B-spline in Tension

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January 21, 1998

Math. Subj. Class.
Keywords: spline, spline in tension, B-spline.
Abstract. **B-spline in Tension.** A construction of B-spline associated with differential operator $D^2(D^2 - p^2)$ is considered. Splines in null-space of constant coefficients differential operators are described in [1] and [2], and in order to construct a B-spline on an equidistant partition of diametar 1 a method in [3] is used. By a simple transformation one easily obtains a B-spline associated with differential operator $D^2(D^2 - p^2/h^2)$ on any equidistant partition.

Definitions. Let $y_0 < y_1 < \cdots < y_k$, $k \in \{1, 2, 3, 4\}$ be the given knot sequence and let the knot y_0 has multiplicity β, and $\beta + k = 5$. By $M_{[y_0,y_i]}$ we denote a B-spline which has the interval $[y_0, y_i]$ as a support.

The function ψ from a null-space of the operator $D^2(D^2 - p^2)$ satisfying

$$\psi^{(k)}(0) = \begin{cases} 0 & k = 0, 1, 2 \\ 1 & k = 3 \end{cases}$$

is called **principal function** of operator $D^2(D^2 - p^2)$. One easily obtains:

$$\psi(x) = \frac{\mathrm{sh}(px) - px}{p^3}.$$

Truncated principal function is

$$\psi(x)_+ = \begin{cases} \psi(x) & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}.$$

Generalized divided difference is a linear functional

$$\Delta f = \sum_{k=0}^{\beta-1} p_k f^{(k)}(y_0) + \sum_{k=\beta}^{4} p_k f(y_{k+1-\beta})$$

where $f \in C^{\beta-1}$, and p_k are solutions to the system

$$\begin{aligned} p_{\beta-1} &= (-1)^{\beta-1} \\ \sum_{k=0}^{\beta-1} p_k(-1)^k u_j^{(k)}(-y_0) + \sum_{k=\beta}^{4} p_k u_j(-y_{k+1-\beta}) &= 0 \quad j=1,\ldots,4 \end{aligned} \tag{1}$$

and $\{u_1, u_2, u_3, u_4\}$ is any basis of the null-space of operator $D^2(D^2-p^2)$.

Construction. B-spline is calculated by formula

$$M_{[0,k]}(x) = \Delta_t \psi(x-t)_+ \tag{2}$$

which is given in [3] where Δ_t is understood as divided difference acting on argument t, and $0, 1, \ldots, k$ are taken as knots, multiplicity of the knot 0 being $5-k$. For the base of the null-space we shall choose

$$u_1(x) = 1, \quad u_2(x) = x, \quad u_3(x) = e^{-px}, \quad u_4(x) = e^{px}.$$

1. In the first step we construct B-spline on interval $[0,4]$. The knots are $0, 1, 2, 3, 4$, so system (1) has the form

$$\begin{aligned} p_0 &= 1 \\ p_0 u_j(0) + p_1 u_j(-1) + p_2 u_j(-2) &+ p_3 u_j(-3) + p_4 u_j(-4) = 0 \quad j=1,\ldots,4. \end{aligned}$$

The solutions of the system are

$$\begin{aligned} p_0 &= p_4 = 1 \\ p_1 &= p_3 = -2\,(\mathrm{ch}p + 1) \\ p_2 &= 2\,(2\mathrm{ch}p + 1)\,. \end{aligned}$$

By substituting in (2) we obtain:

$$M_{[0,4]}(x) =$$

$$\frac{1}{p^3}\begin{cases} \mathrm{sh}\,[p(x-2)]\,\mathrm{ch}(2p)+\mathrm{ch}\,[p(x-2)]\,\mathrm{sh}(2p)-p(x-2)-2p & x\in[0,1] \\ -\left(\mathrm{sh}\,[p(x-2)]-p(x-2)\right)(1+2\mathrm{ch}p)-2\mathrm{ch}\,[p(x-2)]\,\mathrm{sh}p+2p\mathrm{ch}p & x\in[1,2] \\ \left(\mathrm{sh}\,[p(x-2)]-p(x-2)\right)(1+2\mathrm{ch}p)-2\mathrm{ch}\,[p(x-2)]\,\mathrm{sh}p+2p\mathrm{ch}p & x\in[2,3] \\ -\mathrm{sh}\,[p(x-2)]\,\mathrm{ch}(2p)+\mathrm{ch}\,[p(x-2)]\,\mathrm{sh}(2p)+p(x-2)-2p & x\in[3,4] \end{cases}$$

which we can write in a more elegant and symmetrical way

$$M_{[0,4]}(x)=$$

$$\frac{1}{p^3}\begin{cases} \mathrm{sh}\,[p(2-|x-2|)]-p(2-|x-2|) & x\in[0,1]\cup[3,4] \\ \left(\mathrm{sh}\,(p|x-2|)\right)-p|x-2|)\,(1+2\mathrm{ch}p)-2\mathrm{ch}\,(p|x-2|)\,\mathrm{sh}p+2p\mathrm{ch}p & x\in[1,3] \end{cases}.$$

$M_{[0,4]}$ satisfies

$$\begin{aligned} M_{[0,4]}(0) &= M_{[0,4]}(4)=0 \\ M_{[0,4]}(1) &= M_{[0,4]}(3)=\frac{\mathrm{sh}p-p}{p^3} \\ M_{[0,4]}(2) &= -2\frac{\mathrm{sh}p-p\mathrm{ch}p}{p^3}. \end{aligned}$$

It can be easily shown that $M_{[0,4]}$ converge towards cubic B-spline when $p\to 0$. However, when $p\to\infty$ spline $M_{[0,4]}$ grows boundlessly, so it is necessarily to norm it. Since $M_{[0,4]}$ assumes the maximal value in knot 2, B-spline $\bar{M}[0,4]$

$$\bar{M}_{[0,4]}(x)=\frac{M_{[0,4]}(x)}{M_{[0,4]}(2)}$$

will be bounded from above by 1. The other way to norm the spline is that the sum of the values in knots is constant, that is, to divide $M_{[0,4]}$ by

$$\alpha=M_{[0,4]}(1)+M_{[0,4]}(2)+M_{[0,4]}(3)=2\frac{\mathrm{ch}p-1}{p^2}, \tag{3}$$

or, alternately, to divide $M_{[0,4]}$ by

$$I=\int_0^4 M_{[0,4]}(x)dx=\frac{\mathrm{ch}p-1}{p^2}$$

to get the integral of the spline equall to unity.

Lemma 1. B-splines normed as in (3) form a partition of unity.

Proof: Choose arbitrary point y. We may assume, without loss of generality that $y \in [3,4]$. Nontrivial B-splines on interval $[3,4]$ are $\tilde{M}_{[0,4]}$, $\tilde{M}_{[1,5]}$, $\tilde{M}_{[2,6]}$ and $\tilde{M}_{[3,7]}$. Equivalence $\tilde{M}_{[i,i+k]}(y) = \tilde{M}_{[0,k]}(y-i)$ implies that

$$\begin{aligned} S &= \tilde{M}_{[0,4]}(y) + \tilde{M}_{[1,5]}(y) + \tilde{M}_{[2,6]}(y) + \tilde{M}_{[3,7]}(y) = \\ &= \tilde{M}_{[0,4]}(y) + \tilde{M}_{[0,4]}(y-1) + \tilde{M}_{[0,4]}(y-2) + \tilde{M}_{[0,4]}(y-3). \end{aligned}$$

From the expression for $\tilde{M}_{[0,4]}$ we get $S = 1$. •

2. For calculation of spline $M_{[0,3]}$, the multiplicity of knot 0 is 2 and system (1) is:

$$\begin{aligned} p_1 &= -1 \\ p_0 u_j(0) - p_1 u_j'(0) + p_2 u_j(-1) \quad &+ \quad p_3 u_j(-2) + p_4 u_j(-3) = 0 \quad j = 1, \ldots, 4. \end{aligned}$$

Solutions are

$$\begin{aligned} p_0 &= \frac{\mathrm{sh}p - p\mathrm{ch}(2p)}{\mathrm{sh}(2p) - 2\mathrm{sh}p} \\ p_1 &= -1 \\ p_2 &= \frac{2p\mathrm{ch}(2p) - \mathrm{sh}(2p) + p\mathrm{ch}p - \mathrm{sh}p}{\mathrm{sh}(2p) - 2\mathrm{sh}p} \\ p_3 &= \frac{\mathrm{sh}p - p\mathrm{ch}(2p) + \mathrm{sh}(2p) - 2p\mathrm{ch}p}{\mathrm{sh}(2p) - 2\mathrm{sh}p} \\ p_4 &= \frac{p\mathrm{ch}p - \mathrm{sh}p}{\mathrm{sh}(2p) - 2\mathrm{sh}p}, \end{aligned}$$

so

$$M_{[0,3]}(x) = p_0\psi(x)_+ - p_1\psi'(x)_+ + p_2\psi(x-1)_+ + p_3\psi(x-2)_+ + p_4\psi(x-3)_+.$$

Lemma 2. B-splines $\tilde{M}_{[0,3]}$, $\tilde{M}_{[0,4]}$, $\tilde{M}_{[1,5]}$ and $\tilde{M}_{[2,6]}$ form a partition of unity on $[2,3]$, where

$$\tilde{M}_{[0,3]} = \frac{p^2\mathrm{sh}p}{p\mathrm{ch}p - \mathrm{sh}p} M_{[0,3]}.$$

3. For calculation of spline $M_{[0,2]}$ the multiplicity of knot 0 is 3 and system (1) is:

$$\begin{aligned} p_2 &= 1 \\ p_0 u_j(0) - p_1 u_j'(0) + p_2 u_j''(0) \quad &+ \quad p_3 u_j(-1) + p_4 u_j(-2) = 0 \quad j = 1, \ldots, 4. \end{aligned}$$

Solutions are

$$
\begin{aligned}
p_0 &= \frac{p^2\left(p+\mathrm{sh}p-\mathrm{sh}(2p)\right)}{2(\mathrm{sh}p-p\mathrm{ch}p)(\mathrm{ch}p-1)}\\
p_1 &= -\frac{p^2\mathrm{sh}p}{\mathrm{sh}p-p\mathrm{ch}p}\\
p_2 &= 1\\
p_3 &= \frac{p^2(\mathrm{sh}p\mathrm{ch}p-p)}{(\mathrm{sh}p-p\mathrm{ch}p)(\mathrm{ch}p-1)}\\
p_4 &= -\frac{p^2(\mathrm{sh}p-p)}{2(\mathrm{sh}p-p\mathrm{ch}p)(\mathrm{ch}p-1)}.
\end{aligned}
$$

Lemma 3. B-splines $\tilde{M}_{[0,2]}$, $\tilde{M}_{[0,3]}$, $\tilde{M}_{[0,4]}$ and $\tilde{M}_{[1,5]}$ form a partition of unity on $[1,2]$, where

$$\tilde{M}_{[0,2]} = p\frac{\mathrm{ch}p-1}{\mathrm{sh}p-p}M_{[0,2]}.$$

4. For calculation of spline $M_{[0,1]}$ the multiplicity of knot 0 is 3 and system (1) is:

$$
\begin{aligned}
p_3 &= -1\\
p_0u_j(0)-p_1u_j'(0)+p_2u_j''(0) &- p_3u_j'''(0)+p_4u_j(-1)=0 \quad j=1,\ldots,4.
\end{aligned}
$$

Solutions are

$$
\begin{aligned}
p_0 &= p_1 = -\frac{p^3}{\mathrm{sh}p-p}\\
p_2 &= -p\frac{\mathrm{ch}p-1}{\mathrm{sh}p-p}\\
p_3 &= -1\\
p_4 &= \frac{p^3}{\mathrm{sh}p-p}.
\end{aligned}
$$

Lemma 4. B-splines $M_{[0,1]}$, $\tilde{M}_{[0,2]}$, $\tilde{M}_{[0,3]}$ and $\tilde{M}_{[0,4]}$ form a partition of unity on $[0,1]$.

Splines $M_{[0,1]}$, $\tilde{M}_{[0,2]}$, $\tilde{M}_{[0,3]}$ and $\tilde{M}_{[0,4]}$, converge towards normed cubic B-splines when $p\to 0$, and to piecewise linear B-splines when $p\to\infty$.

References

[1] Schweikert, D.G., *'An Interpolating Curve Using a Spline in Tension'*, J.Math.Physics,45,312-317 (1966).

[2] Kozak, J., *'Shape Preserving Approximation'*, Computers in Industry,7,435-440 (1986).

[3] Schmidt, E., Lancaster, P., Watkins, D., *'Bases of Splines Associated with Constant Coefficient Differential Operators'*, SIAM J. Numer. Anal., Vol.12., No. 4, 630-645 (1975).