^{\mathcal{B}} h_q(z) \leq C_{24} z_d^{q-\alpha}.$$

   (c) Let $(\alpha - 1)_+ < q < p < \alpha + \beta_1$. There exist $C_{25} > 0$, $C_{26} > 0$ and $C_{27} > 0$ such that for every $z \in U$,

$$-C_{26} z_d^{q-\alpha} - C_{27} z_d^{\beta_1} |\log z_d|^{\beta_3} \leq L^{\mathcal{B}} h_q(z) \leq -C_{25} z_d^{q-\alpha}.$$

Proof. (a) Let $z \in U$. Then by using that $L^{\mathcal{B}} g_p(x) = 0$, we see that

$$L^{\mathcal{B}} h_p(z) = - \int_{D(1,1)^c \cap \mathbb{R}_+^d} \frac{y_d^p}{|y - z|^{d+\alpha}} \mathcal{B}(z, y) \, dy ,$$

which is negative. Further, if $y \in D(1,1)^c$, then $|y| \geq \sqrt{2} \geq 2|z|$ (since $z \in U = D(1/2, 1/2)$). Thus it follows from Lemma 5.3(b) (with $b = 1$), that

$$|L^{\mathcal{B}} h_p(z)| \leq \int_{|y| \geq (2|z|) \vee (1/2)} \frac{|y|^p}{|y - z|^{d+\alpha}} \mathcal{B}(z, y) \, dy \leq c_1 z_d^{\beta_1} |\log z_d|^{\beta_3} .$$

(b) Let $z \in U$. Then by using Lemma 5.4,

$$L^{\mathcal{B}} h_q(z) = (C(\alpha, q, \mathcal{B}) - C(\alpha, p, \mathcal{B})) z_d^{q-\alpha} - \int_{B(0,1)^c \cap \mathbb{R}_+^d} \frac{y_d^q}{|y - z|^{d+\alpha}} \mathcal{B}(z, y) \, dy \, .$$

Since the second term is non-negative, by removing it we obtain the upper bound. In the same way as before (this uses $q < \alpha + \beta_1$),

$$\left| \int_{B(0,1)^c \cap \mathbb{R}_+^d} \frac{y_d^q}{|y - z|^{d+\alpha}} \mathcal{B}(z, y) \, dy \right| \le c_2 z_d^{\beta_1} |\log z_d|^{\beta_3} \, .$$

Thus, for any $z \in U$,

$$L^{\mathcal{B}} h_q(z) \ge (C(\alpha, q, \mathcal{B}) - C(\alpha, p, \mathcal{B})) z_d^{q-\alpha} - c_2 z_d^{\beta_1} |\log z_d|^{\beta_3} \, .$$

Since $q - \alpha < \beta_1$ and $C(\alpha, q, \mathcal{B}) - C(\alpha, p, \mathcal{B}) > 0$, we can find $r_0 \in (0, 1/2]$ such that the function $t \mapsto (C(\alpha, q, \mathcal{B}) - C(\alpha, p, \mathcal{B})) t^{q-\alpha} - c_3 t^{\beta_1} |\log t|^{\beta_3}$ is positive and bounded away from zero for all $t \in (0, r_0)$. This concludes the proof of the lower bound.

(c) This is almost the same as the proof of (b) except that now $C(\alpha, q, \mathcal{B}) - C(\alpha, p, \mathcal{B}) < 0$ and we keep the term with $z_d^{\beta_1} |\log z_d|^{\beta_3}$ in the lower bound (hence do not have to shrink). □

For $r \in (0, 1)$, let $C(r) := (D(1, 1) \setminus D(\sqrt{r}, \sqrt{r})) \cap \{y \in \mathbb{R}_+^d : y_d \ge |\widetilde{y}|\}$ be the part of the cone in $D(1, 1) \setminus D(\sqrt{r}, \sqrt{r})$. Note that for $z \in U(r)$ and $y \in C(r)$ we have $|z - y| \le 2|y|$, $|y| \le \sqrt{2} y_d$, $|z| \le |y|/\sqrt{2}$, and therefore $y_d/4 \le |y|/4 \le |z - y| \le 2\sqrt{2} y_d$. Now we find a lower bound for $\mathcal{B}(z, y)$, $z \in U(r)$, $y \in C(r)$. By using **(B7)**, the lower bound in (5.2), $z_d < y_d$ and $y_d \asymp |z - y| \asymp |y|$, we get

$$
\begin{aligned}
\mathcal{B}(z, y) &\ge c_1 \left(\frac{z_d \wedge y_d}{|z - y|} \wedge 1 \right)^{\beta_1} \left(\frac{z_d \vee y_d}{|z - y|} \wedge 1 \right)^{\beta_2} \log \left(1 + \frac{(z_d \vee y_d) \wedge |z_d - y_d|}{z_d \wedge y_d \wedge |z - y|} \right)^{\beta_3} \\
&\ge c_2 \left(\frac{z_d}{|y|} \right)^{\beta_1} \log(1 + \frac{y_d}{z_d})^{\beta_3} \, .
\end{aligned}
$$

Since $y_d \ge \sqrt{r}$ and $z_d \le r/2$, we see that $y_d/z_d \ge 2/\sqrt{z_d}$, hence $\log(1 + y_d/z_d) \ge \log(y_d/z_d) \ge \log(1/\sqrt{z_d}) = |\log z_d|/2$. Thus, $\mathcal{B}(z, y) \ge c_3 \left(z_d/|y|^d \right)^{\beta_1} |\log z_d|^{\beta_3}$. By using the Lévy system formula we get for $x \in U(r)$ (with constants c_4, c_5 *not* depending on r),

$$
\begin{aligned}
\mathbb{E}_x[h_p(Y_{\tau_{U(r)}})] &\ge \mathbb{E}_x[h_p(Y_{\tau_{U(r)}}), Y_{\tau_{U(r)}} \in C(r)] \\
&\ge c_4 \mathbb{E}_x \int_{C(r)} \int_0^{\tau_{U(r)}} |Y_t - y|^{-d-\alpha} \left(\frac{Y_t^d}{|y|} \right)^{\beta_1} |\log Y_t^d|^{\beta_3} y_d^p \, dy \, dt \\
&\ge c_5 \int_{C(r)} y_d^p |y|^{-d-\alpha-\beta_1} \, dy \left(\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} \, dt \right) \, .
\end{aligned}
$$

Since $\int_{D(1,1)} y_d^p |y|^{-d-\alpha-\beta_1} \, dy = +\infty$, we can choose r_0 small enough so that

$$c_5 \int_{C(r)} y_d^p |y|^{-d-\alpha-\beta_1} \, dy \ge 1, \qquad \text{for all } r \in (0, r_0].$$

Hence, for all $r \in (0, r_0]$

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} \, dt \le \mathbb{E}_x[h_p(Y_{\tau_{U(r)}})], \quad x \in U(r) \, . \tag{5.10}$$

In the remainder of this section, we will, in several places, need Lemma 9.5, an extension of Proposition 3.7 to functions which are neither smooth nor of compact support. Since its

proof is rather technical, in order not to interrupt the flow of the presentation, we postpone it to Section 9.

Lemma 5.7. *(a) For all $r \in (0, r_0]$ and $x \in U(r)$,*

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} \, dt \leq x_d^p. \tag{5.11}$$

(b) There exist a constant $C_{28} > 0$ such that for all $x \in U$,

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} \, dt \leq C_{28} x_d^p. \tag{5.12}$$

Proof. (a) By Lemma 9.5 it holds that

$$\mathbb{E}_x[h_p(Y_{\tau_{U(r)}})] = x_d^p + \mathbb{E}_x \int_0^{\tau_{U(r)}} L^{\mathcal{B}} h_p(Y_s) \, ds \leq x_d^p,$$

where the inequality follows from Lemma 5.6 (a). The required estimate is a consequence of (5.10).

(b) Let $\tau_{r_0 U}^{(r_0)}$ be the first time that the process $Y^{(r_0)} = (Y^{(r_0),1}, \ldots, Y^{(r_0),d})$ exits $r_0 U = U(r_0)$. Then it is easy to check by definition that $\tau_{r_0 U}^{(r_0)} = r_0^\alpha \tau_U$. Using this, one can easily see that

$$\mathbb{E}_{r_0 x} \int_0^{\tau_{r_0 U}^{(r_0)}} (Y_s^{(r_0),d})^{\beta_1} |\log(Y_s^{(r_0),d})|^{\beta_3} \, ds \;\; = \;\; r_0^{\alpha+\beta_1} \mathbb{E}_x \int_0^{\tau_U} (Y_s^d)^{\beta_1} |\log(r_0 Y_s^d)|^{\beta_3} \, ds$$

$$\geq \;\; r_0^{\alpha+\beta_1} \mathbb{E}_x \int_0^{\tau_U} (Y_s^d)^{\beta_1} |\log Y_s^d|^{\beta_3} \, ds. \tag{5.13}$$

In the last line we used that since $r_0 < 1$, for all $t \in (0, 1/2)$ it holds that $|\log(r_0 t)| \geq |\log t|$. Hence, for any $x \in U$ we have

$$\mathbb{E}_x \int_0^{\tau_U} (Y_s^d)^{\beta_1} |\log Y_s^d|^{\beta_3} \, ds \;\; \leq \;\; r_0^{-\alpha-\beta_1} \mathbb{E}_{r_0 x} \int_0^{\tau_{r_0 U}^{(r_0)}} (Y_s^{(r_0),d})^{\beta_1} |\log(Y_s^{(r_0),d})|^{\beta_3} \, ds$$

$$\leq \;\; r_0^{-\alpha-\beta_1} (r_0 x_d)^p = r_0^{-\alpha-\beta_1+p} x_d^p = C_{28} x_d^p$$

with $C_{28} = r_0^{-\alpha-\beta_1+p}$. $\qquad \qquad \square$

Next we want to prove the opposite inequality under the additional assumption **(B4)**. *In the rest of this section we assume that (B4) also holds.* The key step towards this goal is the following lemma.

Lemma 5.8. *Let $p \in ((\alpha - 1)_+, \alpha + \beta_1)$ and assume that $\theta > (\alpha - 1)_+$.*

(a) *There exist a function $\psi : \mathbb{R}^d \to [0, \infty)$ and a constant $C_{29} > 0$ such that*

$$L^{\mathcal{B}} \psi(x) \leq C_{29} x_d^{\beta_1} |\log x_d|^{\beta_3}, \quad x \in U$$

  and the following assertions hold:

(b) *The function $\phi(x) := h_p(x) - \psi(x)$, $x \in \mathbb{R}_+^d$, satisfies the following properties:*
  (b1) $\phi(x) = x_d^p$ *for all $x = (\widetilde{0}, x_d) \in U$ with $0 < x_d < 1/4$;*
  (b2) $\phi(x) \leq 0$ *for all $x \in U^c \cap \mathbb{R}_+^d$;*
  (b3) *There exists $C_{30} > 0$ such that $L^{\mathcal{B}} \phi(x) \geq -C_{30} x_d^{\beta_1} |\log x_d|^{\beta_3}$ for all $x \in U$.*

(c) *Let $(\alpha - 1)_+ < q < p$. The function $\varphi(x) := h_q(x) - \psi(x)$ satisfies the following properties:*
  (c1) $\varphi(x) = x_d^q$ *for all $x = (\widetilde{0}, x_d) \in U$ with $0 < x_d < 1/4$;*
  (c2) $\varphi(x) \leq 0$ *for all $x \in U^c \cap \mathbb{R}_+^d$;*

(c3) *There exist $C_{31} > 0$ and $C_{32} > 0$ such that $L^{\mathcal{B}}\varphi(x) \geq -C_{31}x_d^{q-\alpha} - C_{32}x_d^{\beta_1}\,|\log x_d|^{\beta_3}$ for all $x \in U$.*

The proof of the lemma is rather involved, we defer it to the Section 8.

Combining Lemma 5.8 (b1) with a formal application of Proposition 3.7 to ϕ, we can get the following inequality opposite to (5.12). However, since ϕ is not smooth, we can not apply Proposition 3.7 to ϕ directly. Therefore we postpone its proof to Section 9.

Lemma 5.9. *Let $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. For any $x = (\widetilde{0}, x_d)$ with $0 < x_d < 1/4$ it holds that*

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1}\,|\log Y_t^d|^{\beta_3}\,dt \geq C_{30}^{-1}x_d^p, \tag{5.14}$$

where C_{30} is the constant from Lemma 5.8.

Note that there exists $c > 0$ such that $z^{\beta_1}|\log z|^{\beta_3} \leq c$ for all $z \in (0, 1)$. Since $Y_t^d \in (0, 1)$ for $t < \tau_U$, (5.14) immediately implies the following estimate for the expected exit time from U:

$$\mathbb{E}_x \tau_U \geq c^{-1}C_{30}^{-1}x_d^p, \quad x = (\widetilde{0}, x_d) \text{ with } 0 < x_d < 1/4. \tag{5.15}$$

One can easily obtain the following exit probability estimate by combining Lemmas 5.7 and 5.9.

Lemma 5.10. *Let $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. There exists $C_{33} > 1$ such that for $x \in D(1/4, 1/4)$,*

$$\mathbb{P}_x(Y_{\tau_U} \in D(1/2, 1) \setminus D(1/2, 3/4)) \leq C_{33}x_d^p, \tag{5.16}$$

and for $x = (\widetilde{0}, x_d) \in D(1/8, 1/8)$,

$$\mathbb{P}_x(Y_{\tau_{D(1/4,1/4)}} \in D(1/4, 1) \setminus D(1/4, 3/4)) \geq C_{33}^{-1}x_d^p. \tag{5.17}$$

Proof. For $y \in U = D(1/2, 1/2)$ and $z \in D(1/2, 1) \setminus D(1/2, 3/4)$, it holds that $y_d < z_d$, $|y - z| \asymp z_d$ and $y_d < 2|y - z|$. Hence, $\mathcal{B}(y, z) \asymp (y_d/|y - z|)^{\beta_1}\log(1 + (z_d/y_d))$. Since $1 \geq z_d \geq (3/2)y_d$, we have that $\log(1/y_d) \leq \log(z_d/y_d) \leq \log(1 + z_d/y_d) \leq c\log(z_d/y_d) \leq c\log(1/y_d)$. We conclude that $\mathcal{B}(y, z) \asymp (y_d/|y - z|)^{\beta_1}\,|\log y_d|^{\beta_3}$. Hence, by using the Lévy system formula, we get

$$\mathbb{P}_x\left(Y_{\tau_U} \in D(1/2, 1) \setminus D(1/2, 3/4)\right)$$
$$\leq c_1\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1}\,|\log Y_t^d|^{\beta_3} \int_{D(1/2,1)\setminus D(1/2,3/4)} \frac{1}{|Y_t - z|^{d+\alpha+\beta_1}}\,dz\,dt$$
$$\leq c_2\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1}\,|\log Y_t^d|^{\beta_3}\,dt$$

and

$$\mathbb{P}_x\left(Y_{\tau_{D(1/4,1/4)}} \in D(1/4, 1) \setminus D(1/4, 3/4)\right)$$
$$\geq c_3\mathbb{E}_x \int_0^{\tau_{D(1/4,1/4)}} (Y_t^d)^{\beta_1}\,|\log Y_t^d|^{\beta_3} \int_{D(1/4,1)\setminus D(1/4,3/4)} \frac{1}{|Y_t - z|^{d+\alpha+\beta_1}}\,dz\,dt$$
$$\geq c_4\mathbb{E}_x \int_0^{\tau_{D(1/4,1/4)}} (Y_t^d)^{\beta_1}\,|\log Y_t^d|^{\beta_3}\,dt\,.$$

The claim now follows from Lemmas 5.7 (b) and 5.9. $\qquad\qquad\square$

Lemma 5.11. *Let $(\alpha - 1)_+ < p < \alpha + \beta_1$ and $\alpha + \beta_2 < p$. There exist $r_1 > 0$ and $C_{34} > 0$ such that for all $r > 0$ and $x = (\widetilde{0}, x_d)$ with $0 < x_d < r_1 r$, it holds that*

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_2}\, dt \geq C_{34} x_d^{\alpha + \beta_2}. \tag{5.18}$$

Proof. Let $q := \alpha + \beta_2$. Note that $q < p$ by assumption. By combining Lemma 5.8 (c3) with a formal application of Proposition 3.7 to φ, we can get the assertion of this lemma. However, since φ is not smooth, we can not apply Proposition 3.7 to φ directly. Therefore, the rigorous proof is postponed to Section 9. $\quad\square$

Lemma 5.12. *Let $(\alpha - 1)_+ < p < \alpha + \beta_1$ and $\alpha + \beta_2 < p$. There exist a constant $C_{35} > 0$ such that for all $r > 0$ and $x \in U(r)$,*

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_2} dt \leq C_{35} x_d^{\alpha + \beta_2}. \tag{5.19}$$

Proof. Let $q = \alpha + \beta_2$. Note that $q < p$ by assumption. By combining Lemma 5.6 with a formal application of Proposition 3.7 to $h_q - h_p$, we can get the assertion of this lemma. However, since $h_q - h_p$ is not smooth, we can not apply Proposition 3.7 to $h_q - h_p$ directly. Therefore, the rigorous proof is postponed to Section 9. $\quad\square$

The following counterpart of (5.15) is considerably simpler than the previous results, but has an additional restriction on the range of p.

Lemma 5.13. *Suppose that $p \in ((\alpha - 1)_+, \alpha)$. There exist a constant $C_{36} > 0$ such that*

$$\mathbb{E}_x \tau_U \leq C_{36} x_d^p, \quad x \in U. \tag{5.20}$$

Proof. Choose $q \in (p, \alpha)$ and let $\eta(x) := h_p(x) - h_q(x)$, $x \in \mathbb{R}_+^d$. By combining Lemma 5.6 with a formal application of Proposition 3.7 to η, we can get the assertion of this lemma. However, since η is not smooth, we can not apply Proposition 3.7 to η directly. Therefore The rigorous proof is postponed to Section 9. $\quad\square$

6. Boundary Harnack principle

In this section we give the proof of Theorem 1.3.

Lemma 6.1. *There exists $C_{37} > 0$ such that for all $0 < 4r \leq R \leq 1$ and $w \in D(r, r)$,*

$$\mathbb{P}_w\left(Y_{\tau_{B(w,r) \cap \mathbb{R}_+^d}} \in A(w, R, 4) \cap \mathbb{R}_+^d\right) \leq C_{37} \frac{r^{\alpha + \beta_1}}{R^{\alpha + \beta_1}} \frac{w_d^p}{r^p}.$$

Proof. Let $y \in B(w, r) \cap \mathbb{R}_+^d$ and $z \in A(w, R, 4) \cap \mathbb{R}_+^d$. By Lemma 5.2(b) we have that

$$\mathcal{B}(y, z) \leq c_1 y_d^{\beta_1} |\log y_d|^{\beta_3} |y - z|^{-\beta_1} \leq c_2 y_d^{\beta_1} |\log y_d|^{\beta_3} |w - z|^{-\beta_1}.$$

Thus for all $0 < 4r \leq R \leq 1$ and $w \in D(r, r)$,

$$\mathbb{P}_w\left(Y_{\tau_{B(w,r) \cap \mathbb{R}_+^d}} \in A(w, R, 4) \cap \mathbb{R}_+^d\right)$$

$$\leq c_2 \mathbb{E}_w \int_0^{\tau_{B(w,r) \cap \mathbb{R}_d^+}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \int_{B(w,R)^c} \frac{1}{|w - z|^{d + \alpha + \beta_1}} dz$$

$$\leq c_3 R^{-\alpha - \beta_1} \mathbb{E}_w \int_0^{\tau_{B(w,r) \cap \mathbb{R}_d^+}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt.$$

Since $B(w, r) \cap \mathbb{R}_d^+ \subset D(2r, 2r)$, applying Lemma 5.7 and scaling, we get that there exists $c_4 > 0$ such that for all $0 < 4r \le R \le 1$ and $w \in D(r, r)$,

$$\mathbb{P}_w \left(Y_{\tau_{B(w,r) \cap \mathbb{R}_d^+}} \in A(w, R, 4) \cap \mathbb{R}_+^d \right)$$

$$\le c_3 R^{-\alpha-\beta_1} \mathbb{E}_w \int_0^{\tau_{D(2r,2r)}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \le c_4 2^{\alpha+\beta_1-p} \frac{r^{\alpha+\beta_1}}{R^{\alpha+\beta_1}} \frac{w_d^p}{r^p}.$$

$\square$

Lemma 6.2. *There exists $C_{38} > 0$ such that for any $x \in D(2^{-5}, 2^{-5})$,*

$$\mathbb{P}_x \left(Y_{\tau_U} \in D(1, 1) \right) \le C_{38} x_d^p.$$

Proof. Let

$$H_2 := \{Y_{\tau_U} \in D(1, 1)\}, \quad H_1 := \{Y_{\tau_U} \in D(1/2, 1) \setminus D(1/2, 3/4)\}.$$

We claim that $\mathbb{P}_x(H_2) \le c\mathbb{P}_x(H_1)$ for all $x \in D(2^{-5}, 2^{-5})$. Combining this claim with (5.16) we arrive at the conclusion of this lemma. Now we prove the claim. Note that by (5.17),

$$\mathbb{P}_w(H_1) \ge \mathbb{P}_w(Y_{\tau_{D_{\widetilde{w}}(1/4,1/4)}} \in D_{\widetilde{w}}(1/4, 1) \setminus D_{\widetilde{w}}(1/4, 3/4)) \ge c_1 w_d^p, \quad w \in D(2^{-3}, 2^{-3}). \quad (6.1)$$

For $i \ge 1$, set

$$s_0 = s_1, \quad s_i = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{50}\sum_{j=1}^i \frac{1}{j^2}\right) \quad \text{and} \quad J_i = D(s_i, 2^{-i-3}) \setminus D(s_i, 2^{-i-4}).$$

Note that $1/(20) < s_i < 1/(16)$. Define for $i \ge 1$,

$$d_i = \sup_{z \in J_i} \frac{\mathbb{P}_z(H_2)}{\mathbb{P}_z(H_1)}, \quad \widetilde{J}_i = D(s_{i-1}, 2^{-i-3}), \quad \tau_i = \tau_{\widetilde{J}_i}. \quad (6.2)$$

Repeating the argument leading to [13, (6.29)], we get that for $z \in J_i$ and $i \ge 2$,

$$\mathbb{P}_z(H_2) \le \left(\sup_{1 \le k \le i-1} d_k\right) \mathbb{P}_z(H_1) + \mathbb{P}_z\left(Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k\right). \quad (6.3)$$

For $i \ge 2$, define $\sigma_{i,0} = 0, \sigma_{i,1} = \inf\{t > 0 : |Y_t - Y_0| \ge 2^{-i-3}\}$ and $\sigma_{i,m+1} = \sigma_{i,m} + \sigma_{i,1} \circ \theta_{\sigma_{i,m}}$ for $m \ge 1$. By Lemma 4.1, we have that there exists $k_1 \in (0, 1)$ such that

$$\mathbb{P}_w(Y_{\sigma_{i,1}} \in \widetilde{J}_i) \le 1 - \mathbb{P}_w(\sigma_{i,1} = \zeta) \le 1 - \mathbb{P}_w(\tau_{B(w,\delta_D(w)/2)} = \zeta) < k_1, \quad w \in \widetilde{J}_i. \quad (6.4)$$

For the purpose of further estimates, we now choose a positive integer l such that $k_1^l \le 2^{-(\alpha+\beta_1)}$. Next we choose $i_0 \ge 2$ large enough so that $2^{-i} < 1/(200li^3)$ for all $i \ge i_0$. Now we assume $i \ge i_0$. Using (6.4) and the strong Markov property we have that for $z \in J_i$,

$$\mathbb{P}_z(\tau_i > \sigma_{i,li}) \le \mathbb{P}_z(Y_{\sigma_{i,k}} \in \widetilde{J}_i, 1 \le k \le li)$$

$$= \mathbb{E}_z \left[\mathbb{P}_{Y_{\sigma_{i,li-1}}}(Y_{\sigma_{i,1}} \in \widetilde{J}_i) : Y_{\sigma_{i,li-1}} \in \widetilde{J}_i, Y_{\sigma_{i,k}} \in \widetilde{J}_i, 1 \le k \le li - 2\right]$$

$$\le \mathbb{P}_z\left(Y_{\sigma_{i,k}} \in \widetilde{J}_i, 1 \le k \le li - 1\right) k_1 \le k_1^{li}. \quad (6.5)$$

Note that if $z \in J_i$ and $y \in D(1, 1) \setminus [\widetilde{J}_i \cup (\cup_{k=1}^{i-1} J_k)]$, then $|y - z| \ge (s_{i-1} - s_i) \wedge (2^{-4} - 2^{-i-3}) = 1/(400i^2)$. Furthermore, since $2^{-i-3} < 1/(400i^2)$ (recall that $i \ge i_0$), if $Y_{\tau_i}(\omega) \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k$ and $\tau_i(\omega) \le \sigma_{i,li}(\omega)$, then $\tau_i(\omega) = \sigma_{i,k}(\omega)$ for some $k = k(\omega) \le li$. Dependence of k on ω will be omitted in the next few lines. Hence on $\{Y_{\tau_i} \in D(1, 1) \setminus \cup_{k=1}^{i-1} J_k, \quad \tau_i \le \sigma_{i,li}\}$ with $Y_0 = z \in J_i$, we have $|Y_{\sigma_{i,k}} - Y_{\sigma_{i,0}}| = |Y_{\tau_i} - Y_0| > \frac{1}{400i^2}$ for some $1 \le k \le li$. Thus for

some $1 \leq k \leq li$, $\sum_{j=0}^{k} |Y_{\sigma_{i,j}} - Y_{\sigma_{i,j-1}}| > (400i^2)^{-1}$ which implies for some $1 \leq j \leq k \leq li$, $|Y_{\sigma_{i,j}} - Y_{\sigma_{i,j-1}}| \geq (k400i^2)^{-1} \geq (li)^{-1}(400i^2)^{-1}$. Thus, we have

$$\{Y_{\tau_i} \in D(1,1) \setminus \cup_{k=1}^{i-1} J_k, \quad \tau_i \leq \sigma_{i,li}\}$$
$$\subset \cup_{j=1}^{li} \{|Y_{\sigma_{i,j}} - Y_{\sigma_{i,j-1}}| \geq 1/(800li^3), Y_{\sigma_{i,j}} \in D(1,1), Y_{\sigma_{i,j-1}} \in \widetilde{J}_i\}.$$

Now, using Lemma 6.1 (with $r = 2^{-i-3}$ and $R = 1/(800li^3)$) (noting that $4 \cdot 2^{-i-2} < 1/(400li^3)$ for all $i \geq i_0$), and repeating the argument leading to [13, (6.34)], we get that for $z \in J_i$,

$$\mathbb{P}_z\left(Y_{\tau_i} \in D(1,1) \setminus \cup_{k=1}^{i-1} J_k, \quad \tau_i \leq \sigma_{i,li}\right) \leq li \sup_{z \in \widetilde{J}_i} \mathbb{P}_z\left(|Y_{\sigma_{i,1}} - z| \geq (800li^3)^{-1}, Y_{\sigma_{i,1}} \in D(1,1)\right)$$

$$\leq \mathbb{P}_z\left(4 > |Y_{\sigma_{i,1}} - z| \geq (800li^3)^{-1}\right) \leq c_{12}li\left(\frac{800li^3}{2^{i+3}}\right)^{\alpha+\beta_1}.$$

By this and (6.5), we have for $z \in J_i$, $i \geq i_0$,

$$\mathbb{P}_z\left(Y_{\tau_i} \in D(1,1) \setminus \cup_{k=1}^{i-1} J_k\right) \leq k_1^{li} + c_2 li\left(\frac{800li^3}{2^{i+3}}\right)^{\alpha+\beta_1}. \tag{6.6}$$

By our choice of l, we have

$$li\left(\frac{800li^3}{2^{i+3}}\right)^{\alpha+\beta_1} = 100^{\alpha+\beta_1} l^{1+\alpha+\beta_1} i^{1+3(\alpha+\beta_1)}\left(2^{-(\alpha+\beta_1)}\right)^i \geq \left(2^{-(\alpha+\beta_1)}\right)^i \geq (k_1^l)^i. \tag{6.7}$$

Thus combining (6.7) with (6.6), and then using (6.1), we get that for $z \in J_i$, $i \geq i_0$,

$$\frac{\mathbb{P}_z(Y_{\tau_i} \in D(1,1) \setminus \cup_{k=1}^{i-1} J_k)}{\mathbb{P}_z(H_1)} \leq c_3 li2^{ip}\left(\frac{800li^3}{2^{i+3}}\right)^{\alpha+\beta_1} \leq c_4 i^{1+3(\alpha+\beta_1)} 2^{(p-\alpha-\beta_1)i}. \tag{6.8}$$

By this and (6.3), for $z \in J_i$, $i \geq i_0$, for all $i \geq i_0$

$$\frac{\mathbb{P}_z(H_2)}{\mathbb{P}_z(H_1)} \leq \sup_{1 \leq k \leq i-1} d_k + \frac{\mathbb{P}_z(Y_{\tau_i} \in D(1,1) \setminus \cup_{k=1}^{i-1} J_k)}{\mathbb{P}_z(H_1)} \leq \sup_{1 \leq k \leq i-1} d_k + c_4 \frac{i^{1+3(\alpha+\beta_1)}}{2^{(\alpha+\beta_1-p)i}}.$$

This implies that for all $i \geq 1$

$$d_i \leq \sup_{1 \leq k \leq i_0-1} d_k + c_4 \sum_{k=1}^{i} \frac{i^{1+3(\alpha+\beta_1)}}{2^{(\alpha+\beta_1-p)i}} \leq \sup_{1 \leq k \leq i_0-1} d_k + c_4 \sum_{k=1}^{\infty} \frac{i^{1+3(\alpha+\beta_1)}}{2^{(\alpha+\beta_1-p)i}} =: c_5 < \infty.$$

Thus the claim above is valid, since $D(2^{-5}, 2^{-5}) \subset \cup_{k=1}^{\infty} J_k$. The proof is now complete. $\qquad\square$

Proof of Theorem 1.3. By scaling, we just need to consider the case $r = 1$. Moreover, by the Harnack inequality, the continuity of harmonic functions and a standard chain argument, it suffices to prove (1.11) for $x, y \in D_{\widetilde{w}}(2^{-8}, 2^{-8})$.

Since f is harmonic in $D_{\widetilde{w}}(2,2)$ and vanishes continuously on $B(\widetilde{w}, 2) \cap \partial\mathbb{R}_+^d$, it is regular harmonic in $D_{\widetilde{w}}(7/4, 7/4)$ and vanishes continuously on $B(\widetilde{w}, 7/4) \cap \partial\mathbb{R}_+^d$. Throughout the remainder of this proof, we assume that $x \in D_{\widetilde{w}}(2^{-8}, 2^{-8})$. Without loss of generality we take $\widetilde{w} = 0$.

Define $x_0 = (\widetilde{x}, 1/(16))$. By the Harnack inequality and Lemma 5.10, we have

$$f(x) = \mathbb{E}_x[f(Y_{\tau_{D(1/2,1/2)}})] \geq \mathbb{E}_x[f(Y_{\tau_{D(1/2,1/2)}}); Y_{\tau_{D(1/2,1/2)}} \in D(1/2,1) \setminus D(1/2,3/4)]$$
$$\geq c_1 f(x_0)\mathbb{P}_x(Y_{\tau_{D_{\widetilde{x}}(1/4,1/4)}} \in D_{\widetilde{x}}(1/4,1) \setminus D(1/4,3/4)) \geq c_2 f(x_0)x_d^p. \tag{6.9}$$

(a) *The case $\beta_1 = \beta_2 = \beta \geq 0$, $\beta_3 = \beta_4 = 0$.* For $z \in U$ and $y \in \mathbb{R}_+^d \setminus D(1,1)$, we have $|y - z| \asymp |y|$. Thus, by using (2.3),

$$\int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|z-y|^{d+\alpha}} \left(\frac{z_d}{|z-y|} \wedge 1 \right)^\beta \left(\frac{y_d}{|z-y|} \wedge 1 \right)^\beta dy$$

$$\asymp \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha}} \left(\frac{z_d}{|y|} \wedge 1 \right)^\beta \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy \asymp z_d^\beta \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha+\beta}} \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy.$$

Hence

$$\mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1) \right] \asymp \mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^\beta dt \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha+\beta}} \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy.$$

Combining the above with Lemmas 5.7.(b) and 5.9 we get

$$\mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1) \right] \asymp x_d^p \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha+\beta}} \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy. \qquad (6.10)$$

On the other hand, by Theorem 1.2 and Lemma 6.2, we have

$$\mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \in D(1,1) \right] \leq c_3 f(x_0) \mathbb{P}_x (Y_{\tau_U} \in D(1,1)) \leq c_4 f(x_0) x_d^p. \qquad (6.11)$$

Combining (6.9), (6.10) and (6.11), we get

$$f(x) = \mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1) \right] + \mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \in D(1,1) \right]$$

$$\leq c_5 x_d^p \left(f(x_0) + \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha+\beta}} \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy \right)$$

and

$$f(x) \geq \frac{1}{2} \mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \in D(1/2,1) \setminus D(1/2,3/4) \right] + \frac{1}{2} \mathbb{E}_x \left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1) \right]$$

$$\geq c_6 x_d^p \left(f(x_0) + \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha+\beta}} \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy \right).$$

Therefore

$$f(x) \asymp x_d^p \left(f(x_0) + \int_{\mathbb{R}_+^d \setminus D(1,1)} \frac{f(y)}{|y|^{d+\alpha+\beta}} \left(\frac{y_d}{|y|} \wedge 1 \right)^\beta dy \right). \qquad (6.12)$$

For any $y \in D(2^{-8}, 2^{-8})$, we have the same estimate with $f(y_0)$ instead of $f(x_0)$, where $y_0 = (\widetilde{y}, 1/(16))$. By the Harnack inequality, we have $f(x_0) \asymp f(y_0)$. Therefore it follows from (6.12) that for all $x, y \in D(2^{-8}, 2^{-8})$,

$$\frac{f(x)}{f(y)} \leq c_7 \frac{x_d^p}{y_d^p},$$

which is same as the conclusion of the theorem.

(b) *The case $p < \alpha$.* Set $w = (\widetilde{0}, 2^{-7})$. Then

$$f(w) \geq \mathbb{E}_w \left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1) \right]$$

$$\geq \mathbb{E}_w \int_0^{\tau_{B(w,2^{-10})}} \int_{\mathbb{R}_+^d \setminus D(1,1)} J^{\mathbb{R}_+^d}(Y_t, y) f(y) dy dt$$

$$\geq c_{10} \mathbb{E}_w \tau_{B(w,2^{-10})} \int_{\mathbb{R}_+^d \setminus D(1,1)} J^{\mathbb{R}_+^d}(w,y) f(y) dy = c_{11} \int_{\mathbb{R}_+^d \setminus D(1,1)} J^{\mathbb{R}_+^d}(w,y) f(y) dy, \quad (6.13)$$

where in the last line we used Proposition 3.11 (a).

We now show that there is a constant $c_{12} > 0$ such that for any $z \in U$ and $y \in \mathbb{R}^d_+ \setminus D(1,1)$ it holds that

$$J^{\mathbb{R}^d_+}(z,y) \leq c_{12} J^{\mathbb{R}^d_+}(w,y). \tag{6.14}$$

To prove this, first note that $|z - y| \asymp |w - y| \asymp |y|$. We consider two cases. The first case is when $y_d \geq 1/2$. Then

$$J^{\mathbb{R}^d_+}(z,y) \preceq \frac{z_d^{\beta_1} y_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \log\left(1 + 1/z_d\right)^{\beta_3} \left(1 + |\log y_d|^{\beta_3}\right) \log\left(1 + \frac{|y|}{y_d}\right)^{\beta_4},$$

$$J^{\mathbb{R}^d_+}(w,y) \asymp \frac{y_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(1 + |\log y_d|^{\beta_3}\right) \log\left(1 + \frac{|y|}{y_d}\right)^{\beta_4},$$

implying (6.14) since $t \mapsto t^{\beta_1} \log(1 + 1/t)^{\beta_3}$ is bounded on $(0, 1/2]$. Here we use notation $\preceq$ to denote that the left-hand side is smaller than the right-hand side multiplied by a constant. The second case is $y_d \leq 1/2$. In this case,

$$\log\left(1 + \frac{|y|}{(w_d \vee y_d) \wedge |y|}\right) \asymp \log|y|,$$

and hence

$$J^{\mathbb{R}^d_+}(w,y) \asymp \frac{y_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log y_d|^{\beta_3} (\log|y|)^{\beta_4}. \tag{6.15}$$

If $z_d \leq y_d$, then $(z_d \vee y_d) \wedge |y| = y_d$, so that

$$\log\left(1 + \frac{|y|}{(z_d \vee y_d) \wedge |y|}\right)^{\beta_4} \asymp (\log|y| + |\log y_d|)^{\beta_4} \preceq (\log|y|)^{\beta_4} + |\log y_d|^{\beta_4}.$$

Therefore

$$J^{\mathbb{R}^d_+}(z,y) \preceq \frac{z_d^{\beta_1} y_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \log\left(1 + \frac{y_d}{z_d}\right)^{\beta_3} \left((\log|y|)^{\beta_4} + |\log y_d|^{\beta_4}\right)$$

$$\leq \frac{z_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log z_d|^{\beta_3} (\log|y|)^{\beta_4} + \frac{z_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log z_d|^{\beta_3} y_d^{\beta_2} |\log y_d|^{\beta_4}$$

$$\preceq \frac{y_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log y_d|^{\beta_3} (\log|y|)^{\beta_4},$$

where the last approximate inequality follows from the fact that $t \mapsto t^{\beta_1} |\log t|^{\beta_3}$ is almost increasing $t \mapsto t^{\beta_2} |\log t|^{\beta_4}$ bounded on $(0, 1/2]$ and $y_d \leq 1/2$. By comparing with (6.15), we see that (6.14) holds true.

If $z_d \geq y_d$, then $(z_d \vee y_d) \wedge |y| = z_d$, so that

$$\log\left(1 + \frac{|y|}{(z_d \vee y_d) \wedge |y|}\right)^{\beta_4} \asymp (\log|y| + |\log z_d|)^{\beta_4} \preceq (\log|y|)^{\beta_4} + |\log z_d|^{\beta_4}.$$

Therefore

$$J^{\mathbb{R}^d_+}(z,y) \preceq \frac{y_d^{\beta_1} z_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \log\left(1 + \frac{y_d}{z_d}\right)^{\beta_3} \left((\log|y|)^{\beta_4} + |\log z_d|^{\beta_4}\right)$$

$$\leq \frac{y_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log y_d|^{\beta_3} (\log|y|)^{\beta_4} + \frac{y_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log y_d|^{\beta_3} z_d^{\beta_2} |\log z_d|^{\beta_4}$$

$$\preceq \frac{y_d^{\beta_1}}{|y|^{d+\alpha+\beta_1+\beta_2}} |\log y_d|^{\beta_3} (\log|y|)^{\beta_4}.$$

Again, by comparing with (6.15), we get (6.14).

Combining (6.14) with (5.20) and (6.13) we now have

$$\mathbb{E}_x\left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1)\right] = \mathbb{E}_x \int_0^{\tau_U} \int_{\mathbb{R}_+^d \setminus D(1,1)} J^{\mathbb{R}_+^d}(Y_t, y) f(y) dy dt$$

$$\leq c_{13} \mathbb{E}_x \tau_U \int_{\mathbb{R}_+^d \setminus D(1,1)} J^{\mathbb{R}_+^d}(w, y) f(y) dy \leq c_{14} x_d^p f(w). \tag{6.16}$$

On the other hand, by the Harnack inequality, Carleson's estimate and Lemma 6.2, we have

$$\mathbb{E}_x\left[f(Y_{\tau_U}); Y_{\tau_U} \in D(1,1)\right] \leq c_{15} f(x_0) \mathbb{P}_x\left(Y_{\tau_U} \in D(1,1)\right) \leq c_{16} f(x_0) x_d^p. \tag{6.17}$$

Combining (6.16) and (6.17), and using the Harnack inequality and Carleson's estimate again, we get

$$f(x) = \mathbb{E}_x\left[f(Y_{\tau_U}); Y_{\tau_U} \in D(1,1)\right] + \mathbb{E}_x\left[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1,1)\right]$$
$$\leq c_{17} x_d^p (f(x_0) + f(w)) \leq c_{18} x_d^p f(x_0).$$

Together with (6.9) we get that

$$f(x) \asymp x_d^p f(x_0). \tag{6.18}$$

Using (6.18) instead of (6.12), we also conclude that the theorem holds by the same argument.
□

7. Proof of Theorem 1.4

In this section we assume that the assumptions of Theorem 1.4 hold. In particular, the parameters satisfy $\alpha + \beta_2 < p < \alpha + \beta_1$.

Suppose that the non-scale invariant BHP holds near the boundary of $\mathbb{R}_+^d$ (see the paragraph before Theorem 1.4).

Note that by taking $g(x) = \mathbb{P}_x(Y_{\tau_U} \in D(1/2,1) \setminus D(1/2, 3/4))$, we see from Lemma 5.10 that there exists $\widehat{R} \in (0,1)$ such that for any $r \in (0, \widehat{R}]$ there exists a constant $C_{39} = C_{39}(r) > 0$ such that for any non-negative function f in $\mathbb{R}_+^d$ which is harmonic in $\mathbb{R}_+^d \cap B(0, r)$ with respect to Y and vanishes continuously on $\partial \mathbb{R}_+^d \cap B(0, r)$,

$$\frac{f(x)}{f(y)} \leq C_{39} \frac{x_d^p}{y_d^p}, \quad \text{for all } x, y \in \mathbb{R}_+^d \cap B(0, r/2). \tag{7.1}$$

Let $r_0 = \widehat{R}/4$ and choose a point $z_0 \in \partial \mathbb{R}_+^d$ with $|z_0| = 4$. For $n \in \mathbb{N}$, $B(z_0, 1/n)$ does not intersect $B(0, 2r_0)$. We define

$$K_n := \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} |\log(y_d)|^{\beta_3 + \beta_4} dy, \qquad f_n(y) := K_n^{-1} y_d^{-\beta_1} \mathbf{1}_{\mathbb{R}_+^d \cap B(z_0, 1/n)}(y),$$

and

$$g_n(x) := \mathbb{E}_x\left[f_n(Y_{\tau_{U(r_0)}})\right] = \mathbb{E}_x \int_0^{\tau_{U(r_0)}} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} J^{\mathbb{R}_+^d}(Y_t, y) f_n(y) dy dt, \quad x \in U(r_0).$$

Lemma 7.1. *There exist $C_{40} > 0$ and $a_1 \in (0,1)$ such that*

$$\liminf_{n \to \infty} g_n(x) \geq C_{40} x_d^{\beta_2 + \alpha} \tag{7.2}$$

for all $x = x^{(s)} = (\widetilde{0}, s) \in \mathbb{R}_+^d$ with $s \in (0, r_0 a_1)$.

Proof. Since

$$6 > |z - y| > 2 > y_d \wedge z_d \quad \text{for } (y, z) \in (\mathbb{R}_+^d \cap B(z_0, 1/n)) \times U(r_0),$$

using **(B7)** we have for $(y, z) \in (\mathbb{R}_+^d \cap B(z_0, 1/n)) \times U(r_0)$,

$$J^{\mathbb{R}_+^d}(z, y) \asymp (z_d \wedge y_d)^{\beta_1} (z_d \vee y_d)^{\beta_2} \log\left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{z_d \vee y_d}\right)^{\beta_4}$$

$$\asymp z_d^{\beta_1} y_d^{\beta_1} (z_d \vee y_d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{z_d \vee y_d}\right)^{\beta_4}.$$

Therefore, for $x \in U(r_0)$,

$$g_n(x) \asymp$$

$$K_n^{-1} \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (Y_t^d \vee y_d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{Y_t^d \vee y_d}{Y_t^d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{Y_t^d \vee y_d}\right)^{\beta_4} dy dt.$$

$$(7.3)$$

Note that, using $\sup_{t \geq 1} t^{-(\beta_1 - \beta_2)} \log(1 + t)^{\beta_3} < \infty$, for $z \in U(r_0)$,

$$K_n^{-1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (z_d \vee y_d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{z_d \vee y_d}\right)^{\beta_4} dy$$

$$\leq K_n^{-1} (z_d)^{-(\beta_1 - \beta_2)} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n) \cap \{z_d \leq y_d\}} (y_d/z_d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{y_d}{z_d}\right)^{\beta_3} \log\left(\frac{1}{y_d}\right)^{\beta_4} dy$$

$$+ K_n^{-1} (z_d)^{-(\beta_1 - \beta_2)} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n) \cap \{z_d > y_d\}} \log\left(1 + \frac{z_d}{y_d}\right)^{\beta_3} \log\left(\frac{1}{z_d}\right)^{\beta_4} dy$$

$$\leq c K_n^{-1} (z_d)^{-(\beta_1 - \beta_2)} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} \log\left(\frac{1}{y_d}\right)^{\beta_3 + \beta_4} dy \leq c z_d^{-(\beta_1 - \beta_2)} \tag{7.4}$$

and

$$\lim_{n \to \infty} K_n^{-1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (z_d \vee y_d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{z_d \vee y_d}\right)^{\beta_4} dy = z_d^{-(\beta_1 - \beta_2)}.$$

Moreover, by Lemma 5.12, $\mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_2} dt < \infty$ for all $x \in U(r_0)$. Thus we can use the dominated convergence theorem to get that for all $x \in U(r_0)$,

$$\lim_{n \to \infty} K_n^{-1} \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (Y_t^d \vee z_d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{Y_t^d \vee y_d}{Y_t^d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{Y_t^d \vee y_d}\right)^{\beta_4} dy dt$$

$$= \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} (Y_t^d)^{-(\beta_1 - \beta_2)} dt = \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_2} dt. \tag{7.5}$$

Combining (7.5) with Lemma 5.11 we conclude that (7.2) holds true. $\square$

Proof of Theorem 1.4. From (7.3), (7.4) and Lemma 5.12, we see that for all $x \in U(r_0)$,

$$g_n(x)$$

$$\leq c K_n^{-1} \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (Y_t^d)^{-(\beta_1 - \beta_2)} \log\left(1 + \frac{Y_t^d \vee y_d}{Y_t^d \wedge y_d}\right)^{\beta_3} \log\left(\frac{1}{Y_t^d \vee y_d}\right)^{\beta_4} dy dt$$

$$\leq c \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_2} dt \leq c x_d^{\beta_2 + \alpha}. \tag{7.6}$$

Thus the g_n's are non-negative functions in $\mathbb{R}^d_+$ which are harmonic in $\mathbb{R}^d_+ \cap B(0, 2^{-2}r_0)$ with respect to $Y^{\mathbb{R}^d_+,\kappa}$ and vanish continuously on $\partial\mathbb{R}^d_+ \cap B(0, 2^{-2}r_0)$. Therefore, by (7.1),

$$\frac{g_n(y)}{g_n(w)} \leq C_{30}\frac{(y_d)^p}{(w_d)^p} \quad \text{for all } y \in D \cap B(0, 2^{-3}r_0),$$

where $w = (\widetilde{0}, 2^{-3}r_0)$ and $C_{30} = C_{30}(2^{-2}r_0)$. Thus by (7.6), for all $y \in \mathbb{R}^d_+ \cap B(0, 2^{-3}r_0)$,

$$\limsup_{n\to\infty} g_n(y) \leq C_{30}\limsup_{n\to\infty} g_n(w)\frac{(y_d)^p}{(w_d)^p} \leq c_4(y_d)^p.$$

This and (7.2) imply that for all $x = x^{(s)} = (\widetilde{0}, s) \in \mathbb{R}^d_+$ with $s \in (0, 2^{-3}a_1 r_0)$, $x_d^{\beta_2+\alpha} \leq cx_d^p$, which gives a contradiction because $\beta_2 + \alpha < p$. $\qquad\square$

8. Proof of Lemma 5.8

In this section we prove Lemma 5.8. *Throughout this section, we assume that **(B1)**, **(B4)**, **(B7)**, **(B8)** and* (1.10) *hold.* Recall that, because of **(B4)**, we can, without loss of generality, assume that $B(x,x) = 1$. We start with three auxiliary lemmas estimating certain integrals, and then state and prove a technical lemma needed later. We then define the function ψ, show that (b1), (b2), (c1) and (c2) are immediate consequences of the definition of ψ, and that (b3) and (c3) follow directly from Lemma 5.6 and Lemma 5.8 (a). The remaining part, namely the proof of (a), is divided into two cases depending on the location of the point $z \in U$.

8.1. **Auxiliary lemmas.** In the next lemma and throughout the section we use the notation $\mathbb{R}^d_- := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d < 0\}$.

Lemma 8.1. *For $\alpha \in [1, 2)$,*

$$\int_{\mathbb{R}^d_- \cap \{|y-z|<6\}} \frac{1}{|y-z|^{d+\alpha-1}}dy \leq C_{41}\begin{cases} z_d^{1-\alpha}, & \alpha \in (1,2); \\ |\log z_d|, & \alpha = 1, \end{cases} \quad z \in U.$$

Proof. Immediate by enlarging the area of integration to $B(z, 6) \setminus B(z, z_d)$. $\qquad\square$

Lemma 8.2. *For $k \in \mathbb{N}$, let $I(k) := \int_{D(7,7)} |y-z|^{-d-\alpha+k}\mathcal{B}(y, z)\, dy$. There exists $C_{42} > 0$ such that*

(a) *if $\alpha < k \leq \alpha + \beta_1$, then $I(k) \leq C_{42}z_d^{k-\alpha}|\log z_d|^{\beta_3+1}$ for $z \in D(2^{-1}, 2^{-1})$;*

(b) *if k is the smallest integer such that $k > \alpha + \beta_1$, then $I(k) \leq C_{42}z_d^{\beta_1}|\log z_d|^{\beta_3}$ for $z \in D(2^{-1}, 2^{-1})$.*

Proof. We write

$$\int_{D(7,7)\cap\{|y-z|<z_d\}} \frac{\mathcal{B}(y,z)dy}{|y-z|^{d+\alpha-k}} + \int_{D(7,7)\cap\{|y-z|\geq z_d\}} \frac{\mathcal{B}(y,z)dy}{|y-z|^{d+\alpha-k}} =: I + II.$$

Then, by using that $\mathcal{B}(y, z) \leq C_1$,

$$I \leq C_1 \int_{D(7,7)\cap\{|y-z|<z_d\}} |y-z|^{-d-\alpha+k}dy \leq c_1 \int_0^{z_d} r^{-1-\alpha+k}dr \leq c_2 z_d^{k-\alpha} .$$

For II we use Lemma 5.2(b) together with the fact that $|y| \leq 7\sqrt{2}$ for $y \in D(7, 7)$, to get the estimate $\mathcal{B}(y, z) \leq c_3 z_d^{\beta_1}|\log z_d|^{\beta_3}|y-z|^{-\beta_1}$. This implies that

$$II \leq \begin{cases} c_3 z_d^{\beta_1}|\log z_d|^{\beta_3}\int_{z_d}^{c_4} r^{-1-\alpha-\beta_1+k}dr \leq c_5 z_d^{k-\alpha}|\log z_d|^{\beta_3+1} & \text{if } \alpha < k < \alpha+\beta_1; \\ c_3 z_d^{\beta_1}|\log z_d|^{\beta_3}\int_{z_d}^{c_4} r^{-1}dr \leq c_5 z_d^{k-\alpha}|\log z_d|^{\beta_3+1} & \text{if } \alpha < k = \alpha+\beta_1; \\ c_3 z_d^{\beta_1}|\log z_d|^{\beta_3}\int_{z_d}^{c_4} r^{-1-\alpha-\beta_1+k}dr \leq c_5 z_d^{\beta_1}|\log z_d|^{\beta_3} & \text{if } k > \alpha+\beta_1. \end{cases}$$

$\square$

Lemma 8.3. *For $\alpha \in [1, 2)$,*

$$\int_{D(7,7)} \frac{1}{|y-z|^{d+\alpha-1}} |1 - \mathcal{B}(y,z)| dy \leq C_{43} \begin{cases} z_d^{1-\alpha}, & \alpha \in (1,2); \\ |\log z_d|, & \alpha = 1, \end{cases} \quad z \in D(2^{-1}, 2^{-1}).$$

Proof. We write

$$\int_{D(7,7)} \frac{|1 - \mathcal{B}(y,z)|}{|y-z|^{d+\alpha-1}} dy = \int_{D(7,7)\cap\{|y-z|<z_d/2\}} + \int_{D(7,7)\cap\{|y-z|\geq z_d/2\}} =: I + II.$$

If $y \in B(z, 2^{-1}z_d)$, then $|y-z| \leq z_d/2 \leq y_d$ and $y_d \asymp z_d$ hence by **(B4)**, we have that

$$I \leq c_1 z_d^{-\theta} \int_{|y-z|<z_d/2} |y-z|^{\theta-d-\alpha+1} dy = c_2 z_d^{-\theta} \int_0^{z_d/2} r^{\theta-\alpha} dr = c_3 z_d^{1-\alpha}.$$

Clearly, $|1 - \mathcal{B}(y,z)| \leq c_4$. Therefore,

$$II \leq \int_{D(7,7)\cap\{|y-z|\geq z_d/2\}} |y-z|^{-d-\alpha+1} dy = c_5 \int_{z_d/2}^K r^{-\alpha} \leq c_6 \begin{cases} z_d^{1-\alpha}, & \alpha \in (1,2); \\ \log\frac{1}{z_d}, & \alpha = 1. \end{cases}$$

$\square$

Lemma 8.4. *Let $N \geq 2$ be an integer and $C > 0$. For $t, s > 0$, let*

$$F(t, s; C, N) := \left(\frac{t}{s}\right)^N - C \sum_{k=1}^{N-1} \left(\frac{t}{s}\right)^{N-k} |\log s|^{1+\beta_3}.$$

There exists $\widetilde{\kappa} = \widetilde{\kappa}(C, N, \beta_3) \in (0, e^{-1})$ such that if $s \leq \widetilde{\kappa}$ and $t \geq s^{1/2}$, then $F(t, s; C, N) \geq 0$.

Proof. By defining $c_1 = C(N-1)$, we can rewrite $F(t, s; C, N)$ in the form

$$F(t, s; C, N) = \frac{1}{N-1} \sum_{k=1}^{N-1} \left(\frac{t}{s}\right)^{N-k} \left(\left(\frac{t}{s}\right)^k - c_1 |\log s|^{1+\beta_3}\right)$$

$$\geq \frac{1}{N-1} \sum_{k=1}^{N-1} \left(\frac{t}{s}\right)^{N-k} \left(\frac{t}{s} - c_1 |\log s|^{1+\beta_3}\right)$$

provided that $t \geq s$. Since $\lim_{s \to 0} s^{1/2} |\log s|^{\beta_3+1} = 0$, there exists $\widetilde{\kappa} = \widetilde{\kappa}(c_1, \beta_3) \in (0, e^{-1})$ such that for $s \leq \widetilde{\kappa}$ we have $c_1 s^{1/2} |\log s|^{\beta_3+1} \leq 1$. If now $t \geq s^{1/2}$, then $t/s^{1/2} \geq 1 \geq c_1 s^{1/2} |\log s|^{\beta_3+1}$, implying $t/s - c_1 |\log s|^{\beta_3+1} \geq 0$. This implies that $F(t, s; C, N) \geq 0$ for $s \leq \widetilde{\kappa}$ and $t \geq s^{1/2}$. $\square$

8.2. Definition of ψ. Let $N := \lfloor \alpha + \beta_1 \rfloor + 1$ (the smallest integer larger than $\alpha + \beta_1$) and assume $\gamma > 2N + 2$. Let ψ be a non-negative C^N function in the upper half space with bounded support such that

$$\psi(y) = \begin{cases} |\widetilde{y}|^\gamma, & y \in D(2^{-2}, 2^{-2}); \\ 1, & y \in D(2,2) \setminus U; \\ 0, & y \in D(3,3)^c, \end{cases}$$

and $\psi(y) \geq 4^{-\gamma}$ for $y \in U \setminus D(2^{-2}, 2^{-2})$. The function ψ in the half space can be constructed so that, for $y = (\widetilde{y}, y_d)$ with $y_d \in (0, \frac{1}{8})$, $\psi(y)$ depends on $\widetilde{y}$ only. We extend ψ to be identically zero in the lower half space.

We recall the definition of the operator $L^{\mathcal{B}}$ from (5.9). For $\alpha \in (0, 1)$, $L_\alpha^{\mathcal{B}}\psi(z)$ is not really a principal value integral. For $\alpha \in [1, 2)$, it follows from Proposition 3.4 (a) that, for $z \in \mathbb{R}_+^d$, the principal value integral $L_\alpha^{\mathcal{B}}\psi(z)$ can be interpreted as the expression after the last equality in the display below

$$\int_{\mathbb{R}^d} \frac{\psi(y) - \psi(z) - \nabla\psi(z)\mathbf{1}_{\{|y-z|<6\}} \cdot (y-z)}{|y-z|^{d+\alpha}} dy + \int_{\mathbb{R}_-^d} \frac{\psi(z)}{|y-z|^{d+\alpha}} dy$$

$$+ \int_{\mathbb{R}_+^d} \frac{\psi(y) - \psi(z)}{|y-z|^{d+\alpha}} (\mathcal{B}(y,z) - 1) dy$$

$$= \int_{\mathbb{R}_+^d} \frac{\psi(y) - \psi(z) - \nabla\psi(z)\mathbf{1}_{\{|y-z|<6\}} \cdot (y-z)}{|y-z|^{d+\alpha}} dy + \int_{\mathbb{R}_+^d} \frac{\psi(y) - \psi(z)}{|y-z|^{d+\alpha}} (\mathcal{B}(y,z) - 1) dy$$

$$- \int_{\mathbb{R}_-^d} \frac{\nabla\psi(z)\mathbf{1}_{\{|y-z|<6\}} \cdot (y-z)}{|y-z|^{d+\alpha}} dy$$

$$= \int_{\mathbb{R}_+^d} \frac{\psi(y) - \psi(z) - \nabla\psi(z)\mathbf{1}_{\{|y-z|<6\}} \cdot (y-z)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d} \frac{\nabla\psi(z)\mathbf{1}_{\{|y-z|<6\}} \cdot (y-z)}{|y-z|^{d+\alpha}} (\mathcal{B}(y,z) - 1) dy - \int_{\mathbb{R}_-^d} \frac{\nabla\psi(z)\mathbf{1}_{\{|y-z|<6\}} \cdot (y-z)}{|y-z|^{d+\alpha}} dy$$

$$= \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\psi(y) - \psi(z) - \nabla\psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$- \psi(z) \int_{\mathbb{R}_+^d \cap \{|y-z|>6\}} \frac{1}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\nabla\psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} (\mathcal{B}(y,z) - 1) dy - \int_{\mathbb{R}_-^d \cap \{|y-z|<6\}} \frac{\nabla\psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} dy.$$

Thus, since $|y| \geq |y - z| - |z| > 5$ for $z \in U$ and $|y - z| > 6$, we have

$$L_\alpha^{\mathcal{B}}\psi(z) \leq \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\psi(y) - \psi(z) - \nabla\psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\nabla\psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} (\mathcal{B}(y,z) - 1) dy - \int_{\mathbb{R}_-^d \cap \{|y-z|<6\}} \frac{\nabla\psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} dy. \quad (8.1)$$

8.3. Proof of Lemma 5.8 (b) and (c). Let $p \in ((\alpha - 1)_+, \alpha + \beta_1)$, and recall that $h_p(x) = x_d^p \mathbf{1}_{D(1,1)}(x)$. Define $\phi := h_p - \psi$. The function ϕ is obviously non-positive on U^c, hence Lemma 5.8 (b2) holds true. Moreover, since $\psi((\widetilde{0}, x_d)) = 0$, we have that $\phi((\widetilde{0}, x_d)) = x_d^p$, for $(\widetilde{0}, x_d) \in U$, which is Lemma 5.8 (b1). Similarly, for $(\alpha - 1)_+ < q < p$, the function $\varphi := h_q - \psi$ satisfies Lemma 5.8 (c1) and (c2). Furthermore Lemma 5.8 (b3) and (c3) follow from Lemma 5.6 and Lemma 5.8 (a). In fact,

$$L^{\mathcal{B}}\phi(z) = L^{\mathcal{B}}h_p(z) - L^{\mathcal{B}}\psi(z) \geq (-C_{22} - C_{29})z_d^{\beta_1} |\log z_d|^{\beta_3}, \qquad x \in U(r_0).$$

By Lemma 5.6 (c) and part (a) of this lemma, for all $x \in U$ we have

$$L^{\mathcal{B}}\varphi(x) \;=\; L^{\mathcal{B}}h_q(x) - L^{\mathcal{B}}\psi(x) \geq -C_{26}z_x^{q-\alpha} - (C_{27} + C_{29})x_d^{\beta_1} |\log x_d|^{\beta_3}.$$

$\square$

8.4. Proof of Lemma 5.8 (a). It remains to prove Lemma 5.8 (a). The proof is split into two cases: (i) $z \in D(2^{-2}, 2^{-2})$, and (ii) $z \in U \setminus D(2^{-2}, 2^{-2})$.

8.4.1. *Proof when $z \in D(2^{-2}, 2^{-2})$.* Throughout this subsection we assume that $z \in D(2^{-2}, 2^{-2})$ so that $\psi(z) = \psi(\widetilde{z}) = |\widetilde{z}|^\gamma$. We will use Taylor's theorem in $d-1$ variables for the function $\psi(\widetilde{z})$ so we first briefly go over the notation. For a mulit-index $\rho = (\rho_1, \ldots, \rho_{d-1})$, we define $|\rho| := \rho_1 + \cdots + \rho_{d-1}$ and

$$\partial^\rho \psi(\widetilde{z}) = \partial_1^{\rho_1} \ldots \partial_{d-1}^{\rho_{d-1}} \psi(\widetilde{z}) = \frac{\partial^{|\rho|} \psi(\widetilde{z})}{\partial z_1^{\rho_1} \cdots \partial z_{d-1}^{\rho_{d-1}}}.$$

Let further $\rho! = \rho_1! \ldots \rho_{d-1}!$ and $\widetilde{z}^\rho = z_1^{\rho_1} \ldots z_{d-1}^{\rho_{d-1}}$. Then (recall that $N = \lfloor \alpha + \beta_1 \rfloor + 1$)

$$\psi(\widetilde{y}) = \sum_{|\rho| \leq N-1} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} (\widetilde{y} - \widetilde{z})^\rho + \sum_{|\rho| = N} \frac{\partial^\rho \psi(c\widetilde{y} + (1-c)\widetilde{z})}{\rho!} (\widetilde{y} - \widetilde{z})^\rho \tag{8.2}$$

for some $c \in (0,1)$. It is easy to see that, for any non-negative integer $k \leq N$,

$$\left| \sum_{|\rho|=k} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} \right| \leq c(\gamma, k) |\widetilde{z}|^{\gamma-k} \tag{8.3}$$

for some $c(\gamma, k) > 0$. Similarly, the terms in Taylor's formula (8.2) are estimated as

$$\left| \sum_{|\rho|=k} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} (\widetilde{y} - \widetilde{z})^\rho \right| \leq C(k) |\widetilde{z}|^{\gamma-k} |\widetilde{y} - \widetilde{z}|^k \leq C |\widetilde{z}|^{\gamma-k} |y - z|^k, \quad k \leq N-1, \tag{8.4}$$

$$\left| \sum_{|\rho|=N} \frac{\partial^\rho \psi(c\widetilde{y} + (1-c)\widetilde{z})}{\rho!} (\widetilde{y} - \widetilde{z})^\rho \right| \leq C(N) |y - z|^N, \tag{8.5}$$

for some constants $C(k) = C(\gamma, k) > 0$, $k = 0, 1, \ldots, N$.

Recall that $N = \lfloor \alpha + \beta_1 \rfloor + 1$ and $\gamma > 2N + 2$. Suppose first that $\alpha \in (0,1)$ and expand $L_\alpha^\mathcal{B} \psi(z)$ further:

$$L_\alpha^\mathcal{B} \psi(z) = \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\psi(y) - \psi(z)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy - \psi(z) \int_{\mathbb{R}_+^d \cap \{|y-z|>6\}} \frac{1}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$\leq \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\psi(y) - \psi(z)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \tag{8.6}$$

$$= \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\psi(y) - \psi(z) - \sum_{1 \leq |\rho| \leq N-1} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} (\widetilde{y} - \widetilde{z})^\rho}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d \cap \{|y-z|\}<6} \sum_{2 \leq |\rho| \leq N-1} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} \frac{(\widetilde{y} - \widetilde{z})^\rho}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d \cap \{|y-z|\}<6} \frac{\partial \psi(\widetilde{z})}{1!} \frac{(\widetilde{y} - \widetilde{z})}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy =: I + II + III.$$

When $\alpha \in [1,2)$, we use (8.1) to expand $L_\alpha^\mathcal{B}$ further to get

$$L_\alpha^\mathcal{B} \psi(z) \leq \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\psi(y) - \psi(z) - \sum_{1 \leq |\rho| \leq N-1} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} (\widetilde{y} - \widetilde{z})^\rho}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d \cap \{|y-z|\}<6} \sum_{2 \leq |\rho| \leq N-1} \frac{\partial^\rho \psi(\widetilde{z})}{\rho!} \frac{(\widetilde{y} - \widetilde{z})^\rho}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$+ \int_{\mathbb{R}_+^d \cap \{|y-z|<6\}} \frac{\nabla \psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}} (\mathcal{B}(y,z) - 1) dy - \int_{\mathbb{R}_-^d \cap \{|y-z|<6\}} \frac{\nabla \psi(z) \cdot (y-z)}{|y-z|^{d+\alpha}}$$

$$=: I + II + IV.$$

By using (8.2) we estimate the numerator in I by (8.5). The obtained integral is then estimated by use of Lemma 8.2 (b) by a constant multiple of $z_d^{\beta_1} |\log z_d|^{\beta_3}$ to get that

$$I \leq C(N) z_d^{\beta_1} |\log z_d|^{\beta_3} .$$

The terms in II are estimated by (8.4), and the obtained integrals by Lemma 8.2 (a) to get that

$$II \leq \sum_{k=2}^{N-1} C(k) |\widetilde{z}|^{\gamma-k} z_d^{k-\alpha} |\log z_d|^{\beta_3+1}.$$

By the same argument we get the estimate

$$III \leq C(1) |\widetilde{z}|^{\gamma-1} z_d^{1-\alpha} |\log z_d|^{\beta_3+1}.$$

The terms in IV are estimated by use Lemmas 8.3 and 8.1 to get that

$$IV \leq c|\widetilde{z}|^{\gamma-1} z_d^{1-\alpha} |\log z_d| \leq c|\widetilde{z}|^{\gamma-1} z_d^{1-\alpha} |\log z_d|^{\beta_3+1}.$$

Combining the estimates for I, II, III and IV, we get that for $\alpha \in (0,2)$ it holds that

$$L_\alpha^B(z) \leq \widehat{C} \left(z_d^{\beta_1} |\log z_d|^{\beta_3} + \sum_{k=1}^{N-1} |\widetilde{z}|^{\gamma-k} z_d^{k-\alpha} |\log z_d|^{\beta_3+1} \right) \tag{8.7}$$

for some constant $\widehat{C} > 0$.

Hence

$$L^{\mathcal{B}} \psi(z)$$

$$\leq \widehat{C} z_d^{\beta_1} |\log z_d|^{\beta_3} + \widehat{C} \sum_{k=1}^{N-1} |\widetilde{z}|^{\gamma-k} z_d^{k-\alpha} |\log z_d|^{\beta_3+1} - C(\alpha, p, \mathcal{B}) |\widetilde{z}|^{\gamma} z_d^{-\alpha} \tag{8.8}$$

$$= \widehat{C} z_d^{\beta_1} |\log z_d|^{\beta_3} - C(\alpha, p, \mathcal{B}) \frac{|\widetilde{z}|^{\gamma-N}}{z_d^{\alpha-N}} \left(\left(\frac{|\widetilde{z}|}{z_d} \right)^N - \frac{\widehat{C}}{C(\alpha, p, \mathcal{B})} \sum_{k=1}^{N-1} \left(\frac{|\widetilde{z}|}{z_d} \right)^{N-k} |\log z_d|^{\beta_3+1} \right)$$

$$= \widehat{C} z_d^{\beta_1} |\log z_d|^{\beta_3} - C(\alpha, p, \mathcal{B}) \frac{|\widetilde{z}|^{\gamma-N}}{z_d^{\alpha-N}} \left(\left(\frac{|\widetilde{z}|}{z_d} \right)^N - C \sum_{k=1}^{N-1} \left(\frac{|\widetilde{z}|}{z_d} \right)^{N-k} |\log z_d|^{\beta_3+1} \right), \tag{8.9}$$

where $C := \widehat{C}/C(\alpha, p, \mathcal{B})$,

Now we take $z_d \leq \widetilde{\kappa}(C, N)$ (where $\widetilde{\kappa}$ comes from Lemma 8.4) and $|\widetilde{z}| \geq z_d^{1/2}$. Then it follows from (8.9) and Lemma 8.4 that

$$L^{\mathcal{B}} \psi(z) \leq \widehat{C} z_d^{\beta_1} |\log z_d|^{\beta_3} - C(\alpha, p, \mathcal{B}) \frac{|\widetilde{z}|^{\gamma-N}}{z_d^{\alpha-N}} F(|\widetilde{z}|, z_d; C, N) \leq \widehat{C} z_d^{\beta_1} |\log z_d|^{\beta_3}.$$

In case when $|\widetilde{z}| \leq z_d^{1/2}$ and $z_d \leq \widetilde{\kappa}$, we estimate the middle term in (8.8) as

$$\sum_{k=1}^{N-1} |\widetilde{z}|^{\gamma-k} z_d^{k-\alpha} |\log z_d|^{\beta_3+1} = \frac{|\widetilde{z}|^{\gamma-N}}{z_d^{\alpha-N}(z_d^{1/2})^N} \sum_{k=1}^{N-1} \left(\frac{|\widetilde{z}|}{(z_d)^{1/2}} \right)^{N-k} (z_d^{1/2})^k |\log z_d|^{\beta_3+1}$$

$$= \left(\frac{|\widetilde{z}|}{z_d^{1/2}} \right)^{\gamma-N} z_d^{\frac{\gamma}{2}-\frac{1}{2}-\alpha} |\log z_d|^{\beta_3+1} \left(\sum_{k=1}^{N-1} \left(\frac{|\widetilde{z}|}{z_d^{1/2}} \right)^{N-k} (z_d^{1/2})^{k+1} \right)$$

$$\leq z_d^{\frac{\gamma}{2}-\frac{1}{2}-\alpha} |\log z_d|^{\beta_3+1} (N-1) \leq c z_d^{\beta_1}.$$

In the last line we first used that $|\widetilde{z}|/z_d^{1/2} \leq 1$ and $z_d \leq 1$ and then that $\gamma/2 - 1/2 - \alpha > \beta_1$. Thus, in this case we can disregard the last term in (8.8) and obtain again that $L^{\mathcal{B}}\psi(z) \leq C_{44}z_d^{\beta_1}|\log z_d|^{\beta_3}$ with $C_{44} = c\widehat{C}$.

Finally, it follows from (8.7) that for $z \in U$ with $z_d \geq \kappa := \widetilde{\kappa}$ it holds that $L^{\mathcal{B}}\psi(z) \leq C_{45}$ for some constant $C_{45} = C_{45}(\kappa) > 0$.

Set $C_{46} := C_{45}((\kappa^{\beta_1}(\log 2)^{\beta_3})) \vee 1)^{-1}$. Then $L^{\mathcal{B}}\psi(z) \leq (C_{44} \vee C_{46})z_d^{\beta_1}|\log z_d|^{\beta_3}$ for all $z \in D(2^{-2}, 2^{-2})$.

8.4.2. *Proof when* $z \in U \setminus D(2^{-2}, 2^{-2})$. Throughout this subsection we assume that $z \in U \setminus D(2^{-2}, 2^{-2})$. We show that there exist constants $C_{47} > 0$ and $\kappa \in (0, 1/2)$ such that (i) for $z_d \leq \kappa$ and $|\widetilde{z}| \in (1/4, 1)$ it holds that $L^{\mathcal{B}}\psi(z) \leq 0$; (ii) For $z_d \in [\kappa, 1/2)$ and $|\widetilde{z}| \in (1/4, 1)$ it holds that $L^{\mathcal{B}}\psi(z) \leq C_{47}$.

When $\alpha \in (0, 1)$ we see from (8.6) that

$$L_\alpha^{\mathcal{B}}\psi(z) \leq c \int_{D(7,7)} \frac{1}{|y - z|^{d+\alpha-1}}\mathcal{B}(y, z)\, dy \,.$$

Combining this with Lemma 8.2 we get that

$$L_\alpha^{\mathcal{B}}\psi(z) \leq C_{48}\left(z_d^{\beta_1} + z_d^{1-\alpha}\right)|\log z_d|^{\beta_3+1}, \quad z \in U\,. \tag{8.10}$$

When $\alpha \in [1, 2)$ we use (8.1) to estimate

$$L_\alpha^{\mathcal{B}}\psi(z) \leq c\left(\int_{D(7,7)} \frac{1}{|y - z|^{d+\alpha-2}}\mathcal{B}(y, z)\, dy + \int_{D(7,7)} \frac{1}{|y - z|^{d+\alpha-1}}|1 - \mathcal{B}(y, z)|\, dy \right.$$
$$\left. + \int_{\mathbb{R}_-^d \cap \{|y-z|<6\}} \frac{1}{|y - z|^{d+\alpha-1}}\mathcal{B}(y, z)\, dy\right)\,.$$

Combining this with Lemmas 8.1–8.3 we get that there exists a positive constant, which by slight abuse of notation we denote by $C_{48} > 0$, such that

$$L_\alpha^{\mathcal{B}}\psi(z) \leq C_{48}\left(z_d^{2-\alpha} + z_d^{\beta_1} + z_d^{1-\alpha}\right)|\log z_d|^{\beta_3+1}, \quad z \in U\,. \tag{8.11}$$

Let

$$f(z_d) := \left(z_d^2 + z_d^{\beta_1+\alpha} + z_d\right)|\log z_d|^{\beta_3+1}\,.$$

Assume that $|\widetilde{z}| \in (4^{-1}, 1)$. By the assumption on ψ, we have that $\psi(z) = \psi(\widetilde{z}, z_d) \geq 4^{-\gamma}$. Since $\lim_{z_d \to 0} f(z_d) = 0$, we can choose $\kappa > 0$ so that

$$f(z_d) - \frac{C(\alpha, p, \mathcal{B})4^{-\gamma}}{C_{48}} \leq 0$$

for all $z_d \in (0, \kappa)$. Then,

$$L^{\mathcal{B}}\psi(z) = L_\alpha^{\mathcal{B}}\psi(z) - C(\alpha, p, \mathcal{B})z_d^{-\alpha}\psi(z) \leq L_\alpha^{\mathcal{B}}\psi(z) - C(\alpha, p, \mathcal{B})4^{-\gamma}z_d^{-\alpha}$$
$$\leq C_{48}z_d^{-\alpha}\left((f(z_d) - \frac{C(\alpha, p, \mathcal{B})4^{-\gamma}}{C_{48}}\right) \leq 0$$

for all $z \in U \setminus D(2^{-2}, 2^{-2})$ with $|\widetilde{z}| \in (4^{-1}, 1)$ and $z_d \in (0, \kappa)$.

Finally, it follows from (8.10) and (8.11) that there exists $C_{47} = C_{47}(\kappa)$ such that $L^{\mathcal{B}}\psi(z) \leq L_\alpha^{\mathcal{B}}\psi(z) \leq C_{47}$ for all $z \in U$ with $z_d \geq \kappa$. $\qquad\square$

9. Proofs of Lemmas 5.9 and 5.11–5.13

Before we give the proofs of these lemmas, we first do some preparation.

Let $v \in C_c^\infty(\mathbb{R}^d)$ be a non-negative smooth radial function such that $v(y) = 0$ for $|y| \geq 1$ and $\int_{\mathbb{R}^d} v(y)\,dy = 1$. For any $a > 0$, set $D(a) := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > a\}$ and $U_a(r) := \{y \in U(r) : \delta_{U(r)} > a\}$. We write U_a for $U_a(1)$ and recall that $U = D(1/2, 1/2)$. For $b \geq 10$ and $k \geq 1$, set $v_k(y) := b^{kd} v(b^k y)$. Next we define $g_k := v_k * (g\mathbf{1}_{D(5^{-k})})$ for a bounded function g vanishing on $\mathbb{R}^d \setminus \mathbb{R}_+^d$. Since $b^{-k} < 5^{-k}$, we have $g_k \in C_c^\infty(\mathbb{R}_+^d)$ and hence $L^{\mathcal{B}} g_k$ is defined everywhere. Also note that $v_k * g \in C_c^\infty(\mathbb{R}_+^d; \mathbb{R}^d)$ and thus $L^{\mathcal{B}}(v_k * g)$ is well defined (cf. Subsection 3.2).

Let $(a_k)_{k \geq 1}$ be a decreasing sequence of positive numbers such that $\lim_{k \to \infty} a_k = 0$ and

$$a_k \geq 2^{-k(\beta_1/2+1)/(1+\alpha+3\beta_1/2)} \geq 2^{-k}.$$

Lemma 9.1. *Let* $g : \mathbb{R}^d \to [0,1]$ *be a Borel function vanishing on* $\mathbb{R}^d \setminus \mathbb{R}_+^d$. *For any* $z \in U$ *it holds that*

$$\lim_{k \to \infty} L^{\mathcal{B}}(v_k * g - g_k)(z) = 0. \tag{9.1}$$

Moreover, there exists $C_{49} > 0$ *independent of* g *such that for all* $k \geq 2$ *and* $z \in U_{a_k}$

$$0 \leq L^{\mathcal{B}}(v_k * g - g_k)(z) \leq C_{49}(2/3)^{k(\beta_1/2+1)} z_d^{\beta_1}. \tag{9.2}$$

Proof. Let $z \in U_{a_k}$. We first estimate the difference

$$L^{\mathcal{B}}(v_k * g - g_k)(z) = \lim_{\epsilon \to 0} \int_{\mathbb{R}_+^d, |y-z|>\epsilon} \frac{\big((v_k * g)(y) - g_k(y)\big) - \big((v_k * g)(z) - g_k(z)\big)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z)\,dy$$
$$- C(\alpha, p, \mathcal{B}) z_d^{-\alpha}(v_k * g - g_k)(z).$$

Note that for $k \geq 2$, $u \in B(0, b^{-k})$ and $y \in \mathbb{R}_+^d$ with $y_d > 3^{-k}$, $y_d - u_d > 3^{-k} - 10^{-k} > 5^{-k}$. Therefore

$$1 - \mathbf{1}_{D(5^{-k})}(y - u) = 0. \tag{9.3}$$

Since v_k is supported in $B(0, b^{-k})$, for all $k \geq 2$ and $z \in \mathbb{R}_+^d$ with $z_d > a_k > 2^{-k}$,

$$\int_{\mathbb{R}^d} (1 - \mathbf{1}_{D(5^{-k})}(z - u)) h_p(z - u) v_k(u)\,du = 0.$$

Thus $(v_k * g - g_k)(z) = 0$. Because of the same reason we have that for $z \in U_{a_k}$,

$$\int_{\mathbb{R}_+^d, |y-z|>\epsilon} \frac{\big((v_k * g)(y) - g_k(y)\big) - \big((v_k * g)(z) - g_k(z)\big)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z)\,dy$$

$$= \int_{\mathbb{R}_+^d, |y-z|>\epsilon, y_d \leq 3^{-k}} \int_{\mathbb{R}^d} v_k(u) \frac{(1 - \mathbf{1}_{D(5^{-k})})(y-u)g(y-u)}{|y-z|^{d+\alpha}}\,du \mathcal{B}(y,z)\,dy$$

$$\leq \int_{\mathbb{R}^d} v_k(u)\,du \int_{\mathbb{R}^d, y_d \leq 3^{-k}} \frac{\mathcal{B}(y,z)}{|y-z|^{d+\alpha}}\,dy$$

$$\leq c_1 \int_{y_d \leq 3^{-k}} \frac{1}{|y-z|^{d+\alpha}} \left(\frac{y_d}{|y-z|}\right)^{\beta_1/2}\,dy$$

$$\leq c_2 (3^{-k})^{\beta_1/2+1} \int_0^\infty \frac{t^{d-2}}{(t^2 + c z_d^2)^{(d+\alpha+\beta_1/2)/2}}\,dt$$

$$= c_3 (3^{-k})^{\beta_1/2+1} z_d^{-1-\alpha-\beta_1/2} \int_0^\infty \frac{s^{d-2}}{(s^2+1)^{(d+\alpha+\beta_1/2)/2}}\,ds$$

$$\leq c_4 (2/3)^{k(\beta_1/2+1)} z_d^{\beta_1}.$$

In the third line we used that $0 \leq g \leq 1$, in the fourth the fact that (together with (5.2))

$$\left(\frac{y_d \wedge z_d}{|y - z|} \wedge 1 \right)^{\beta_1} \log \left(1 + \frac{(y_d \vee z_d) \wedge |y - z|}{y_d \wedge z_d \wedge |y - z|} \right)^{\beta_3} \leq c \left(\frac{y_d}{|y - z|} \wedge 1 \right)^{\beta_1/2},$$

in the fifth integration in polar coordinates in $\mathbb{R}^{d-1}$, in the sixth the change of variables $t = c^{1/2} z_d s$, and in the last line the fact that $2^{-k(\beta_1/2+1)} z_d^{-1-\alpha-3\beta_1/2} \leq 1$ which follows from $z_d \geq a_k$ and the choice of a_k. Note also that it is clear from the second line that the first line is non-negative. Thus by letting $\epsilon \to 0$ we get for $z \in U_{a_k}$,

$$0 \leq L^{\mathcal{B}}(v_k * g - g_k)(z) \leq c_4 (2/3)^{k(\beta_1/2+1)} z_d^{\beta_1}.$$

Now take $z \in U$. Then there exists $k_0 \geq 1$ such that $z \in U_{a_k}$ for all $k \geq k_0$, and it follows from above that

$$\lim_{k \to \infty} L^{\mathcal{B}}(v_k * g - g_k)(z) = 0.$$

$\square$

Lemma 9.2. *Assume that $g : \mathbb{R}_+^d \to [0, 1]$ is a function which is C^2 on $D(1, 1)$. For any $k \geq 2$, $z \in U_{a_k}$ and $|u| < b^{-k}$,*

$$\text{p.v.} \int_{\mathbb{R}_+^d} \frac{g(y - u) - g(z - u)}{|y - z|^{d+\alpha}} \mathcal{B}(y, z) \, dy \tag{9.4}$$

is well defined. Moreover, for $z \in U_{a_k}$,

$$L_\alpha^{\mathcal{B}}(v_k * g)(z) = \int_{\mathbb{R}^d} v_k(u) \left(\text{p.v.} \int_{\mathbb{R}_+^d} \frac{g(y - u) - g(z - u)}{|y - z|^{d+\alpha}} \mathcal{B}(y, z) \, dy \right) du, \tag{9.5}$$

and there exists $C_{50}(z) > 0$ such that $|L_\alpha^{\mathcal{B}}(v_k * g)(z)| \leq C_{50}(z)$ for all $k \geq 2$.

Proof. Let $z \in U_{a_k}$ and $|u| < b^{-k}$. Let $G(y, z, u) := (g(y - u) - g(z - u))|y - z|^{-d-\alpha}$. For $0 < \epsilon < \eta < z_d/10$, consider

$$\int_{\mathbb{R}_+^d, \epsilon < |y-z|} G(y, z, u) \mathcal{B}(y, z) \, dy - \int_{\mathbb{R}_+^d, \eta < |y-z|} G(y, z, u) \mathcal{B}(y, z) \, dy$$

$$= \int_{\mathbb{R}_+^d, \epsilon < |y-z| < \eta} G(y, z, u) \mathcal{B}(y, z) \, dy$$

$$= \int_{\epsilon < |y-z| < \eta} G(y, z, u) \, dy + \int_{\epsilon < |y-z| < \eta} G(y, z, u)(\mathcal{B}(y, z) - 1) \, dy$$

$$=: I + II.$$

Since g is C^2 on $D(1, 1)$ and $y - u, z - u \in D(1, 1)$, we see that

$$|I| \leq \int_{\epsilon < |(y-u)-(z-u)| < \eta} \frac{|g(y - u) - g(z - u) - \nabla g(z - u) \mathbf{1}_{(|(y-u)-(z-u)|<1)} \cdot (y - z)|}{|(y - u) - (z - u)|^{d+\alpha}} \, dy$$

$$\leq c_1 \sup_{w \in B(z, z_d/5)} |\partial^2 g(w)| \int_{\epsilon < |y-z| < \eta} |y - z|^{-d-\alpha+2} \, dy = c_2(z)(\eta^{2-\alpha} - \epsilon^{2-\alpha}).$$

Further, by using the mean value theorem in the first line and **(B4)** in the second, we get

$$|II| \leq \sup_{w \in B(z, z_d/5)} |\nabla g(w)| \int_{\epsilon < |y-z| < \eta} \frac{|\mathcal{B}(y, z) - 1|}{|y - z|^{d+\alpha-1}} \, dy$$

$$\leq c_3(z) \int_{\epsilon < |y-z| < \eta} |y - z|^{-d-\alpha} \left(\frac{|y - z|}{y_d \wedge z_d} \right)^\theta \, dy$$

$$\leq c_4 c_3(z) z_d^{-\theta} \int_{\epsilon < |y-z| < \eta} |y-z|^{-d-\alpha+\theta}\, dy = c_5(z)(\eta^{\theta-\alpha+1} - \epsilon^{\theta-\alpha+1})\,.$$

The estimates for I and II imply that the principal value integral in (9.4) is well defined.

Let $z \in U_{a_k}$. For $\epsilon < z_d/10$ and $|u| < b^{-k}$, we have

$$\left| \int_{\mathbb{R}_+^d, |y-z|>\epsilon} G(y,z,u)\mathcal{B}(y,z)\, dy \right|$$

$$\leq \left| \int_{\mathbb{R}_+^d, |y-z|\geq z_d/10} G(y,z,u)\mathcal{B}(y,z)\, dy \right| + \left| \int_{\mathbb{R}_+^d, \epsilon<|y-z|<z_d/10} G(y,z,u)\mathcal{B}(y,z)\, dy \right|$$

$$=: III + IV\,.$$

By estimating g by 1, we get that

$$III \leq 2 \int_{|y-z|\geq z_d/10} |y-z|^{-d-\alpha} dy \leq c_6 z_d^{-\alpha} = c_7(z)\,.$$

The integral in IV is estimated in I and II with $\eta = z_d/10$, so we have

$$IV \leq c_2(z)(z_d/10)^{2-\alpha} + c_5(z)(z_d/10)^{\theta-\alpha+1} = c_8(z)\,.$$

Thus we have that

$$\left| \int_{\mathbb{R}_+^d, |y-z|>\epsilon} \frac{g(y-u)-g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z)\, dy \right| \leq c_9(z)\,. \tag{9.6}$$

Hence we can use the dominated convergence theorem to conclude that

$$L_\alpha^{\mathcal{B}}(v_k * g)(z) = \lim_{\epsilon \to 0} \int_{\mathbb{R}_+^d, |y-z|>\epsilon} \frac{(v_k*g)(y)-(v_k*g)(z)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z)\, dy$$

$$= \lim_{\epsilon \to 0} \int_{|u|<b^{-k}} v_k(u) \int_{\mathbb{R}_+^d, |y-z|>\epsilon} \frac{g(y-u)-g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z)\, dy\, du$$

$$= \int_{|u|<b^{-k}} v_k(u) \left(\lim_{\epsilon \to 0} \int_{\mathbb{R}_+^d, |y-z|>\epsilon} \frac{g(y-u)-g(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z)\, dy \right) du\,,$$

which is (9.5). The last statement follows from (9.6) . □

We note also that since g is continuous in $D(1,1)$, it holds that $\lim_{k\to\infty}(v_k * g)(z) = g(z)$ for all $z \in D(1,1)$.

Recall that the function h_q was defined in Section 5: $h_q(x) = x_d^q \mathbf{1}_{D(1,1)}$.

Lemma 9.3. *Let $q \in (\alpha-1)_+, \alpha+\beta_1)$ and set*

$$a_k := 2^{-k(q+1+\frac{1}{2}\beta_1)/(\alpha+1+\frac{3}{2}\beta_1)} \vee 2^{-k(2+\beta_1)/(1+\alpha+\frac{3}{2}\beta_1-q)}\,.$$

There exists a constant $C_{51} > 0$ such that for any $k \geq 1$ and $z \in U_{a_k}$,

$$|L^{\mathcal{B}}(v_k * h_q)(z) - L^{\mathcal{B}}h_q(z)| \leq C_{51}\left(\frac{4}{5}\right)^k\,. \tag{9.7}$$

In particular, the functions $z \to |L^{\mathcal{B}}(v_k * h_q)(z) - L^{\mathcal{B}}h_q(z)|$ are all bounded by the constant C_{51} on U, and for any $z \in U$, $\lim_{k\to\infty}\left|L^{\mathcal{B}}(v_k * h_q)(z) - L^{\mathcal{B}}h_q(z)\right| = 0$.

Proof. First note that $a_k \geq 2^{-k}$ since the first term in its definition is larger than 2^{-k}. Next, by using Lemma 9.2 with $g = h_q$ in the third line below we see that

$$
\begin{aligned}
L^{\mathcal{B}}&(v_k * h_q)(z) \\
&= L_\alpha^{\mathcal{B}}(v_k * h_q)(z) - C(\alpha, p, \mathcal{B}) z_d^{-\alpha} (v_k * h_q)(z) \\
&= \int_{\mathbb{R}^d} v_k(u) \left(\text{p.v.} \int_{\mathbb{R}_+^d} \frac{h_q(y-u) - h_q(z-u)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right) du - C(\alpha, p, \mathcal{B}) z_d^{-\alpha}(v_k * h_q)(z) \\
&= \int_{\mathbb{R}^d} v_k(u) \left(\text{p.v.} \int_{\mathbb{R}_+^d} \frac{h_q(y-u) - h_q(z-u) - (h_q(y) - h_q(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right) du \\
&\quad + C(\alpha, p, \mathcal{B}) z_d^{-\alpha}(h_q(z) - (v_k * h_q)(z)) + L^{\mathcal{B}} h_q(z).
\end{aligned}
$$

Note that for $z \in U_{a_k}$,

$$
|h_q(z) - (v_k * h_q)(z)| = \left| \int_{B(0, b^{-k})} v_k(u)(h_q(z) - h_q(z-u)) du \right| \leq c_0 (1/5)^k z_d^q. \tag{9.8}
$$

Set $b = 10 \vee 2^{4(q-2)_- + 3}$. Now we write, for $u \in B(0, b^{-k})$,

$$
\begin{aligned}
\int_{\mathbb{R}_+^d} & \frac{h_q(y-u) - h_q(z-u) - (h_q(y) - h_q(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \\
&= \int_{D(1+b^{-k}, 1+b^{-k}) \setminus U, y_d > 5^{-k}} + \int_{D(1+b^{-k}, 1+b^{-k}), y_d < 5^{-k}} \\
&\quad + \int_{U, y_d > 5^{-k}, |y-z| > 2^{-1} z_d} + \int_{U, y_d > 5^{-k}, B(z, z_d/2)} =: I + II + III + IV.
\end{aligned}
$$

We deal with I first. For $u \in B(0, b^{-k})$,

$$
I = \int_{D(1+b^{-k}, 1+b^{-k}) \setminus D(1-b^{-k}, 1-b^{-k}), y_d > 5^{-k}} + \int_{D(1-b^{-k}, 1-b^{-k}) \setminus U, y_d > 5^{-k}} =: I_1 + I_2.
$$

Obviously, we have $|I_1| \leq c_1 b^{-k}$.

Let $A_k := (D(1 - b^{-k}, 1 - b^{-k}) \setminus U) \cap \{y : y_d > 5^{-k}\}$. Then, we have

$$
\begin{aligned}
|I_2| &= \left| \int_{A_k} \frac{(y_d - u_d)^q - y_d^q - ((z_d - u_d)^q - z_d^q)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right| \\
&= \left| \int_{A_k} \frac{q(y_d - z_d) \cdot \int_0^1 ((z_d - u_d + \theta(y_d - z_d))^{q-1} - (z_d + \theta(y_d - z_d)^{q-1})) \, d\theta}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \right| \\
&\leq c_2 b^{-k} 2^{3k(q-2)_-} \leq c_2 2^{-k((q-2)_- + 3)},
\end{aligned}
$$

where in the first inequality we use mean value theorem inside the integral in the numerator.

For II, we have

$$
|II| \leq c_3 \int_{D(1+b^{-k}, 1+b^{-k}), y_d < 5^{-k}} \frac{y_p^q + z_d^q + b^{-kq}}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy.
$$

Similarly as in the proof of Lemma 9.1 we first estimate

$$
\begin{aligned}
\int_{D(1+b^{-k}, 1+b^{-k}), y_d < 5^{-k}} & \frac{y_d^q + b^{-kq}}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \\
&\leq c_4 \int_{\mathbb{R}^{d-1}} \int_0^{5^{-k}} \frac{y_d^{q+\beta_1/2} + 10^{-kq} y_d^{\beta_1/2}}{|y-z|^{d+\alpha+\frac{1}{2}\beta_1}} dy_d d\widetilde{y}
\end{aligned}
$$

$$\leq c_5 5^{-k(q+1+\frac{1}{2}\beta_1)} \int_0^\infty \frac{t^{d-2}}{(t^2 + cz_d^2)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} dt$$

$$= c_6 5^{-k(q+1+\frac{1}{2}\beta_1)} z_d^{-1-\alpha-\frac{1}{2}\beta_1} \int_0^\infty \frac{s^{d-2}}{(s^2 + 1)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} ds$$

$$\leq c_7 (2/5)^{k(q+1+\frac{1}{2}\beta_1)} z_d^{\beta_1}.$$

For the remaining part, we use a similar argument:

$$\int_{D(1+b^{-k}, 1+b^{-k}), y_d < 5^{-k}} \frac{z_d^q}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy$$

$$\leq c_8 z_d^q \int_{\mathbb{R}^{d-1}} \int_0^{5^{-k}} \frac{y_d^{\beta_1/2}}{|y-z|^{d+\alpha+\frac{1}{2}\beta_1}} dy_d d\widetilde{y}$$

$$\leq c_9 z_d^q 5^{-k(1+\frac{1}{2}\beta_1)} \int_0^\infty \frac{t^{d-2}}{(t^2 + cz_d^2)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} dt$$

$$= c_{10} z_d^q 5^{-k(1+\frac{1}{2}\beta_1)} z_d^{-1-\alpha-\frac{1}{2}\beta_1} \int_0^\infty \frac{s^{d-2}}{(s^2 + 1)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} ds$$

$$\leq c_{11}(4/5)^{k(1+\frac{1}{2}\beta_1)} 2^{-k(2+\beta_1)} z_d^{q-1-\alpha-\frac{1}{2}\beta_1} \leq c_{12}(4/5)^{k(1+\frac{1}{2}\beta_1)} z_d^{\beta_1}.$$

Thus

$$|II| \leq c_{13}\big((2/5)^{k(q+1+\frac{1}{2}\beta_1)} + (4/5)^{k(1+\frac{1}{2}\beta_1)}\big) z_d^{\beta_1}.$$

Let $B_k := U \cap \{y_d > 5^{-k}\} \cap \{y : |y-z| > 2^{-1} z_d\}$. Then, we have

$$|III| = \left| \int_{B_k} \frac{(y_d - u_d)^q - y_d^q - ((z_d - u_d)^q - z_d^q)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right|$$

$$= \left| \int_{B_k} \frac{q(y_d - z_d) \cdot \int_0^1 \big((z_d - u_d + \theta(y_d - z_s))^{q-1} - (z_d + \theta(y_d - z_d))^{q-1}\big) d\theta}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right|$$

$$\leq c_{14} b^{-k} 2^{k3(q-2)-} \int_{U, |y-z| > 2^{-1} z_d} \frac{1}{|y-z|^{d+\alpha-1}} dy$$

$$\leq c_{15}(4/5)^k 2^{-3k}(z_d^{1-\alpha} \vee \log \frac{1}{z_d}) \leq c_{16}(4/5)^k z_d^3(z_d^{1-\alpha} \vee \log \frac{1}{z_d}),$$

where in the first inequality we use mean value theorem inside the integral in the numerator and the fact the derivative of the integrand is bounded above by $c(5^{-k} - b^{-k})^{-(q-2)-} \leq c2^{3k(p-2)-}$.

Let $F(y_d, z_d, u_d) := q(q-1) \int_0^1 \big((z_d - u_d + \theta(y_d - z_d))^{q-2} - (z_d + \theta(y_d - z_d))^{q-2}\big)(1-\theta) d\theta$. For $\theta \in [0, 1]$, $u \in B(0, b^{-k})$ and $y \in B(z, \frac{1}{2}z_d)$, $z_d - u_d + \theta(y_d - z_d)$ and $z_d + \theta(y_d - z_d)$ are both comparable with z_d. Thus, for IV, we have for large k,

$$|IV| = \left| \int_{U, y_d > 5^{-k}, B(z, 2^{-1} z_d)} \frac{(y_d - u_d)^q - y_d^q - ((z_d - u_d)^q - z_d^q)}{|y-z|^{d+\alpha}} \mathcal{B}(y, z) dy \right|$$

$$= \left| \int_{U, y_d > 5^{-k}, B(z, 2^{-1} z_d)} \frac{q(q-1)(y_d - z_d)^2 F(y_d, z_d, u_d)}{|y-z|^{d+\alpha}} B(y, z) dy \right|$$

$$\leq c_{17} b^{-k} z_d^{q-3} \int_{B(z, 2^{-1} z_d)} \frac{1}{|y-z|^{d+\alpha-2}} dy = c_{18} b^{-k} z_d^{q-1-\alpha} \leq c_{18}(4/5)^k z_d^{q+1-\alpha},$$

where in the first inequality we used the mean value theorem inside the integral in the numerator.

By putting together (9.8) and the estimates for I, II, III and IV we see that (9.7) is true for some constant C (independent of z and $k \geq 1$). □

Recall that the function ψ in the next lemma was introduced in Section 8.

Lemma 9.4. *Let $(a_k)_{k \geq 1}$ be a decreasing sequence of positive numbers such that $\lim_{k \to \infty} a_k = 0$ and $a_k \geq 2^{-k(2+\beta_1)/(2+\alpha+\frac{3}{2}\beta_1)}$. There exists a constant $C_{52} > 0$ such that for every $k \geq 1$ and $z \in U_{a_k}$*

$$|L^{\mathcal{B}}(v_k * \psi)(z) - L^{\mathcal{B}}\psi(z)| \leq C_{52}\left(\frac{4}{5}\right)^k. \tag{9.9}$$

In particular, for any $z \in U$, $\lim_{k \to \infty}\left|L^{\mathcal{B}}(v_k * \psi)(z) - L^{\mathcal{B}}\psi(z)\right| = 0$.

Proof. The proof is similar to the proof of Lemma 9.3. By using Lemma 9.2 with $g = \psi$ in the third line below we see that

$$L^{\mathcal{B}}(v_k * \psi)(z)$$
$$= L^{\mathcal{B}}_\alpha(v_k * \psi)(z) - C(\alpha, p, \mathcal{B})z_d^{-\alpha}(v_k * \psi)(z)$$
$$= \int_{\mathbb{R}^d} v_k(u)\left(\text{p.v.}\int_{\mathbb{R}^d_+}\frac{\psi(y-u)-\psi(z-u)}{|y-z|^{d+\alpha}}\mathcal{B}(y,z)dy\right)du - C(\alpha, p, \mathcal{B})z_d^{-\alpha}(v_k * \psi)(z)$$
$$= \int_{\mathbb{R}^d} v_k(u)\left(\text{p.v.}\int_{\mathbb{R}^d_+}\frac{\psi(y-u)-\psi(z-u)-(\psi(y)-\psi(z))}{|y-z|^{d+\alpha}}\mathcal{B}(y,z)dy\right)du$$
$$+ C(\alpha, p, \mathcal{B})z_d^{-\alpha}(\psi(z) - (v_k * \psi)(z)) + L^{\mathcal{B}}\psi(z).$$

Set $b = 10 \vee 2^{4(q-2)_- + 3}$. Note that for $z \in U_{a_k}$,

$$|\psi(z) - (v_k * \psi)(z)| = \left|\int_{B(0,b^{-k})} v_k(u)(\psi(z) - \psi(z-u))du\right| \leq cb^{-k}. \tag{9.10}$$

Now we write, for $u \in B(0, b^{-k})$,

$$\int_{\mathbb{R}^d_+}\frac{\psi(y-u)-\psi(z-u)-(\psi(y)-\psi(z))}{|y-z|^{d+\alpha}}\mathcal{B}(y,z)dy$$
$$= \int_{D(3+b^{-k},3+b^{-k})\setminus U, y_d > 5^{-k}} + \int_{D(3+b^{-k},3+b^{-k}), y_d < 5^{-k}}$$
$$+ \int_{U, y_d > 5^{-k}, |y-z| > 2^{-1}z_d} + \int_{U, y_d > 5^{-k}, B(z, z_d/2)} =: I + II + III + IV.$$

We deal with I first. For $u \in B(0, b^{-k})$,

$$I = \int_{D(3+b^{-k},3+b^{-k})\setminus D(3-b^{-k},3-b^{-k}), y_d > 5^{-k}} + \int_{D(3-b^{-k},3-b^{-k})\setminus U, y_d > 5^{-k}} =: I_1 + I_2.$$

Obviously, we have $|I_1| \leq cb^{-k}$. Let $A_k := (D(3-b^{-k}, 3-b^{-k}) \setminus U) \cap \{y : y_d > 5^{-k}\}$. Then, we have

$$|I_2| = \left|\int_{A_k}\frac{\psi(y-u)-\psi(y)-(\psi(z-u)-\psi(z))}{|y-z|^{d+\alpha}}\mathcal{B}(y,z)dy\right|$$
$$= \left|\int_{A_k}\frac{(y-z)\cdot\int_0^1\left(\nabla\psi(z-u+\theta(y-z))-\nabla\psi(z+\theta(y-z))\right)d\theta}{|y-z|^{d+\alpha}}\mathcal{B}(y,z)dy\right| \leq cb^{-k},$$

where in the first inequality we use mean value theorem inside the integral in the numerator.

For II, we have

$$|II| \leq c \int_{D(3+b^{-k}, 3+b^{-k}), y_d < 5^{-k}} \frac{1}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy$$

$$\leq c \int_{\mathbb{R}^{d-1}} \int_0^{5^{-k}} \frac{y_d^{\beta_1/2}}{|y-z|^{d+\alpha+\frac{1}{2}\beta_1}} dy_d d\widetilde{y}$$

$$\leq c 5^{-k(1+\frac{1}{2}\beta_1)} \int_0^\infty \frac{t^{d-2}}{(t^2 + cz_d^2)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} dt$$

$$= c 5^{-k(1+\frac{1}{2}\beta_1)} z_d^{-1-\alpha-\frac{1}{2}\beta_1} \int_0^\infty \frac{s^{d-2}}{(s^2+1)^{(d+\alpha+\frac{1}{2}\beta_1)/2}} ds$$

$$\leq c(4/5)^{k(1+\frac{1}{2}\beta_1)} 2^{-k(2+\beta_1)} z_d^{-1-\alpha-\frac{1}{2}\beta_1} \leq c(4/5)^{k(1+\frac{1}{2}\beta_1)} z_d^{1+\beta_1},$$

since $a_k \geq 2^{-k(2+\beta_1)/(2+\alpha+\frac{3}{2}\beta_1)}$.

Let $B_k := U \cap \{y_d > 5^{-k}\} \cap \{y : |y-z| > 2^{-1}z_d\}$. Then, we have

$$|III| = \Big| \int_{B_k} \frac{\psi(y-u) - \psi(y) - (\psi(z-u) - \psi(z))}{|y-z|^{d+\alpha}} B(y,z) dy \Big|$$

$$= \Big| \int_{B_k} \frac{(y-z) \cdot \int_0^1 (\nabla\psi(z-u+\theta(y-z)) - \nabla\psi(z+\theta(y-z))) \, d\theta}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \Big|$$

$$\leq c b^{-k} \int_{U, |y-z|>2^{-1}z_d} \frac{1}{|y-z|^{d+\alpha-1}} dy$$

$$\leq c(4/5)^k 2^{-3k} (z_d^{1-\alpha} \vee \log \frac{1}{z_d}) \leq c(4/5)^k z_d^3 (z_d^{1-\alpha} \vee \log \frac{1}{z_d}),$$

where in the first inequality we use mean value theorem inside the integral in the numerator and the fact the derivative of the integrand is bounded.

Let $F(y,z,u) := \int_0^1 \left(\nabla^2\psi(z-u+\theta(y-z)) - \nabla^2\psi(z+\theta(y-z)) \right)(1-\theta) d\theta$. For IV, we have

$$|IV| = \Big| \int_{U, y_d > 5^{-k}, B(z,2^{-1}z_d)} \frac{\psi(y-u) - \psi(y) - (\psi(z-u) - \psi(z))}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \Big|$$

$$= \Big| \int_{U, y_d > 5^{-k}, B(z,2^{-1}z_d)} \frac{(y-z) \otimes (y-z) \cdot F(y,z,u)}{|y-z|^{d+\alpha}} \mathcal{B}(y,z) dy \Big|$$

$$\leq c b^{-k} \int_{B(z,2^{-1}z_d)} \frac{1}{|y-z|^{d+\alpha-2}} dy = c b^{-k} z_d^{2-\alpha},$$

where in the first inequality we use mean value theorem inside the integral in the numerator.

By putting together (9.10) and estimates for I, II, III and IV we see that (9.9) is true for some constant C (independent of z and $k \geq 1$). $\qquad\qquad\square$

Lemma 9.5. *Let $r \leq 1$. For every $x \in U(r)$ it holds that*

$$\mathbb{E}_x[h_p(Y_{\tau_{U(r)}})] = h_p(x) + \mathbb{E}_x \int_0^{\tau_{U(r)}} L^{\mathcal{B}} h_p(Y_s) \, ds. \tag{9.11}$$

Proof. Set $g_k := v_k * (h_p \mathbf{1}_{D(5^{-k})})$. By combining Lemmas 9.1 and 9.3 (with $q = p$), we see that for every $z \in U(r)$ with $r \leq 1/2$,

$$\lim_{k \to \infty} |L^{\mathcal{B}} g_k(z) - L^{\mathcal{B}} h_p(z)| = 0$$

and $|L^{\mathcal{B}}g_k(z) - L^{\mathcal{B}}h_p(z)|$ is bounded by the constant $C_{51} > 0$. Let $x \in U(r)$, $r \leq 1$. There is $k_0 \geq 1$ such that $x \in U_{a_k}(r)$ for all $k \geq k_0$. Note that since $g_k \in C_c^\infty(\mathbb{R}_+^d)$, it follows from Proposition 3.7 that for all $t \geq 0$,

$$\mathbb{E}_x[g_k(Y_{t \wedge \tau_{U_{a_k}(r)}})] = g_k(x) + \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L^{\mathcal{B}} g_k(Y_s) \, ds \,.$$

Clearly, $\lim_{k \to \infty} \tau_{U_{a_k}(r)} = \tau_{U(r)}$. Since $g_k \to h_p$ as $k \to \infty$, we get that the left-hand side above converges to $\mathbb{E}_x[h_p(Y_{t \wedge \tau_{U(r)}})]$. Further, by Lemma 9.3,

$$\lim_{k \to \infty} \left(\mathbf{1}_{s < \tau_{U_{a_k}(r)}} L^{\mathcal{B}} g_k(Y_s) - \mathbf{1}_{s < \tau_{U(r)}} L^{\mathcal{B}} h_p(Y_s) \right) = 0,$$

and $\left| \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L^{\mathcal{B}} g_k(Y_s) - \mathbf{1}_{s < \tau_{U(r)}} L^{\mathcal{B}} h_p(Y_s) \right|$ is bounded by the constant $C_{51} > 0$. By Lemma 5.6 (a), $0 \geq L^{\mathcal{B}} h_p(z) \geq -C_{22} z_d^{\beta_1} |\log z_d|^{\beta_3}$, and hence that

$$\mathbb{E}_x \int_0^t |\mathbf{1}_{s < \tau_{U(r)}} L^{\mathcal{B}} h_p(Y_s)| \, ds < \infty \,.$$

Thus we can use the dominated convergence theorem to conclude that

$$\lim_{k \to \infty} \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L^{\mathcal{B}} g_k(Y_s) \, ds = \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U(r)}} L^{\mathcal{B}} h_p(Y_s) \, ds \,.$$

By letting $t \to \infty$ we obtain (9.11). $\qquad \square$

Proof of Lemma 5.9. From Lemma 5.8 (b) we know that $\phi(x) = x_d^p$ for all $x = (\widetilde{0}, x_d)$, $\phi(x) \leq 0$ for $x \in U^c \cap \mathbb{R}_+^d$ and $L^{\mathcal{B}} \phi(x) \geq -C_{30} x_d^{\beta_1} |\log x_d|^{\beta_3}$ for all $x \in U$. In particular, $L^{\mathcal{B}} \phi(x) \geq -C_{30}$ for all $x \in U$.

Let $g_k = v_k * (\phi \mathbf{1}_{D(5^{-k})})$. It follows from Lemmas 9.1, 9.3 and 9.4 that $\lim_{k \to \infty} |L^{\mathcal{B}} g_k(z) - L^{\mathcal{B}} \phi(z)| = 0$ for every $z \in U$, and the sequence of functions $|L^{\mathcal{B}} g_k(z) - L^{\mathcal{B}} \phi(z)|$ on U is bounded by some constant C. Then $L^{\mathcal{B}} g_k(z) \geq L^{\mathcal{B}} \phi(z) - C \geq -C_{30} - C$ for every $z \in U$.

Note that since $g_k \in C_c^\infty(\mathbb{R}_+^d)$, it follows from Proposition 3.7 that for all $t \geq 0$,

$$\mathbb{E}_x[g_k(Y_{t \wedge \tau_{U_{a_k}}})] = g_k(x) + \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}}} L^{\mathcal{B}} g_k(Y_s) \, ds \,.$$

Clearly, $\lim_{k \to \infty} \tau_{U_{a_k}} = \tau_U$. Since $g_k \to \phi$ as $k \to \infty$, we get that the left-hand side above converges to $\mathbb{E}_x[\phi(Y_{t \wedge \tau_U})]$. Further

$$\lim_{k \to \infty} \left(\mathbf{1}_{s < \tau_{U_{a_k}}} L^{\mathcal{B}} g_k(Y_s) \right) = \mathbf{1}_{s < \tau_U} L^{\mathcal{B}} \phi(Y_s)$$

and $\mathbf{1}_{s < \tau_{U_{a_k}}} L^{\mathcal{B}} g_k(Y_s) \geq -C_{30} - C$. Thus we may use Fatou's lemma to conclude that

$$\mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_U} L^{\mathcal{B}} \phi(Y_s) \, ds \leq \liminf_{k \to \infty} \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}}} L^{\mathcal{B}} g_k(Y_s) \, ds.$$

By using that the right-hand side above is equal to $\mathbb{E}_x[\phi(Y_{t \wedge \tau_U})] - \phi(x)$ and then letting $t \to \infty$, we arrive at

$$\mathbb{E}_x[\phi(Y_{\tau_U})] - \phi(x) \geq \mathbb{E}_x \int_0^{\tau_U} L^{\mathcal{B}} \phi(Y_s) \, ds \geq -C_{30} \mathbb{E}_x \int_0^{\tau_U} (Y_s^d)^{\beta_1} |\log Y_s^d|^{\beta_3} \, ds \,.$$

Since $\mathbb{E}_x[\phi(Y_{\tau_U})] \leq 0$ and $\phi(x) = x_d^p$, this concludes the proof. $\qquad \square$

Proof of Lemma 5.11. Let $q := \alpha + \beta_2$. Note that $q < p$ by assumption. By repeating the proof of Lemma 5.9 (with h_p replaced by h_q, and Lemma 5.8.(b) replaced by Lemma 5.8.(c)), we get that $x = (\widetilde{0}, x_d) \in U(r)$,

$$C_{31}\mathbb{E}_x \int_0^{\tau_U(r)} (Y_t^d)^{\beta_2}\, dt + C_{32}\mathbb{E}_x \int_0^{\tau_U(r)} (Y_t^d)^{\beta_1}|\log Y_t^d|^{\beta_3}\, dt \geq x_d^q.$$

By Lemma 5.7, we have $\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1}|\log Y_t^d|^{\beta_3}\, dt \leq C_{28}x_d^p$ for $x \in U$. Thus for $x \in U$,

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_2}\, dt \geq C_{31}^{-1}(x_d^q - C_{32}C_{28}x_d^p)\,.$$

There exist $a_1 > 0$ and $C_{34} > 0$ so that the last term is greater than $C_{34}x_d^q$ whenever $0 < x_d < a_1$. By scaling, we can replace U by $U(r)$. $\qquad\square$

Proof of Lemma 5.12. Let $q := \alpha + \beta_2$ and $\eta(x) := h_q(x) - h_p(x)$. Note that $q < p$ by assumption. For $x \notin D(1,1)$, $\eta(x) = 0$, while if $x \in D(1,1)$ we have $\eta(x) = x_d^q - x_d^p > 0$. Since $(\alpha - 1)_+ < q < p$, by Lemma 5.6 (a) and (c), for all $x \in U$ we have that $L^{\mathcal{B}}h_p(x) \geq -C_{22}x^{\beta_1}|\log x_d|^{\beta_3}$ and $L^{\mathcal{B}}h_q(x) \leq -C_{25}x^{q-\alpha} = -C_{25}x^{\beta_2}$. Thus, since $\beta_2 < \beta_1$, we can find $r_1 \in (0, 1/2]$ such that

$$L^{\mathcal{B}}\eta(x) = L^{\mathcal{B}}h_q(x) - L^{\mathcal{B}}h_p(x) \leq -C_{25}x_d^{\beta_2} + C_{22}x_d^{\beta_1}|\log x_d|^{\beta_3} \leq -2^{-1}C_{25}x_d^{\beta_2}, \qquad x \in U(r_1)$$

and, also by Lemma 5.6 (c),

$$L^{\mathcal{B}}\eta(x) = L^{\mathcal{B}}h_q(x) - L^{\mathcal{B}}h_p(x) \geq -C_{26}z_d^{q-\alpha} - C_{27}z_d^{\beta_1}|\log z_d|^{\beta_3}, \qquad x \in U(r_1).$$

Repeating the proof of Lemma 9.5, we get that for $x \in U(r_1)$,

$$\mathbb{E}_x[\eta(Y_{\tau_{U(r_1)}})] = \eta(x) + \mathbb{E}_x \int_0^{\tau_{U(r_1)}} L^{\mathcal{B}}\eta(Y_s)\, ds\,.$$

Thus for $x \in U(r_1)$,

$$\mathbb{E}_x[\eta(Y_{\tau_{U(r_1)}})] \leq \eta(x) - 2^{-1}C_{25}\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_2}dt\,.$$

Since $\eta \geq 0$ everywhere, we get $0 \leq x_d^q - 2^{-1}C_{25}\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_2}dt$ for all $x \in U(r_1)$. Now (5.19) follows from this and the scaling argument in (5.13). $\qquad\square$

Proof of Lemma 5.13. Choose $q \in (p, \alpha)$ and let $\eta(x) := h_p(x) - h_q(x)$, $x \in \mathbb{R}_+^d$. For $x \notin D(1,1)$, $\eta(x) = 0$, while if $x \in D(1,1)$ we have $\eta(x) = x_d^p - x_d^q > 0$. By Lemma 5.6, for all $x \in U(r_0)$ we have that $L^{\mathcal{B}}h_p(x) \leq 0$ and $L^{\mathcal{B}}h_q(x) \geq C_{23}x^{\alpha-q}$. Thus we can find $r_1 \in (0, r_0]$ such that

$$L^{\mathcal{B}}\eta(x) = L^{\mathcal{B}}h_p(x) - L^{\mathcal{B}}h_q(x) \leq -C_{23}x_d^{\alpha-q} \leq -1, \qquad x \in U(r_1). \tag{9.12}$$

Let $g_k = v_k * (\eta \mathbf{1}_{D(5^{-k})})$. It follows from Lemmas 9.1 and 9.3 applied to h_g and h_p that $L^{\mathcal{B}}g_k \to L^{\mathcal{B}}\eta$ on U and the sequence of functions $|L^{\mathcal{B}}g_k - L^{\mathcal{B}}\eta|$ is bounded by some constant $C > 0$. In particular,

$$-L^{\mathcal{B}}g_k(z) \geq -L^{\mathcal{B}}\eta(z) - C \geq 1 - C, \qquad z \in U(r), r \leq r_1. \tag{9.13}$$

It follows from Proposition 3.7 that for all $t \geq 0$,

$$\mathbb{E}_x[g_k(Y_{t \wedge \tau_{U_{a_k}(r)}})] - g_k(x) = \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L^{\mathcal{B}}g_k(Y_s)\, ds\,.$$

As $k \to \infty$, the left-hand side converges to $\mathbb{E}_x[\eta(Y_{t \wedge \tau_{U(r)}})] - \eta(x)$. For the right-hand side we can use Fatou's lemma (justified because of (9.13)) to conclude that

$$\limsup_{k \to \infty} \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U_{a_k}(r)}} L^{\mathcal{B}} g_k(Y_s) \, ds \leq \mathbb{E}_x \int_0^t \mathbf{1}_{s < \tau_{U(r)}} L^{\mathcal{B}} \eta(Y_s) \, ds \leq -\mathbb{E}_x(t \wedge \tau_{U(r)}).$$

Thus we get that $\mathbb{E}_x[\eta(Y_{t \wedge \tau_{U(r)}})] - \eta(x) \leq -\mathbb{E}_x(t \wedge \tau_{U(r)})$, and by letting $t \to \infty$,

$$-\eta(x) \leq \mathbb{E}_x[\eta(Y_{\tau_{U(r)}})] - \eta(x) \leq -\mathbb{E}_x \tau_{U(r)}.$$

Thus we get $\mathbb{E}_x \tau_{U(r)} \leq \eta(x) \leq x_d^p$. By using that $U(r_1) = r_1 U$ and (5.1), for any $x \in U$,

$$\mathbb{E}_x \tau_U = r_1^{-\alpha} \mathbb{E}_{r_1 x} \tau_{r_1 U} \leq r_1^{-\alpha} (r_1 x_d)^p = r_1^{p-\alpha} x_d^p.$$

This proves the claim with $C_{36} = r_1^{p-\alpha}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

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