Collocation by Tchebycheffian splines of order 3

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ABSTRACT:
A differential equation of the form $Dp(x)Du(x) = f(x, u(x))$ *is solved by collocation, using a class of spline functions of order 3. A restriction of each spline function to any subinterval of the partition is piecewisely in the kernel of the operator* $DDp(x)D$. *A coefficient* $p(x)$ *may be singular. An algorithm is described by means of which a B- spline basis could be efficiently constructed. Some numerical experiments are considered.*

KOLOKACIJA ČEBIŠEVLJEVIM SPLINE FUNKCIJAMA REDA 3.
Diferencijalna jednadžba oblika $Dp(x)Du(x) = f(x, u(x))$ *rješava se kolokacijom spline funkcijama reda 3. Restrikcija svake spline funkcije na podintervale je po dijelovima element jezgre operatora* $DDp(x)D$. *Koeficijent* $p(x)$ *može biti i singularan. Opisan je algoritam kojim se može efektivno konstruirati baza B-spline-ova. Promatraju se i neki numerički eksperimenti.*

1 Introduction

We want to solve the boundary value problem (or maybe an initial value one) of the second order:

$$\frac{d}{dx}p(x)\frac{d}{dx}u(x) = f(x, u(x)) \tag{1}$$

$$x \in (0, 1)$$

together with suitable boundary conditions, such that (1) has a unique (isolated) solution. We shall further suppose that $p(x)$ is such that $p(x) \geq 0$ and $p(x)\frac{d}{dx}u(x) \in C^1[0, 1]$. The idea is to project both sides of the equation to some space of polynomial splines, the most simple space being the space of piecewise constants. For $p(x) \equiv 1$ the problem was extensively studied by [3, deBoor] and many others. An important case arises when an equation can not be easily converted to an equation of the form

$$\frac{d^2}{dx^2}u(x) = f(x, u(x))$$

and thus we can not use polynomial splines for collocation. This situation frequently arises in solving ordinary differential equations having their origin in elliptic-parabolic partial differential equations, such as in [9]. Usually $p(x)$ exhibits singular behavior near one or both end points of the interval, such as for instance $p(0) = 0$. A typical example is the equation

$$\begin{aligned} D(x^{0.5}Du) - x^{0.5}u &= 0 \\ 0 < x &< 1 \\ u(0) = 1 \qquad u(1) &= 0 \end{aligned} \tag{2}$$

which we shall later use for numerical purposes, the example itself being borrowed from [5, Jamet], where a central difference scheme to approximate the solution of (1) is constructed in the linear case. A variational approach may be used in solving (1), as in [4], where in fact a special kind of spline space was used (Tchebycheffian "hat" functions). In order to apply collocation techniques to a singular problem, one must first find means of constructing the singular splines effectively, this task being already achieved in the most satisfactory manner in case of polynomial splines in the form of the well known recurrence [2]. In what follows we propose a way of calculating with the Tchebycheff splines of a very special kind (the polynomial analogue would be the space of parabolic splines). Singular splines have been treated in the most general way in [6] as solutions of certain minimization problems, and in connection with collocation they have been used by Reddien [10].

2 Tchebycheff Systems

Let us choose the knot sequence $t_1 \dots t_{m+2}$ in such a way that

$$0 = t_1 = t_2 = t_3 < t_4 \dots < t_m = t_{m+1} = t_{m+2} = 1. \tag{3}$$

We shall seek for an approximate solution of (1) in the space of functions satisfying

$$Dp(x)Du(x) = const.$$

for $x \in (t_i, t_{i+1})$, $i = 3 \dots m-1$. This is equivalent to a condition:

$$DDp(x)Du(x) = 0 \qquad x \in (t_i, t_{i+1}). \tag{4}$$

There are three basis functions in $Ker(DDp(x)D)$, namely:

$$\begin{aligned} u_1 &= 1 \\ u_2 &= \int_0^x \frac{dt_2}{p(t_2)} \\ u_3 &= \int_0^x \frac{dt_2}{p(t_2)} \int_0^{t_2} dt_3 \end{aligned} \tag{5}$$

to be used in construction of the local basis. We shall further need reduced systems obtained by taking generalized derivatives of the function in (5). The following notation is useful:

$$\begin{array}{ll} L_1 := p(x)D & L_1^* := D \\ L_2 := Dp(x)D & L_2^* := p(x)DD. \end{array}$$

The first reduced system is therefore:

$$\begin{aligned} u_{1,1} &= 1 \\ u_{1,2} &= x \end{aligned} \tag{6}$$

and the second one consists only of unity. We shall further need dual systems of functions:

$$\begin{aligned} u_1^* &= 1 \\ u_2^*(y) &= y \\ u_3^*(y) &= \int_0^y dt_3 \int_0^{t_3} \frac{dt_2}{p(t_2)} \end{aligned} \tag{7}$$

and the reduced dual systems, obtained by applying L^* operators to system (7):

$$\begin{aligned} u_{1,1}^* &= 1 \\ u_{1,2}^*(y) &= \int_0^y \frac{dt_3}{p(t_3)} \end{aligned} \tag{8}$$

$$u_{2,1}^* = 1.$$

Let $d\sigma := \frac{1}{p(t)}dt$, and let us suppose that all integrals in (5) exist. Then the following Lemma holds true:

Lemma 2.1 *If $p(x) \geq 0$ then functions in system (5) form a* Canonical Complete Tchebycheff system *(CCT-system) on* $(0, 1)$.

Proof. Since σ is monotone by assumption, the conclusion follows form [12].•

Note. We must be cautious not to employ anything from the Tchebycheff system apparatus involving differentiability of the functions in (5).

3 Tchebycheff Splines

We may now construct the Tchebycheff splines as it is done in [12, 13]. Let us define two important Green' s functions:

$$g_3(x,y) = \int_y^x \frac{dt_2}{p(t_2)} \int_y^{t_2} dt_3 \quad x > y$$

$$g_3(x,y) = 0 \quad x \leq y$$

and

$$g_2(x,y) = \int_y^x dt_3 \quad x > y$$
$$g_2(x,y) = 0 \quad x \leq y.$$

The first one is a Green's function of the system (5) and the second one of the system (6).

Note. We have departed here from the standard definition by which $g_2(x,y)$ is a Green's function in CCT-system consisting of the first two functions in (5).

Lemma 3.1

$$L_1 g_3(x,y) = g_2(x,y)$$

where the derivative is taken with respect to the first variable.

Proof. If $x > y$ we have by [12, Lemma 4.1]:

$$L_1 g_3(x,y) = L_1 \sum_{i=1}^{i=3} (-1)^{3-i} u_i(x) u^*_{4-i}(y) = \sum_{i=2}^{i=3} (-1)^{3-i} u_{1,i-1}(x) u^*_{4-i}(y).$$

The term on the right-hand side we may recognize as $g_2(x,y)$ if we apply the quoted Lemma to the system (6).•

In what follows we shall compute the Tchebycheff (TB-) splines as certain weighted integrals of polynomial splines. To distinguish between them, we shall denote the polynomial B-splines of order k vanishing at t_i and t_{i+k} as B_i^k and the TB-spline as TB_i^k. The following Theorem is based on Lemma (2.1):

Theorem 3.1

$$TB_i^3(x) = \frac{1}{C(i)} \int_{t_i}^x B_i^2 d\sigma - \frac{1}{C(i+1)} \int_{t_{i+1}}^x B_{i+1}^2 d\sigma \tag{9}$$

where $C(i)$ is defined as:

$$C(i) := (t_{i+2} - t_i) \Delta^2 u_3^*(t_i, t_{i+1}, t_{i+2}) \tag{10}$$

Δ^2 *being the ordinary divided difference.*

Proof. By standard definition a normalized TB-spline is a generalized divided difference of Green's function $g_3(x, y)$ (see [13, page 382]). By Sylvester's determinant identity (see [7]) we have:

$$TB_i^3(x) = (-1)^3(<< t_{i+1}, t_{i+2}, t_{i+3} >> g_3(x, \cdot) - << t_i, t_{i+1}, t_{i+2} >> g_3(x, \cdot) \tag{11}$$

where the differences $<<>>$ are with respect to the dual system (7). We can now reduce the order of difference by applying Mülbach's *recursion [8] to each member of (11). The first part of the formula is:

$$\frac{< t_{i+2}, t_{i+3} > g_3(x, \cdot) - < t_{i+1}, t_{i+2} > g_3(x, \cdot)}{< t_{i+2}, t_{i+3} > u_3^* - < t_{i+1}, t_{i+2} > u_3^*} \tag{12}$$

and the second one is obtained by shifting index i. Differences are now with respect to the first two functions of (7). But since these are 1 and y, the differences are up to a constant multiplier the standard polynomial ones. By *recursion for ordinary divided differences (12) is equal to:

$$\frac{(t_{i+3} - t_{i+1})\Delta^2 g_3(x; t_{i+1}, t_{i+2}, t_{i+3})}{(t_{i+3} - t_{i+1})\Delta^2 u_3^*(t_{i+1}, t_{i+2}, t_{i+3})}. \tag{13}$$

We may apply the L_1 operator to TB_i^3. By Lemma (3.1) we obtain polynomial differences of $g_2(x, \cdot)$ on the right-hand side. But g_2 is a well known function $(x - \cdot)_+^1$. Thus by Definition of a linear B-spline and (10) we conclude that:

$$L_1 TB_i^3 = \frac{1}{C(i)} B_i^2(x) - \frac{1}{C(i+1)} B_{i+1}^2(x). \tag{14}$$

The conclusion of the Theorem follows on integration of (14) with respect to $d\sigma$.•

The following Lemma may be used to calculate (10) in a stable way:

Lemma 3.2

$$C(i) := \int_{t_i}^{t_{i+2}} B_i^2 d\sigma \tag{15}$$

Proof. We may explicitly calculate all differences in (10). An easier proof follows from Peano representation Theorem [12, Theorem 4.23].•

At the end, we summarize all the preceding facts in the following

Theorem 3.2

$$TB_i^3(x) = \frac{1}{C(i)}\int_{t_i}^{x} B_i^2 d\sigma - \frac{1}{C(i+1)}\int_{t_{i+1}}^{x} B_{i+1}^2 d\sigma$$

$$L_1 TB_i^3(x) = \frac{1}{C(i)}B_i^2(x) - \frac{1}{C(i+1)}B_{i+1}^2(x).$$

$$L_2 TB_i^3(x) = \frac{B_i^1(x)}{C(i)(t_{i+1}-t_i)} - \frac{B_{i+1}^1(x)}{t_{i+2}-t_{i+1}}\left(\frac{1}{C(i+1)} + \frac{1}{C(i)}\right) + \frac{B_{i+2}^1(x)}{C(i+1)(t_{i+3}-t_{i+2})}$$

Proof. We have only to prove the last equation. Since $L_2 = Dp(x)D$, we just take a derivative of the second relation and then apply a recurrence for derivatives of B- splines [2, page 138]•

4 Collocation

Equation (1) may be written in the form $L_2 u(x) = f(x, u(x))$. We shall assume that boundary conditions are certain linear functional of function values and their L_1- derivatives, independent over $Ker(Dp(x)Du)$. The collocation solution u_c to (1) is a linear combination of TB- splines:

$$u_c(x) = \sum_{i=1}^{m-1} c_i TB_i^3(x)$$

if we assume the knot sequence (3).
Note. Interior knots are all of multiplicity 1, implying therefore L_1- differentiability of the spline space. For a singular coefficient the splines need not be "smooth" in the classical sense.

There is only one collocation point per interval, say x_j, the total of $m-3$ points. The collocation matrix is:

$$L_2 TB_i^3(x_j).$$

Since TB-splines have a support of three intervals, the matrix is three diagonal, if the missing two boundary equations are carefully placed. The nonlinear problem may be solved by quadratically convergent Newton's method (given sufficiently good initial guess), in case of an isolated solution, as in [3, 1]. The convergence in linear case may be proved to be $\Delta d\sigma(\Delta)$. See [11] for an idea of proof.

The placement of collocations points is of interest. We have used midpoints of subintervals as collocation points. There seems to be no better choice for collocation with splines of order three; the similar situation arising in the parabolic spline collocation for a second order differential equation. The situation changes with higher order splines. Let us consider the model example (2). It is a special case of a hypergeometric equation, and may be easily solved via functions of parabolic cylinder. The solution has a large endslope near zero, so polynomial splines are inapplicable. The collocation matrix is a sum of two matrices: the matrix of L_2 derivatives and the matrix of function values of TB-splines. The former is easily evaluated by Theorem 3.2. However, we must calculate integrals for TB- splines itself. This is done in the easiest way by some simple Gaussian formula, such as:

$$\int_a^b \frac{f(x)}{\sqrt{x}} dx \simeq 2(\sqrt{b}-\sqrt{a}) f(\frac{b+\sqrt{ab}+a}{3}) \tag{16}$$

the formula being exact on linear functions. By (16) we have constructed an algorithm similar to deBoor's BSPLVD in [2, page 287]. Since TB-spline values are obtained by subtraction there is a question of numerical stability. By (15) both terms in (9) are less then unity, and we have found by numerical experiment that they are never comparable, regardless of the knots, as long as $p(x)$ is not too oscillatory.

At the end, let us briefly compare some results in approximating the model example (2). Let $f_n(x)$ denote various approximation as follows. Then at $x = 0.5$ we have:

Δ_n	Jamet[5]	Reddien[11]	TB_3
1/8	0.29308	0.25305	0.25206
1/16	0.27826	0.25223	0.252054
1/8192	0.25310		

Analytic solution to (2) at $x = 0.5$ is 0.252042. The error behaves in the same way throughout the interval , and has two sign changes, due to the variation diminishing property of TB-splines and monotonicity of the solution.

References

[1] R. E. Bellman and R. E. Kalaba: Quasilinearisation and Nonlinear Boundary-Value Problems, American Elsevier Publ. Co., N.Y., (1965)

[2] C. deBoor: A Practical Guide to Splines, Springer-Verlag, N.Y., (1978)

[3] C. deBoor, B. Swartz: Collocation at Gaussian points, SIAM J. Numer. Anal. Vol. 10 No 4., 582-606 (1973)

[4] P. G. Ciarlet, F. Natterer, R. S. Varga: Numerical Methods of High-Order Accuracy for Singular Boundary Value Problem, Numer. Math. 15, 87-99 (1970.)

[5] P. Jamet: On the Convergence of Finite-Difference approximations to One-Dimensional Singular Boundary value Problems, Numer. Math. 14, 87-99 (1970)

[6] J. Jerome: On Spline functions Determined by Singular Self-Adjoint Differential Operators, J. of Approx. Theory 5, 15-40 (1972)

[7] T. Lyche: A Recurrence Relation for Chebyshevian B-splines, Constr. Approx. 155-173, (1985)

[8] G. Mülbach: A Recurrence Formula for Generalized Divided Differences and Some Applications, J. of Approx. Theory 9, 165-172 (1973)

[9] S. V. Parter: Numerical Methods for Generalized Axially Symmetric potentials, J. SIAM ser. B Numer. Anal. 2, 500-516 (1965)

[10] G. W. Reddien: Projection Methods and Singular Two Point Boundary Value Problems, Numer. Math. 21, 193-205 (1973)

[11] G. W. Reddien, L. L. Schumaker: On a Collocation Method for Singular Two Point Boundary Value Problems, Numer. Math. 25, 427-432 (1976)

[12] L. L. Schumaker: On Tchebycheffian Spline Functions, J. of Approx. Theory 18, 278-303, (1976)

[13] L. L. Schumaker: Spline Functions: Basic Theory, John Wiley and Sons, N.Y., (1981)