SOLVING 2^{nd} ORDER PARABOLIC SYSTEM BY SIMULATIONS OF MARKOV JUMP PROCESSES

Nedžad Limić and Mladen Rogina,

Department of mathematics, Faculty of Natural Sciences,
University of Zagreb

ABSTRACT:
There are known methods of approximating the solution of parabolic 2^{nd} order systems by solving stochastic differential equations instead. The main idea is based on the fact that stochastic differential equation defines a diffusion process, generated by an elliptic differential operator on $\mathcal{R}^{\mathrm{d}}$. We propose a difference scheme for the elliptic operator, which possesses the structure of Markov (jump) process. The existence of such a scheme is proved, the proof relying on the choice of new coordinates in which the elliptic operator is "almost" Laplacian, and has the properties necessary for discretization.

Time discretization, which involves difference schemes for parabolic equations with known stability difficulties, can thus be replaced by space discretization and simulation of the associated Markov (jump) process.

1. INTRODUCTION

There is a well known statistical method for estimating solutions of the initial value problem of parabolic system of PDE of the 2^{nd} order [KP1, LPS1]. The method is based on statistical estimation of sample paths of simulated solutions of the associated stochastic differential equation.

Here, another statistical method for estimating solutions of the parabolic system is proposed. It consists of two steps. In the first step the elliptic operator of the system is discretized on a space grid so that the resulting stiffness matrix is the generator of a Markov jump process on grid knots. In this way the parabolic system is approximated by initial value problem for a system of ODE. In the second step solutions of ODE at the knots are statistically estimated by simulating the Markov jump process.

The proposed method is advantageous in comparison to other methods only if the diffusion tensor and drift velocity are t-independent, while the extinction coefficient may be t-dependent. Due to this fact the time discretization is avoided. The proposed method simplifies the approach, which is extensively considered by Kuchner [KD1] in which Markov chains are used.

Results described here are more or less expectable. The necessary steps are gathered from various disciplines, and some lacking links are proved in order to get a consistent presentation. In Section 2 a proof is given that there exists a discretization of elliptic differential operator having the structure of generator of a Markov jump process. In Section 3 the result about approximation is extended to the initial value problem for PDE. Statistical estimation of solutions by using Monte Carlo methods is described in Section 4. The initial value problem with homogeneous Dirichlet conditions is discussed in Section 5.

2. NUMERICAL SCHEME FOR AN ELLIPTIC OPERATOR HAVING MJP–STRUCTURE.

Let the functions a_{ij}, b_i and c be defined on $\mathcal{R}^d, i, j = 1, 2, ..., d$, have the following properties:

$$a_{ij}, b_i, c \in C^\alpha(\mathcal{R}^d), c \leq 0,$$
$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) z_i \bar{z}_j \geq \mu \mid z \mid^2, \mu > 0, \mathbf{x} \in \mathcal{R}^d, \tag{1}$$

where $C^\alpha(\mathcal{R}^d)$ is the Hölder class of functions with the parameter $\alpha \in (0, 1)$. Functions of $C^{k+\alpha}(\mathcal{R}^d)$ naturally define a real or a complex linear space. When this space is endowed with the norm $\| \cdot \|_{k+\alpha}$ (see [LU1]), it becomes a Banach space, for which the same symbol is used. For the elliptic differential operator

$$A(\mathbf{x}) = \sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) \partial_i \partial_j + \sum_{i=1}^{d} b_i(\mathbf{x}) \partial_i + c(\mathbf{x}), \tag{2}$$

the following result of classical or Shauder theory of PDE is used (see [LU1]). If (1) is valid and $\mathrm{Re}\,\lambda > 0$, then: a) the differential operator $\lambda I - A$ is one-to-one map between the Banach spaces $C^{2+\alpha}(\mathcal{R}^d)$ and $C^\alpha(\mathcal{R}^d)$, and b) for each $u \in C^{2+\alpha}(\mathcal{R}^d)$ the following inequality $\| (\lambda I - A)u \|_0 \leq \| u \|_0 / \mathrm{Re}\,\lambda$ holds.

For each natural n the set of points $\mathbf{x} = (k_1 h, k_2 h, \ldots, k_d h) \in \mathcal{R}^d, h = 2^{-n}$ defines the numerical grid G_n on $\mathcal{R}^d$. Let $E_2 = C^{2+\alpha}(\mathcal{R}^d), E_2(n)$ be the closed subspace of all elements in E_2 vanishing at G_n. If the space $F_2 = E_2/E_2(n)$ is endowed with a quotient norm, it becomes isomorphic to the subspace of l_∞ spanned by sequences converging to zero. Analogously, $E_0 = C^\alpha(\mathcal{R}^d), E_0(n)$ and F_0 are defined. Because of $F_2 \subset F_0$, and the mentioned isomorphism, we identify F_2 and F_0. The natural embedding from E_k into $F_k, k = 2, 0$, is denoted by $\phi_k(n)$.

We say that an element $u \in E_k$ is approximated with the order $\beta > 0$ by elements $u_n \in F_k$ if there exists an n-independent positive number κ such that $\| \phi_k(n)u - u_n \|_k < \kappa h^\beta$. In this case, we say that u_n factor converges to u with the order β. Also we say that an operator $A \in \mathbf{L}(E_2, E_0)$ is approximated by the operators $A_n \in \mathbf{L}(F_2, F_0)$ with the order β if there exists an n-independent positive number κ such that

$$\| \phi_0(n)A - A_n \phi_2(n) \|_{\mathbf{L}(E_2, F_0)} \leq \kappa h^\beta. \tag{3}$$

The number β is also called the order of approximation. The approximation is called stable if there exists an n-independent $\rho > 0$ such that $\| (A_n)^{-1} \| \leq \rho$.

For $\mathrm{Re}\,\lambda > 0$, solutions u and u_n of respective equations,

$$\begin{gathered} (\lambda I - A)u = f, \\ (\lambda I - A_n)u_n = f_n, n \in \mathcal{N}, \end{gathered} \tag{4}$$

will be compared. For this purpose the following result [Kr1] is used:

THEOREM **1** *Let $A \in \mathbf{L}(E_2, E_0), A_n \in \mathbf{L}(F_2, F_0)$ be approximations of A with the order β. Let $f \in E_0$, and let f_n be approximations of f with the order β. If u and u_n are the unique solutions of (4), and $\lambda I - A_n$ are stable, then u_n factor converges to u with the order equal to β.*

For an elliptic operator A satisfying (1), an approximation A_n is said to posses MJP-structure if A_n are generators of (regular or irregular) Markov jump processes. Of course, the property is assumed to be valid for all natural n up to a finite number of them. These exceptional natural numbers are disregarded in the following.

The basic result of this section is the following theorem:

THEOREM **2** *Let A be defined by (1) and (2). Then there exist approximations $A_n \in \mathbf{L}(F_2, F_0)$ such that:*

(a) each A_n possesses MJP-structure and approximates A with the order α,

(b) there exists $m_B \in \mathcal{N}$ such that A_n have the band structure with the band-width not larger than $1 + 2m_B$,

(c) equations $(\lambda I - A_n)u_n = f_n, \mathrm{Re}\,\lambda > 0$, have unique solutions in F_0 with the property $\| u_n \|_\infty < \| f_n \|_\infty / \mathrm{Re}\lambda$,

(d) If f_n factor converges to f with the order α, then the sequence of solutions u_n of (4) also factor converges to u with the order α.

Proof. The statement (c) is a consequence of the MJP-structure of A_n. This property ensures the stability of operators $\lambda I - A_n$. The statement (d) follows from (a), (c) and Theorem 1.

To prove (a), let the approximation A_n of A have the property $Ap(\mathbf{x}) - A_n p(\mathbf{x}) = 0$ at $\mathbf{x} \in G_n$ whenever p is a polynomial of the second order. Then, by using standard methods of numerical analysis for PDE, the expression (3) can be easily derived with $\beta = \alpha$. Hence, to finish the proof, we have to construct a sequence A_n having the mentioned property and MJP-structure.

The partial differential operators of the first and second orders are approximated in the usual way:

$$\begin{aligned} \partial_i f(\mathbf{x}) \to \Box_i f(\mathbf{x}) &= \frac{1}{2h}\left[f(\mathbf{x}+\mathbf{e}_i h) - f(\mathbf{x}-\mathbf{e}_i h)\right], \\ \partial_i^2 f(\mathbf{x}) \to \Box_i^2 f(\mathbf{x}) &= \frac{1}{h^2}\left[f(\mathbf{x}+\mathbf{e}_i h) - 2f(\mathbf{x}) + f(\mathbf{x}-\mathbf{e}_i h)\right], \end{aligned}$$

where $\mathbf{e}_i$ are unit vectors in the i^{th} direction. The mixed partial differential operator can be approximated by one of the following four possibilities:

$$\partial_i\partial_j f(\mathbf{x}) \to \Box_i\Box_j f(\mathbf{x}) = \frac{1}{h^2}\left\{ \begin{array}{l} f(\mathbf{x} \pm \mathbf{e}_i h \pm \mathbf{e}_j h) - f(\mathbf{x} \pm \mathbf{e}_i h) - f(\mathbf{x} \pm \mathbf{e}_j h) + f(\mathbf{x}), \\ -f(\mathbf{x} \pm \mathbf{e}_i h \mp \mathbf{x}_j h) + f(\mathbf{x} \pm \mathbf{e}_i h) + f(\mathbf{x} \mp \mathbf{e}_j h) - f(\mathbf{x}). \end{array} \right.$$

In this way, the quadratic operator $a_{ii}(\partial_i)^2 + a_{jj}(\partial_j)^2$ is approximated by the matrix $a_{ii}(\Box_i)^2 + a_{jj}(\Box_j)^2$ with negative diagonal elements and non-negative off-diagonal elements. If $a_{ij} \leq 0$ then $a_{ij}\partial_i\partial_j$ is approximated by the half sum of the first two possibilities, otherwise by the half sum of the second two possibilities. Let us assume that there exists $0 < \rho < 1$, such that

$$| a_{ij}(\mathbf{x}) | \leq \rho \min\{a_{ii}(\mathbf{x}), a_{jj}(\mathbf{x})\}, i \neq j, \tag{5}$$

at the considered knot. Then the result, $\sum_{ij} a_{ij}\Box_i\Box_j$, is the matrix having MJP-structure. The approximation of $\sum b_i\partial_i$ by $\sum b_i\Box_i$ can violate the MJP-structure only for larger values of h. Hence, if (5) is valid for all $\mathbf{x} \in \mathcal{R}^d$, the approximations A_n have MJP-structure for large values of n.

If there is no $\mathbf{x} \in \mathcal{R}^d$ with the property (5), a transformation of coordinates can be performed, and a redefinition of grids G_n, so that the set $D_0 \subset \mathcal{R}^d$ for which (5) holds is nonempty. To each $\mathbf{x} \in \mathcal{R}^d$ violating (5) corresponds a new coordinate system centred at $\mathbf{x}$, and denoted by $T(\mathbf{x})$, such that in the new coordinates the operator $A(\mathbf{x})$ has the expression $A(\mathbf{x}) = \mu\Delta + \sum b_i\partial_i + c$. The coordinate transformation $T(\mathbf{x})$ is defined by $d(d-1)$ angles in the set S^d, where $S \subset \mathcal{R}^d$ is the unit spherical surface. These angles are uniquely defined in case when $T(\mathbf{0})$ is specified in advance, and if we demand the continuity of the map $\mathbf{x} \to T(\mathbf{x})$. By using the compactness of S^d and the uniform continuity of a_{ij}, one proves that the set $\mathcal{R}^d$ can be divided into a finite number of subsets $D_m, m = 0, 1, 2, \ldots, M$, with the following properties: for each D_m there is a corresponding coordinate transformation $\hat{T}_m$ in which the transformed operator A has the property (5) for all $\mathbf{x} \in D_m$, and positive number $\rho/2$ instead ρ. A coordinate axes of this system may miss knots of all $G_n, n \in \mathcal{N}$. Therefore, by a slight adjustment of axes, there can be constructed another set of M coordinate systems T_m, such that the transformed operators $A(\cdot)$ have the property (5) on D_m. The approximations A_n in the transformed systems have the MJP structure. After turning to the original system, i.e. to the system T_0, the obtained approximations preserve the MJP-structure. In this way A can be approximated by matrices A_n having the MJP-structure.

In order to prove (b) we point out that only a finite number of coordinate transformations are used for the discretization. Hence, for each grid-knot, irrespectively of grid-step, only a finite number of various numerical neighbourhoods can be chosen. This proves the theorem.

The matrix A_n may be reducible for each n. However, by omitting certain grid knots the irreducibility can be achieved:

LEMMA **1** *There exist subgrids $\tilde{G}_n \subset G_n$ such that $\bigcup \tilde{G}_n$ is dense in $\mathcal{R}^d$, and the approximations $\tilde{A}_n$ associated with $\tilde{G}_n$ are irreducible matrices.*

3. INITIAL VALUE PROBLEM FOR PARABOLIC SYSTEMS ON $\mathcal{R}^d$.

Let the functions a_{ij}, b_i and c have the following properties (the extinction coefficient c is generally t-dependent):

$$\begin{gathered} a_{ij}, b_i, \in C^{1+\alpha}(\mathcal{R}^d) \\ c \in C([0,\infty) \times C^\alpha(\mathcal{R}^d)), c \leq 0, \end{gathered} \tag{6}$$

and consider the elliptic operator B on $\mathcal{R}^d$ defined by the expression:

$$B(t, \mathbf{x}) = \sum_{i,j=1}^{d} \partial_i a_{ij}(\mathbf{x})\partial_j - \sum_{i=1}^{d} b_i(\mathbf{x})\partial_i + c(t, \mathbf{x}). \tag{7}$$

Both differential operators, $B(t)$ and its formal adjoint $A(t) = B^+(t)$, have the form (2) with properties (1) for each $t \geq 0$. Therefore, the approximations $A_n(t) \in \mathbf{L}(F_2, F_0)$ of the previous section can be constructed.

The objects of interest in this section is the following initial value problem:

$$\left(\frac{\partial}{\partial t} - B(t)\right) u(t) = f(t), u(0) = u_0, \tag{8}$$

where $u_0 \in L_1(\mathcal{R}^d)$ and $f \in L_1([0,\infty), \mathcal{R}^d)$. Its approximations are:

$$\left(\frac{\partial}{\partial t} - B_n(t)\right) \mathbf{u}_n(t) = \mathbf{f}(t), \mathbf{u}_n(0) = \mathbf{u}_{0n}, \tag{9}$$

where $B_n(t) = A_n^+(t)$.

The existence of a solution of (8), with appropriate conditions on u_0 and f, can be obtained from the classical result [Fi1]. Let $u_0 \in \dot{C}^{2+\alpha},(\mathcal{R}^d)$ and $f \in C([0,\infty), \dot{C}^{\alpha}(\mathcal{R}^d))$. Then the problem (8) has a unique solution u with the following properties. For each $t \geq 0$, $u(t) \in \dot{C}^{2+\alpha}(\mathcal{R}^d)$, $u'(t) \in \dot{C}^{\alpha}(\mathcal{R}^d)$, and for each $T > 0$, there exists $\kappa(T)$ such that $\parallel u(t) \parallel_{2+\alpha} \leq \kappa(T) \parallel f \parallel_{\alpha}$, $t \in [0,T]$.

The existence of the solutions of (8) in $C([0,\infty), L_1(\mathcal{R}^d))$ can be proved if we assume additional properties on the coefficients. Let $u_0 \in L_1(\mathcal{R}^d)$ and $f \in L_1([0,\infty) \times \mathcal{R}^d))$. It suffices to assume that the coefficients (6) differ from constant values on a compact set. Then, there exists an evolution family $U(t,s), 0 \leq s \leq t$, of positive and contraction operators, such that $u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds$ (basic steps of a proof can be found in [T1]).

For the initial value problems for the column-valued functions $\mathbf{u}_n(\cdot)$ in (9), the following result is valid. The problems (9) are solvable in l_1. The bounded operator B_n on l_1 generates an evolution family of operators $U_n(t,s), s,t \in [0,\infty)$. Solutions of the considered problems can be written as in the previous case.

In the remaining part of exposition it is assumed that (8) has a unique solution in $C([0,\infty), L_1(\mathcal{R}^d))$, and for each $T > 0$, there exists an n-independent positive number $\kappa(T)$ such that the following error estimate is valid:

$$\parallel \phi_2(n)u(t) - \mathbf{u}_n(t) \parallel_{l_1} \leq \kappa(T)h^{\alpha}, t \leq T. \tag{10}$$

It is understood that the estimate (10) is valid for a subspace of initial states and nonhomogeneous terms $f(\cdot)$.

4. ESTIMATION OF SOLUTIONS BY MONTE CARLO SIMULATION

The space discretization of an initial value problem for PDE results in a sequence of initial value problems for ODE associated with Markov jump processes. Therefore, only the basic facts of construction of relevant Markov jump processes on finite or denumerable state space will be described. All constructions must have the following structure. Let $S \subset G_n$ be a subset with the index set $\Re$ and $\mathcal{P}(S)$ be the set of all probability densities on S. Let G be a matrix with indices in $\Re$, having the structure of generator of a regular Markov process. The Markov process on S with initial distribution $\nu \in \mathcal{P}(S)$ and generator G is defined by $\{S, \nu, G\}$ and denoted by $t \to \xi(t)$. Its representation by means of an infinite sequence of independent standard exponential random variables and the corresponding Markov chain can directly be used for simulations.

A discretization of the initial value problem for parabolic system in terms of the resulting initial value problem for ODE has the form

$$\begin{array}{c} \frac{d}{dt}\mathbf{u}(t) = B(t)\mathbf{u}(t) + \mathbf{f}(t), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{array} \tag{11}$$

The matrix $A(t) = B(t)^+$ has MJP-structure for each $t \geq 0$. We say that $A(\cdot)$ is conservative if the sum of each of its rows is zero. In order to describe M.C. (Monte Carlo) method for estimating the j^{th} component of the solution $u_j(t)$, it is useful to distinguish the following three cases:

• t-independent generator and $\mathbf{f} = \mathbf{0}$. A problem with a non-conservative generator can be transformed to a problem with a conservative one by adding an abstract absorbing state a to the state space S and defining a new generator for $S_a = S \cup \{a\}$. Therefore, it suffices to only the consider conservative generators. If the initial state $\nu = \mathbf{u}_0$ of (11) is non-negative and $\| \mathbf{u}_0 \|_1 = 1$, the object $\{S, \nu, a\}$ defines the Markov jump process to be used for simulations. The estimator of $u_j(t)$ is defined by the statistics:

$$s_N(u_j(t)) = \frac{1}{N}\sum_{m=1}^{N} \chi_j(\xi_m(t)), \tag{12}$$

where χ_j is the function on S defined by $\chi_j(\mathbf{x}_i) = \delta_{ij}, \mathbf{x}_j \in G_n$, and $\xi_m(\cdot)$ are independent Markov processes defined by $\{S, \nu, A\}$. The first two statistical moments of (12) are $\mathbf{E}[s_N(u_j(t)] = u_j(t)$ and $\mathbf{Var}[s_N(u_j(t)] = N^{-1}u_j(t)(1 - u_j(t))$, so that the following upper bound is valid:

$$\mathbf{Var}[s_N(u_j(t))] \leq \frac{1}{4N}. \tag{13}$$

If the initial state $\nu = \mathbf{u}_0$ is non-negative and $\| \mathbf{u}_0 \|_1 \neq 1$, an appropriate scaling of initial state must be performed first. If the components of $\mathbf{u}_0$ have both signs, a decomposition $\mathbf{u}_0 = \mathbf{u}_+ - \mathbf{u}_-$ can be made, where $\mathbf{u}_\pm$ have non-negative components. The previous procedure can now be applied to the two processes defined by $\{S, \mathbf{u}_+, A\}$ and $\{S, \mathbf{u}_-, A\}$, and the results must be subtracted.

In the case of $\mathbf{u}_0 \in \mathcal{P}(S)$, the error of the estimate consists of two parts. The first is caused by a space discretization and given by (10), and the second is the statistical error defined by (13). In other cases the scaling must be included in (13).

• t-independent generator and $\mathbf{u}_0 = \mathbf{0}$. Let the column-valued function $\mathbf{f}(\cdot)$ have non-trivial elements only for indices $i \in \Im \subset \Re$. The problem (11) has the solution:

$$u_j(t) = \sum_{i\in\Im}\int_0^t w_{ij}(t-s))f_i(s)ds, j \in \Re,$$

where $W(t) = \exp{(B^+t)}$.

Let the following objects be given: 1) Markov jump processes $\xi_i, i \in \Im$, on S, starting at $\mathbf{x}_i \in S$; 2) continuous random variables $\tau_i, i \in \Im$, with values in $[0, T]$, and defined by means of densities $t \to p_i(t)$; 3) functions $g_i : [0, T] \to \mathcal{R}, g_i = f_i/p_i$, and the function χ_j defined as in (12). Then the random variable

$$\eta_j(T) = \sum_{i\in\Im}\chi_j(\xi_i(T-\tau_i))g(\tau_i),$$

has the moments $u_j(T) = \mathbf{E}[\eta_j(T)]$ and:

$$\mathbf{Var}[\eta_j(T)] = \sum_{i\in\Im}\int_0^T u_{ij}(T-s)g_i(s)^2 p_i(s)ds - \mathbf{E}[\eta_j(T)]^2.$$

The following steps give rise to a simulation of η_j: for each $i \in \Im$ a simulation of τ_i results in t_i; for each $i \in \Im$ a simulation of ξ_i on $[0, T - t_i)$ results in $\mathbf{x}_i$; then a simulation of $\eta_j(T)$ equals

$$\eta_j(T) = \sum_{i\in\Im}\chi_j(\mathbf{x}_i)f_i(\tau_i).$$

• t-dependent generator. It suffices to consider the case with $\mathbf{u}_0 \in \mathcal{P}(S)$ and $\mathbf{f} = \mathbf{0}$. In accordance with the basic problem of Section 3, the generator is non-conservative and can be represented in the following form:

$$A(t) = \begin{bmatrix} 0_{11} & O_{1\Im} \\ O_{\Im 1} & G \end{bmatrix} + \begin{bmatrix} O_{11} & O_{1\Im} \\ R(t) & H(t) \end{bmatrix},$$

where $O_{11} = 0, O_{1\Im}$ is the zero row of card($\Im$) - length, $O_{\Im 1}$ the zero column of the same length, G a conservative generator, $H(\cdot)$ a diagonal matrix-valued function with elements $h_{ij}(t) = c(t, \mathbf{x}_i)$, and $R(\cdot)$ a column-valued function, $r_{i1}(t) = -h_{ii}(t), i \in \Im$. The abstract state associated with the first index of the matrix $A(\cdot)$ is denoted by a. We assume that $c(t, \mathbf{x}) < -c_0$, where c_0 is a positive number.

Let $\nu \in \mathcal{P}(S)$ and $\{S, \nu, A(\cdot)\}$ define a non-homogeneous Markov jump process $\xi(\cdot)$, and $\{S, \nu, G\}$ a homogeneous Markov jump process $\theta(\cdot)$. In addition, we need to define the stopping time τ. Let $\omega \to \theta(\omega, t)$ be a sample path of the process $\theta(\cdot)$. The function

$$F_\omega(t) = 1 - \exp\left(\int_0^t c(s, \theta(\omega, s)ds\right)$$

is a distribution of a random variable, denoted by τ_ω. It gives rise to the following stopping time τ, of the process $\xi(\cdot)$:

$$\xi(t) = \begin{cases} \theta(t) & \text{for} \quad t < \tau, \\ a & \text{for} \quad t \geq \tau. \end{cases}$$

A simulation of a sample path of $\xi(\cdot)$ consists of two steps. Let a simulated sample path of $\theta(\cdot)$ be $\theta(\omega, \cdot)$, and T be a simulated value of τ_ω. Then $\xi(\omega, t) = \theta(\omega, t)$ for $t < T$, and $\xi(\omega, t) = a$ for $t \geq T$.

5. PARABOLIC SYSTEM WITH DIRICHLET BOUNDARY CONDITION.

Let $D \subset \mathcal{R}^d$ be an open, bounded and connected set with the boundary ∂D consisting of regular points with respect to the Laplacian. The so-called cone criterion is one amongst familiar criteria ensuring the regularity of a boundary. Then the regularity with respect to elliptic operators is considered in [Kl1]. We consider the parabolic system on D with the homogeneous Dirichlet boundary conditions on D:

$$\begin{aligned} \left(\tfrac{\partial}{\partial t} - B(t, \mathbf{x})\right) u(t, \mathbf{x}) &= f(t, \mathbf{x}), \quad \mathbf{x} \in D, \\ u(t, \mathbf{x}) &= 0, \quad \mathbf{x} \in \partial D, \\ u(0, \mathbf{x}) &= u_0(\mathbf{x}), \end{aligned} \tag{14}$$

where $u_0 \in L_1(D)$ and $f \in L_1([0,\infty), D)$. The problem can be solved if the conditions (6) are valid. In the case of t-independent coefficients, a contraction semigroup $t \to U_D(t)$ on $L_1(D)$ can be constructed, and solutions can be represented by it in the same way as in Section 3. An extension to a t-depending coefficient c is straightforward [T1].

Solutions of (8) and (14) can be related by means of the diffusion process generated by the generator $A_{\text{dif}} = \sum_{ij} a_{ij}\partial_i\partial_j + \sum_j b_j\partial_j$ [Fe1] in terms of the first exit time from D.

Let $\xi(\cdot)$ be a diffusion process generated by A_{dif} and $\xi_n(\cdot)$ the family of Markov jump processes from the previous section. The following result is valid.

THEOREM **3** *Let (6) be valid and coefficients of A_{dif} be different from constant values on a compact set. Then the sequence of processes ξ_n converges weakly to the process ξ.*

This result enables us to estimate solutions of (14) by simulating the first exit time of Markov jump processes ξ_n from the transient set $D \cap G_n$.

6. CONCLUSION

The initial value problem for parabolic system on $\mathcal{R}^d$ and on a bounded domain with Dirichlet condition is considered. Its efficiency is demonstrated in [SL1]. To estimate solution by using M.C. simulation of associated Markov jump processes seams to be very attractive. The error of the estimate consists of two parts, one being the standard error of numerical analysis of PDE, and the other is a statistical one.

References

[Fe1] Freidlin M., Functional integration and partial differential equations, Princeton University Press, 1985.

[Fi1] Friedman A. Partial differential equations of parabolic type, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[LU1] Ladyzhenskaya O.A. and Ural'tseva N.N., Linear and quasilinear elliptic partial diferential equations, Academic, New York, 1968.

[LPS1] B. Lapeyre, E. Pardoux and R. Sentis: Mèthodes de Monte Carlo pour les èquation de transport et de diffusion, Springer 1998.

[KP1] P.E. Kloeden and E. Platen: Numerical solutions of stochastic differential equations, Springer, Applications of mathematics Vol **23**, 1992.

[Kr1] S.G. Krein: Linear differential equations in Banach spaces (Russ.), Nauka, Moskva, 1967.

[Kl1] Krylov N.V.: On the first boundary value problem for elliptic equations of the 2^{nd} order, Diff. Urav., **3**, 315-326 (1967)

[KD1] H.J. Kushner and P.G. Dupuis: Numerical methods for stochastic control problems in continuous time, Springer,1991.

[SL1] Škunca O. and Limić: Evaluation of stochastic dynamics and estimation of search path quality by Monte Carlo simulation, this journal.

[T1] Tanabe H.: Equations of evolution, Pitman, 1975