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Stable Algorithm for Calculating with Q-splines

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Abstract.

We are using a technique to calculate with Chebyshevian splines of order ≤ 4, based on the known derivative formula for Chebyshevian splines and Oslo type algorithm, to produce simple formulæ for qB-splines developed by Kulkarni and Laurent. Starting with the known fact that local basis for q-splines of order 3 and 4 can be evaluated by making positive linear combinations of less smooth, one order higher polynomial B-splines, we deduce a simple and stable algorithm for such splines.

It is an interesting fact in itself, that the coefficients in such linear combinations are discrete Chebyshevian splines, and therefore make a partition of unity. The same is true for qB-splines themselves.

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1. Introduction

The q-splines have an analogue in the beam theory. Consider a simply supported beam with supports $\{(x_i, f_i)\}_{i=1}^{n+1}$; then the deflection of the beam between successive supports is the solution $s(x)$ to the differential equation $[E \cdot I \cdot D^2]s = M$. Here E denotes Young's modulus of elasticity, I is the cross-sectional moment of inertia, and M is the bending moment. We suppose that $E \cdot I = \frac{1}{q}$, $q > 0$, where q and, under assumption of weightlessness, M, are piecewise-linear continuous functions with break points at the supports. Differentiating the above equation twice, we arrive at the two-point boundary value problem on $[x_i, x_{i+1}]$ $(i = 1, \ldots, n)$:

$$D^2 \frac{1}{q} D^2 s = 0, \quad s(x_i) = f_i,\ s(x_{i+1}) = f_{i+1},\ s''(x_i) = s_i'',\ s(x_{i+1}) = s_{i+1}''.$$

where s_i'' and s_{i+1}'' are chosen so as to ensure that $s \in C^2[x_1, x_{n+1}]$.

The aim is to construct a stable algorithm for calculating with *q-splines*, based on the known derivative formula for Chebyshevian splines and Oslo type algorithm. To this end, we will use one special *Canonical Complete Chebyshev* (CCT)-system, and

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some general techniques from the well known *Chebyshevian Splines Theory*. Instead of calculating directly with q-splines, we propose to write such splines as linear combinations of locally supported ones, which can be expressed as linear combinations of ordinary polynomial B-splines.

Q-splines were introduced by Kulkarni and Laurent [1], and have meanwhile been used in various applications in computer aided geometric design.

2. Chebyshev theory preliminaries

Let $t_{-2} \leq t_{-1} \leq t_0 \leq a = t_1 < t_2 < \cdots < t_{n+1} = b \leq t_{n+2} \leq t_{n+3} \leq t_{n+4}$ be an extended partition of the interval $[a, b]$, and let $q(x)$ be a continuous, piecewise-linear function defined by $q(x)|_{[t_i,t_{i+1}]} = \frac{q_{i+1}-q_i}{h_i}(x - t_i) + q_i$, where $h_i = t_{i+1} - t_i,\ q_i > 0$. Consider the CCT-system $\{u_1, u_2, u_3, u_4\}$:

$$\begin{aligned}
u_1(x) &= 1 \\
u_2(x) &= \int_a^x dt_2 \\
u_3(x) &= \int_a^x dt_2 \int_a^{t_2} q(t_3)\, dt_3 \\
u_4(x) &= \int_a^x dt_2 \int_a^{t_2} q(t_3)\, dt_3 \int_a^{t_3} dt_4.
\end{aligned} \tag{1}$$

We wish to construct a local basis for the spline space spanned piecewisely by these functions, that is, B-splines in $\mathcal{S}(4, \mathbf{m}, d\boldsymbol{\sigma}, \Delta)$, where $\mathbf{m}$ is the *multiplicity vector*, $\mathbf{m} = (1, \ldots, 1)^T$, $d\boldsymbol{\sigma} := (dt_2, q(t_3)\, dt_3, dt_4)^T$ is the *measure vector*, and $\Delta = \{t_i\}_{i=-2}^{n+4}$ (see [4] for details of the notation). An important role is played by the associated generalized derivatives:

$$\begin{aligned}
L_{1,d\boldsymbol{\sigma}} &= D \\
L_{2,d\boldsymbol{\sigma}} &= \frac{1}{q} D^2 \\
L_{3d\boldsymbol{\sigma}} &= D \frac{1}{q} D^2 \\
L_{4d\boldsymbol{\sigma}} &= D^2 \frac{1}{q} D^2.
\end{aligned}$$

To begin with, we focus on the *reduced system* $\{u_{1,1}, u_{1,2}, u_{1,3}\}$, spanning the space $\mathcal{S}(3, \mathbf{m}, d\boldsymbol{\sigma}^{(1)}, \Delta)$, $d\boldsymbol{\sigma}^{(1)} = (q(t_3)\, dt_3, dt_4)^T$ on each interval. The CCT-system is:

$$\begin{aligned}
u_{1,1}(x) &= 1 \\
u_{1,2}(x) &= \int_a^x q(t_3)\, dt_3 \\
u_{1,3}(x) &= \int_a^x q(t_3)\, dt_3 \int_a^{t_3} dt_4.
\end{aligned} \tag{2}$$

Next consider less smooth B-splines $\widetilde{T}_j^3$ from the space $\mathcal{S}(3, \tilde{\mathbf{m}}, d\boldsymbol{\sigma}^{(1)}, \Delta)$, with the multiplicity vector $\tilde{\mathbf{m}} = (2, \ldots, 2)^T$ on the same knot sequence. For the fixed index i, we denote the points in the new extended partition as $t_i = \tilde{t}_{r-1} = \tilde{t}_r < \tilde{t}_{r+1}$, and polynomial B-splines on this partition simply as $\widetilde{B}_j^n$. It is easily seen from the definition of the basis (2) that we can write $\widetilde{T}_j^3$ as

$$\widetilde{T}_{r-1}^3(x) = \sum_{j=s-3}^{s} a_j \widehat{B}_j^4(x) \tag{3}$$

$$\widetilde{T}_r^3(x) = \sum_{j=s-3}^{s+3} a_j \widehat{B}_j^4(x). \tag{4}$$

$\widehat{B}_j^4 \in \mathcal{S}(4, \hat{\mathbf{m}}, d\boldsymbol{\lambda}, \Delta)$, where $\hat{\mathbf{m}} = (3, \ldots, 3)^T$ on the same knot sequence Δ, and $d\boldsymbol{\lambda}$ is the measure vector determined by from Lebesgue measures only. Points in this partition will be $t_i = \hat{t}_{s-2} = \hat{t}_{s-1} = \hat{t}_s < \hat{t}_{s+1}$.

We will use the following general theorem, which is a generalization to Chebyshevian splines of the well known derivative formula for polynomial B-splines:

Theorem 1 *Let $L_{1,d\boldsymbol{\sigma}}$ be the first generalized derivative with respect to CCT-system $\mathcal{S}(n, d\boldsymbol{\sigma})$, and let the multiplicity vector $\mathbf{m} = (n_1, \ldots, n_k)^T$ satisfy $n_i < n - 1$ for $i = 1, \ldots, k$. Then for $x \in [a, b]$ and $i = 1, \ldots, n + \sum_{i=1}^{k} n_i$ the following derivative formula holds:*

$$L_{1,d\boldsymbol{\sigma}} T_{i,d\boldsymbol{\sigma}}^n(x) = \frac{T_{i,d\boldsymbol{\sigma}^{(1)}}^{n-1}(x)}{C_{n-1}(i)} - \frac{T_{i+1,d\boldsymbol{\sigma}^{(1)}}^{n-1}(x)}{C_{n-1}(i+1)} \tag{5}$$

where

$$C_{n-1}(i) := \int_{t_i}^{t_{i+n-1}} T_{i,d\boldsymbol{\sigma}^{(1)}}^{n-1} \, d\sigma_2, \tag{6}$$

with measure vectors $d\boldsymbol{\sigma} = (d\sigma_2(t), \ldots, d\sigma_n(t))^T \in \mathbf{R}^{n-1}$ and $d\boldsymbol{\sigma}^{(1)} := (d\sigma_3(t), \ldots, d\sigma_n(t))^T \in \mathbf{R}^{n-2}$.

3. Construction of the local basis for Q-spline spaces

It is obvious from (2) and Theorem 1 that

$$\bar{L}_1 \widetilde{T}_{r-1}^3(x) = \frac{\widetilde{B}_{r-1}^2(x)}{\widetilde{C}_2(r-1)} - \frac{\widetilde{B}_r^2(x)}{\widetilde{C}_2(r)},$$

where $\bar{L}_1 := L_{1,d\boldsymbol{\sigma}^{(1)}} = \frac{1}{q} D$ is the generalized derivative for the reduced CCT-system, and

$$\widetilde{C}_2(j) = \int_{\tilde{t}_j}^{\tilde{t}_{j+2}} \widetilde{B}_j^2(t)\, q(t)\, dt.$$

In particular,

$$\widetilde{C}_2(r-1)=\frac{(2q_i+q_{i+1})h_i}{6},\quad \widetilde{C}_2(r)=\frac{(q_i+2q_{i+1})h_i}{6}.$$

From the simple properties of B-splines:

$$\widetilde{T}^3_{r-1}(t_i)=0,\ \bar{L}_1\widetilde{T}^3_{r-1}(t_i^+)=\frac{1}{\widetilde{C}_2(r-1)},\ \widetilde{T}^3_{r-1}(t_{i+1})=0,\ \bar{L}_1\widetilde{T}^3_{r-1}(t_{i+1}^-)=-\frac{1}{\widetilde{C}_2(r)},$$

we get the coefficients in (3):

$$a_{s-3}=a_s=0,\ a_{s-2}=\frac{2q_i}{2q_i+q_{i+1}},\ a_{s-1}=\frac{2q_{i+1}}{q_i+2q_{i+1}},$$

whence

$$\widetilde{T}^3_{r-1}(x)=\frac{2q_i}{2q_i+q_{i+1}}\widehat{B}^4_{s-2}(x)+\frac{2q_{i+1}}{q_i+2q_{i+1}}\widehat{B}^4_{s-1}(x). \tag{7}$$

To calculate $\widetilde{T}^3_r$, we use the equations

$$\widetilde{T}^3_r(t_i)=\bar{L}_1\widetilde{T}^3_r(t_i)=\widetilde{T}^3_r(t_{i+2})=\bar{L}_1\widetilde{T}^3_r(t_{i+2})=0,$$

$$\widetilde{T}^3_r(t_{i+1})=1,\ \bar{L}_1\widetilde{T}^3_r(t_{i+1}^-)=\frac{1}{\widetilde{C}_2(r)},\ \bar{L}_1\widetilde{T}^3_r(t_{i+1}^+)=-\frac{1}{\widetilde{C}_2(r+1)},$$

to get the coefficients in (4):

$$a_{s-3}=a_{s-2}=a_{s+2}=a_{s+3}=0,\ a_s=1,$$

$$a_{s-1}=\frac{q_i}{2q_{i+1}+q_i},\quad a_{s+1}=\frac{q_{i+2}}{2q_{i+1}+q_{i+2}},$$

and, finally

$$\widetilde{T}^3_r(x)=\frac{q_i}{q_i+2q_{i+1}}\widehat{B}^4_{s-1}(x)+\widehat{B}^4_s(x)+\frac{q_{i+2}}{2q_{i+1}+q_{i+2}}\widehat{B}^4_{s+1}(x). \tag{8}$$

By integrating (5) in Theorem 1, we can further calculate splines of higher order. We start with the equation

$$\widetilde{T}^4_{r-1}(x)=\frac{1}{\widetilde{C}_3(r-1)}\int_{\tilde{t}_{r-1}}^{x}\widetilde{T}^3_{r-1}(t)\,dt-\frac{1}{\widetilde{C}_3(r)}\int_{\tilde{t}_r}^{x}\widetilde{T}^3_r(t)\,dt. \tag{9}$$

It is easy to see from (7) and (8) that

$$\widetilde{C}_3(r-1) = \frac{h_i}{4}\left[\frac{2q_i}{2q_i+q_{i+1}}+\frac{2q_{i+1}}{q_i+2q_{i+1}}\right] \tag{10}$$

$$\widetilde{C}_3(r) = \frac{1}{4}\left[\frac{q_ih_i}{q_i+2q_{i+1}}+h_i+h_{i+1}+\frac{q_{i+2}h_{i+1}}{2q_{i+1}+q_{i+2}}\right]. \tag{11}$$

From (9) using (7), (8) and the well known recurrence for integrals of the polynomial B-splines

$$\int_{-\infty}^{x} B_i^k(t)\,dt = \frac{t_{i+k}-t_i}{k}\sum_{j=i}^{i+k-1} B_j^{k+1}(x),$$

where $\{t_i\}$ is know any extended partition, we obtain (by looking separately at x from each of the subintervals $[t_i, t_{i+1}]$ and $[t_{i+1}, t_{i+2}]$), that

$$\begin{aligned}\widetilde{T}_{r-1}^4(x) &= \frac{1}{\widetilde{C}_3(r-1)}\frac{2q_i}{2q_i+q_{i+1}}\frac{h_i}{4}\widehat{B}_{s-2}^5(x)+\frac{1}{\widetilde{C}_3(r)}\left(\frac{h_i+h_{i+1}}{4}+\frac{q_{i+2}}{2q_{i+1}+q_{i+2}}\frac{h_{i+1}}{4}\right)\\ &\quad \widehat{B}_{s-1}^5(x)+\frac{1}{\widetilde{C}_3(r)}\frac{q_{i+2}}{2q_{i+1}+q_{i+2}}\frac{h_{i+1}}{4}\widehat{B}_s^5(x). \qquad (12)\end{aligned}$$

In the same way,

$$\begin{aligned}\widetilde{T}_{r}^4(x) &= \frac{1}{\widetilde{C}_3(r)}\frac{q_i}{q_i+2q_{i+1}}\frac{h_i}{4}\widehat{B}_{s-1}^5(x)+\frac{1}{\widetilde{C}_3(r)}\left(\frac{q_i}{q_i+2q_{i+1}}\frac{h_i}{4}+\frac{h_i+h_{i+1}}{4}\right)\\ &\quad \widehat{B}_{s}^5(x)+\frac{1}{\widetilde{C}_3(r+1)}\frac{2q_{i+2}}{q_{i+1}+2q_{i+2}}\frac{h_{i+1}}{4}\widehat{B}_{s+1}^5(x). \qquad (13)\end{aligned}$$

The following Lemma and Theorem are connecting general T-splines of orders 3 and 4 with less smooth ones, which are simpler to calculate and have in the case of Q-splines alredy been constructed by the explicit formulæ:

Lemma 2 *Let $T^3_{i,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3,\mathbf{m},d\boldsymbol{\sigma}^{(1)},\Delta)$ be a Chebyshevian 3^{rd} order spline associated with the multiplicity vector $\mathbf{m}=(1,\ldots,1)^T$, and let us assume that $\widetilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3,\tilde{\mathbf{m}},d\boldsymbol{\sigma}^{(1)},\Delta)$ are B-splines associated with multiplicity vector $\tilde{\mathbf{m}}=(2,\ldots,2)^T$ on the same knot sequence. If $\{t_1,\ldots,t_{k+6}\}$ and $\{\tilde{t}_1,\ldots,\tilde{t}_{2k+6}\}$ are the associated extended partitions, and r an index such that $t_i=\tilde{t}_r<\tilde{t}_{r+1}$, then for $i=1,\ldots,k+3$:*

$$T^3_{i,d\boldsymbol{\sigma}^{(1)}} = T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\widetilde{T}^3_{r,d\boldsymbol{\sigma}^{(1)}} + \widetilde{T}^3_{r+1,d\boldsymbol{\sigma}^{(1)}} + T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\widetilde{T}^3_{r+2,d\boldsymbol{\sigma}^{(1)}}.$$

Theorem 3 *Let $T^4_{i,d\boldsymbol{\sigma}} \in \mathcal{S}(4,\mathbf{m},d\boldsymbol{\sigma},\Delta)$, $\widetilde{T}^4_{i,d\boldsymbol{\sigma}} \in \mathcal{S}(4,\tilde{\mathbf{m}},d\boldsymbol{\sigma},\Delta)$, the multiplicity vectors $\mathbf{m}$, $\tilde{\mathbf{m}}$ being as in Lemma 2. Then positive $\delta_i^4(j)$ exist such that*

$$T^4_{i,d\boldsymbol{\sigma}} = \sum_{j=r}^{r+3}\delta_i^4(j)\widetilde{T}^4_{j,d\boldsymbol{\sigma}},$$

where $r=r_i$ satisfies

$$t_i=\tilde{t}_{r_i}<\tilde{t}_{r_i+1}.$$

Let the extended partitions be $\{t_1,\ldots,t_{k+8}\}$ and $\{\tilde{t}_1,\ldots,\tilde{t}_{2k+8}\}$. Then $\delta_i^4(j), j=r,\ldots,r+3$ are determined by the formulæ:

$$\delta_i^4(r) = \frac{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\widetilde{C}(r)}{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\widetilde{C}(r)+\widetilde{C}(r+1)+T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\widetilde{C}(r+2)}$$

$$\delta_i^4(r+1) = \frac{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\widetilde{C}(r)+\widetilde{C}(r+1)}{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\widetilde{C}(r)+\widetilde{C}(r+1)+T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\widetilde{C}(r+2)}$$

$$\delta_i^4(r+2) = \frac{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\widetilde{C}(r+4)+\widetilde{C}(r+3)}{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\widetilde{C}(r+2)+\widetilde{C}(r+3)+T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\widetilde{C}(r+4)}$$

$$\delta_i^4(r+3) = \frac{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\widetilde{C}(r+4)}{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\widetilde{C}(r+2)+\widetilde{C}(r+3)+T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\widetilde{C}(r+4)}$$

where, as in (6)

$$\widetilde{C}(i) = \int_{\text{support}} \widetilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}}\, d\sigma_2.$$

The proofs may be found in [2]. To use Lemma 2 and Theorem 3, it remains to calculate $T_i^3(t_{i+1})$ and $T_i^3(t_{i+2})$. The derivative formula in Theorem 1 implies

$$T_i^3(x) = \frac{1}{C_2(i)}\int_{t_i}^{x} B_i^2(t)q(t)\,dt - \frac{1}{C_2(i+1)}\int_{t_{i+1}}^{x} B_{i+1}^2(t)q(t)\,dt,$$

where

$$C_2(i) = \int_{t_i}^{t_{i+2}} B_i^2(t)q(t)\,dt = \frac{1}{6}[(q_i+2q_{i+1})h_i + (2q_{i+1}+q_{i+2})h_{i+1}].$$

One finds easily that

$$T_i^3(t_{i+1}) = \frac{h_i(q_i+2q_{i+1})}{6C_2(i)}, \quad T_i^3(t_{i+2}) = \frac{h_{i+2}(2q_{i+2}+q_{i+3})}{6C_2(i+1)},$$

and we have everything that is needed for the evaluation of T_i^4 by means of Theorem 3.

4. Conclusion

We have constructed formulæ for calculating with q-splines as linear combination of polynomial B-splines. Moreover, all the coefficients involved are positive, and thus we have to calculate scalar products of positive quantities only, guaranteeing a stability of such an algorithm.

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