# Nove rekurentne relacije za Čebiševljeve spline funkcije i njihove primjene


## Summary

A recurrence relation for derivatives of Chebyshev splines is developed. In case of piecewise constant weights, it is shown that an algorithm for computing local basis of B–splines can be deduced for parabolic and cubic splines with jumps in derivatives. For hermitic splines a most general two–term recursive relation of deBoor–Cox type is obtained. A stable knot insertion algorithm is given for cubic splines with jumps in second derivatives, and a problem of computing a local basis thus solved completely. Some Chebyshev and polynomial splines are used in methods of collocation type for solving boundary value problems. Algorithms for solving these problems numerically, and an optimal choice of collocation points are discussed.


## Introduction

The popularity of polynomial splines in various applications like data fitting, interpolation, least squares, numerical solution of initial and boundary value problems in differential equations is due to u number of facts. One of the most important seems to be the "locality" property: each spline can be represented as a linear combination of functions with relatively small support, the so called B-splines. This feature immediately implies the band structure of matrices that appear in interpolation and collocation problems; moreover some of the properties of B-splines imply total positivity. In consequence, these matrices can be efficiently stored in a computer and inverted by Gaussian elimination without pivoting. The problem of stable calculation of polynomial B–splines and their derivatives has been completely solved since 1975 and plays an important part in almost every software concerned with data fitting, interpolation etc. The amazing fact that a polynomial B-spline can be computed as convex combination of two lower order B-splines has motivated an extensive research in this area. Most of the effort has gone in finding other spaces of functions for which similar basis exist, and to compute these basis via stable recurrence formulae. Some non-polynomial basis of B-splines have been succesfully constructed by two term recurrences resembling closely the polynomial ones, namely trigonometric and hyperbolic spline basis. They appear to be special cases of some more general L–splines. The negative result by Schumaker proves that no other two term recurrences exist, at least with analytic coefficients. It is possible, however, to find, at least in theory, quite general two-term recurrences for Chebyshevian splines; the problem is to find the coefficients numericaly, since they are certain quotients of determinants which can even be 0/0 at the knots. Some other basis can also be constructed with slight assumptions on the underlying Chebshev system, but not necessarily of the minimal support.

In the thesis I have tried to find some new recurrencies for Chebyshev and

polynomial splines, and to illustrate their use in solving boundary value problems, which has motivated the whole research. The main idea is contained in a simple theorem, which implies that derivatives of Chebyshev splines are piecewisely in a certain Chebyshev space of a simpler structure (possessing one weight less). Thus it is possible to deduce a recurrence relation for derivatives of most general Chebyshev splines, which for special weights gives all the existing ones. For one–weight splines derivatives are ordinary polynomial splines, which can be used to calculate Chebyshev B-splines by integrating with respect to the missing weight. Though the integrals can be computed in a stable way by appropriate Gaussian formulae, the remaining question is how to subtract the integrals stably, and this can not be answered easily.

There are cases in which integration can be performed analyticaly, which leads to a stable recurrence for B-splines. One is of course the well known polynomial case, the so called deBoor–Cox recurrence. There is yet another interesting polynomial case in which jumps of derivatives are matched at the knots, instead of just derivatives. I have treated this splines like Chebyshevian multi-weighted splines in the chapter on g–splines. In classical theory, g–splines are usually defined to be piecewise polynomials with certain functionals prescribed at the knots, and so the name seems appropriate though the functionals dealt with are not of the common EHB–type. Two important cases are discussed. The first is of hermitic g–splines, in which case a deBoor–Cox like recurrence is developed, and thus the problem solved completely. The other one is of complete cubic g–spline, which is defined to be a piecewise cubic with prescribed jumps in second derivatives (first derivative can have a jump as well). Since I have been unable to find any two–term recurrence relations, I have adopted another approach which is similar to discrete spline techniques in polynomal case. The complete cubic gB–spline can be represented as a linear combination of hermitic gB–splines with known coefficients. Since these coefficients are positive, and can be easily computed, the computation of gB–splines amounts to taking scalar products of positive vectors. The same procedure can be generalized to g–splines of arbitrary order, if we do not require the mathcing of derivative jumps of order greater than two; the general algorithm for arbitrary g–splines has not yet been discovered. The coefficients in expansion of complete cubic splines in terms of hermitic ones are of interest in themselves. They form a partition of unity, and can be treated as a kind of discrete cubic splines; I have used the name Chebyshev discrete splines.

In the chapter on collocation I have applied some of the basis of spline functions mentioned above to numerical solution of singular boundary value problems. The Chebyshev splines associated in a natural way with the dominant part of the differential operator are best suited for approximation. The projectors I have used have been known before. That is also true of the optimal choice of collocation points which imply superconvergence at the knots. The superconvergence phenomena were known to hold for collocation by polynomial, trigonometric and some singular splines. There is a case when a two–point boundary value problem can be formulated, but the derivative grows boundlesly near the endpoint, which is sufficiently general to justify the effort of applying the apparatus of Chebyshev and

collocation theory. No attempt has been made to find a general theory of singular spline collocation, partially because it has already been done in most important cases, and partially due to the lack of motivation provided by physics. The examples also suggest that splines associated with the dominant part of the operator can be insufficient to approximate the solution. It is also of interest that g–splines can be used in collocation, if the leading coefficient in differential equation is approximated by a piecewise constant. The idea of approximating the differential operator by approximating its coefficients has also been known before, but it is of interest that it can be extended to a dominant coefficient, even if the convergence is unsatisfactory.

I have tried to present the concrete examples of the theory whenever it was possible. Instead of writing algorithms I've chosen to write FORTRAN programs and explain the method in comments. Not all routines have been written in order to achieve maximum performance, but to illustrate and support the theory. The programs have been compiled by Microsoft Fortran 5.1 and run on a PC; some of the tests including the calculation of convergence order for various collocation methods have been run on HP 700 series.

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