Numerical solutions of Dirichlet problem for elliptic operator in divergence form with a right-hand side measure

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Abstract. We consider a 2^{nd}-order elliptic operator $A = -\sum_{ij} \partial_i a_{ij} \partial_j$ on $\mathbf{R}^d$ in divergence form from the point of view of its numerical approximations in terms of matrices A_n having compartmental structure, that is $(A_n)_{ii} < 0$, $(A_n)_{ij} \geq 0$, $i \neq j$, $\sum_i (A_n)_{ij} \leq 0$. We solve numerically the corresponding Dirichlet problem on a bounded domain $D \subset \mathbf{R}^d$ $(d = 2, 3)$, for which the right hand side is a probability measure with support in D. Numerical solutions on grids are non-positive, and can be naturally embedded into linear spaces of 'hat' functions approximating the original solution in $\dot{W}_1^1(D)$. Numerical solutions converge in $L_1(D)$, while its first partial derivatives converge weakly in a certain way.

The Dirichlet problem with $A(\mathbf{x}) = \Delta$ is numerically solved in [Cl] by finite elements, and to our knowledge this is the only publication about numerical approximations for elliptic equations with a measure on the right hand side. The novelty is, that we consider general operator $A(\mathbf{x})$ and its grid approximations A_n that possess compartmental structure. In the construction of such approximations A_n of the elliptic operator A, we have to overcome two major difficulties: (a) $a_{ij}(\mathbf{x})$, $i \neq j$, are large in comparison with the pair of values $a_{ii}(\mathbf{x})$, $a_{jj}(\mathbf{x})$; (b) the coefficients a_{ij}, $i \neq j$, change sign on $\mathbf{R}^d$. Difficulty (a) is common for operators A in divergence form, as well as for analogous problems for the classical elliptic operators $A = \sum_{ij} a_{ij} \partial_i \partial_j + \sum_i b_i \partial_i + c$ [MW, LR1, LR2], whereas (b) arises for operators in divergence form due to the requirement of compartmental structure for approximations.

The construction of our approximations is valid for general dimension, but the convergence proof essentially needs $d = 2, 3$. We end by a nontrivial example that illustrates the obtained results.

References

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