Monotone Numerical Schemes for a Dirichlet problem for Elliptic Operators in Divergence Form

Mladen Rogina[*1] and **Nedžad Limić**[**1]

[1] Dept. of Mathematics, University of Zagreb, Bijenička 30, Zagreb

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The object of present analysis are numerical solutions of the elliptic boundary value problems in terms of monotone schemes. We assume that the elliptic differential operator has the divergence form, with measurable coefficients satisfying the strict ellipticity condition. The basic idea of monotone schemes can be found in [MW], without the analysis of convergence of approximate solutions. Convergence proofs for C spaces can be found in [SMMM], and for $L_1(\mathbb{R}^d)$-spaces in [LR2] with the restriction on dimension ($d = 2$ and $d = 3$); an extension for $d > 3$ can be found in [LR3]. Here we consider schemes possessing stencils enclosed by rectangles with vertices at grid-knots, and extend published results by constructing schemes with stencils stretching far from basic grid-rectangles, so being conceptually closer to the original idea in [MW]. The schemes are not derived from finite difference operators approximating differential operators, but rather from a general principle which ensures the convergence of approximate solutions. In the case of the classical elliptic problem, this general principle is necessary and sufficient to prove convergence in Hölder spaces.

1 Discretization schemes

We study elliptic operators on $\mathbb{R}^2$ in divergence form defined by

$$A(\boldsymbol{x}) = -\sum_{i,j=1}^{2} \partial_i a_{ij}(\boldsymbol{x})\partial_j \;+\; \sum_{j=1}^{2} \partial_j\big(b_j(\boldsymbol{x})\,\cdot\,\big) \;+\; c(\boldsymbol{x}), \tag{1}$$

assuming that the functions $a_{ij} = a_{ji}$, $b_i, i, j = 1, 2$ and c are measurable on $\mathbb{R}^2$, $c \geq 0$ and $a_{ij}(\boldsymbol{x})$ converge to constant values as $|\boldsymbol{x}|$ increases. We require the existence of $\underline{M}, \overline{M}$, $0 < \underline{M} \leq \overline{M}$, such that $\underline{M}\,|\boldsymbol{x}|^2 \leq \sum_{i,j=1}^{2} a_{ij}(\boldsymbol{x}) z_i \bar{z}_j \leq \overline{M}\,|\boldsymbol{x}|^2$ holds for $\boldsymbol{x} \in \mathbb{R}^2$, $\boldsymbol{z} \in \mathbb{C}$ (strong ellipticity). Let the orthogonal coordinate system in $\mathbb{R}^2$ be determined by unit vectors $\boldsymbol{e}_i$, and let us, for each $n \in \mathbb{N}$, define a numerical grid G_n on $\mathbb{R}^2$ by vectors $\boldsymbol{x} = \sum_{l=1}^{2} h_i(n)\, k_l \boldsymbol{e}_l$, where the grid-steps $h_i(1)$ are fixed and $h_i := h_i(n) = 2^{-n} h_i(1)$. For a domain D with Lipshitz boundary we denote discretizations of D as $G_n(D) = G_n \cap D$.

Standard approach to discretizations of differential operators is based on finite differences approximating $\partial_i, \partial_i\partial_j$. Instead, we can use constructions avoiding finite difference operators altogether. Let $\boldsymbol{r} \in \mathbb{N}^2$. In order to write the entries of the system matrix A_n in the concise form, we use the abbreviations:

$$\begin{aligned} a_{ij}^{\pm +}(\boldsymbol{r}) &= a_{ij}\big(\boldsymbol{x} + \frac{1}{2}(\pm r_1 h_1 \boldsymbol{e}_1 + r_2 h_2 \boldsymbol{e}_2)\big), \\ a_{ij}^{\pm -}(\boldsymbol{r}) &= a_{ij}\big(\boldsymbol{x} + \frac{1}{2}(\pm r_1 h_1 \boldsymbol{e}_1 - r_2 h_2 \boldsymbol{e}_2)\big), \end{aligned}$$

* Corresponding author: e-mail: rogina@math.hr, Phone: +385 1 460 5745, Fax: +385 1 468 0335
** e-mail: nlimic@math.hr, Phone: +385 1 460 5740, Fax: +385 1 468 0335

and $a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(\mathbf{1})$ by convention. Non-trivial off-diagonal entries to A_n have the form:

$$\begin{aligned}(A_n)_{\boldsymbol{x}\boldsymbol{x}\pm h_1\boldsymbol{e}_1} &= -\tfrac{1}{h_1}\left(\tfrac{1}{h_1}\,a_{11}^{\pm+} - \tfrac{r_1}{r_2h_2}\,|a_{12}^{\pm+}(\boldsymbol{r})|\right),\\ (A_n)_{\boldsymbol{x}\boldsymbol{x}\pm h_2\boldsymbol{e}_2} &= -\left(\tfrac{1}{h_2}\,a_{22}^{+\pm} - \tfrac{r_2}{r_1h_1}\,|a_{12}^{+\pm}(\boldsymbol{r})|\right),\\ (A_n)_{\boldsymbol{x}\boldsymbol{x}\pm(r_1h_1\boldsymbol{e}_1-r_2h_2\boldsymbol{e}_2)} &= -\tfrac{1}{r_1r_2h_1h_2}\,a_{12}^{\pm\mp}(\boldsymbol{r}),\end{aligned} \tag{2}$$

while diagonal elements are negative sums of the off-diagonal ones. A matrix is said to have the compartmental structure if it has positive diagonal entries, non-positive off-diagonal entries and positive or zero column sums. If A is compartmental, then $B = A^T$ is a matrix of positive type [Yo].

If $a_{12} \le 0$ and

$$\inf_{\boldsymbol{x}\in G_n}\left\{\frac{1}{r_ih_i}\,a_{ii}^{\alpha\beta} - \frac{1}{r_jh_j}\,|a_{12}^{\gamma\delta}(\boldsymbol{r})|\right\} > 0, \qquad \alpha,\beta,\gamma,\delta \in \{+,-\},$$

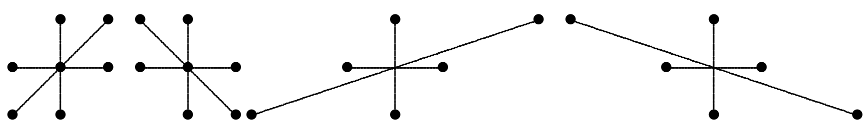
the constructed matrix A_n is compartmental; we call it the first extended scheme. The second extended scheme can be constructed analogously, and it has compartmental structure for $a_{12} \ge 0$.

If A_n is compartmental, or of positive type, then the obtained numerical scheme is monotone. For a matrix A_n on G_n we define numerical neighborhoods

$$\mathcal{N}(\boldsymbol{x}) = \{\boldsymbol{y}\in G_n \,:\, \boldsymbol{x} = h\boldsymbol{k},\ \boldsymbol{y} = h\boldsymbol{l},\ (A_n)_{\boldsymbol{kl}} \ne 0\}.$$

Numerical neighborhoods for two extended schemes are depicted in Figure 1.

Fig. 1 Possible numerical neighborhoods: (a),(b) classical schemes; (c),(d) extended schemes



(a): $a_{12} \ge 0$, *(b):* $a_{12} \le 0$ *(c):* $a_{12} \ge 0$ *(d):* $a_{12} \le 0$
$h_1 = h_2, r_1 = r_2 = 1$ $h_1 = h_2, r_1 = 3, r_2 = 1$

2 Convergence of numerical solutions

Differential operator (1) is associated with a bilinear form a on $\dot{W}_2^1(D) \times \dot{W}_2^1(D)$. To avoid unessential technical complexity, we assume here that the differential operator is $A = -\sum_{ij}\partial_i a_{ij}\partial_j$. The standard variational formulation for a weak $\dot{W}_2^1(D)$-solution has the form:

$$a(v,u) = \langle\, v\,|\,\mu\,\rangle, \quad \text{for any } v \in \dot{W}_2^1(D). \tag{3}$$

where μ is a linear functional on $\dot{W}_2^1(D)$. By using discretizations A_n of the differential operator and discretizations of μ, we can approximate (3) by linear systems of the form

$$A_n\mathbf{u}_n = \boldsymbol{\mu}_n. \tag{4}$$

Each grid-solution $\mathbf{u}_n$ on $G_n(D)$ is embedded into the linear space of hat functions on $G_n(D)$ as in [LR3], thus leading to a continuous approximate solution $u(n)$. The sequence $\mathfrak{U} = \{u(n) : n \in \mathbb{N}\}$ converges strongly in $W_2^1(\mathbb{R}^2)$ to the unique solution $u \in W_2^1(\mathbb{R}^2) \cap \dot{W}_2^1(D)$ of (3).

Regularity properties of approximate solutions of the converging sequences $\mathfrak{U}$ follow from the well-known results of DeGiorgi type. We can use criteria developed by [LU] as in our approach [LR3], to get the following expected result: if μ is a bounded function on D, then the sequence $\mathfrak{U}$ has convergent subsequences in the Hölder space $C^{(\alpha)}(\overline{D})$ with certain $\alpha > 0$. This important result, and the fact that the spaces of hat functions with centers at grid-knots of $G_n(D)$ are finite-dimensional, makes it possible to prove the main convergence result [LR3]. If μ is a Radon measure on Borel subsets of D, then the sequence $\mathfrak{U}$ converges strongly in the Banach space $\dot{L}_1^{(\alpha)}(D)$ [St].

(5)

3 Example

We consider the differential operator $A = -\sum_{ij=1}^{2} \partial_i a_{ij} \partial_j$ with the diffusion tensor

$$a = \begin{bmatrix} \sigma^2 & \alpha(\boldsymbol{x}) \\ \alpha(\boldsymbol{x}) & 1 \end{bmatrix}, \quad \alpha(\boldsymbol{x}) = \rho \mathbb{1}_{D_0}(\boldsymbol{x}), \quad \rho^2 < \sigma^2,$$

where σ^2 is a positive number, ρ is a real number and $D_0 = (1/4, 3/4)^2$.

Let $D = (0,1)^2 \subset \mathbb{R}^2$ and ∂D be its boundary. The function $\boldsymbol{x} \mapsto u^*(x_1, x_2) = x_1 x_2$ is the unique solution to the boundary value problem

$$\begin{aligned} \big(Au\big)(\boldsymbol{x}) &= f(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in D, \\ u|\partial D &= u^*|\partial D, \end{aligned}$$

where

$$f(\boldsymbol{x}) = -2\,\rho\,\mathbb{1}_{D_0}(\boldsymbol{x}) - \frac{\rho}{4}\Big[\delta\Big(x_1 - \frac{1}{4}\Big) - 3\delta\Big(x_1 - \frac{3}{4}\Big) + \delta\Big(x_2 - \frac{1}{4}\Big) - 3\delta\Big(x_2 - \frac{3}{4}\Big)\Big].$$

The set $\mathbb{R}^2$ is discretized by the grid G_n of grid-knots $\boldsymbol{x}_{kl} = hk\boldsymbol{e}_1 + hl\boldsymbol{e}_2, k, l \in \mathbb{Z}$, where h is a grid-step. In order to get a discretization of D suitable for numerical handling, we assume $h = 1/N$ where $N = 4M$. In this way we have discretizations $G_n(D)$ of the open square $D = (0,1)^2$ defined by grid-knots $\boldsymbol{x}_{kl} = (x_k, y_l), 1 \leq k, l < N$.

If $|\rho| > 1$ then the first and second standard schemes do not provide us with system matrices having the compartmental structure. By insisting on an equal step size h in both directions, we are forced to use the first or second classical scheme. To demonstrate the efficiency of the proposed schemes, we choose $\rho = 2$ and $r_1 = 1, r_2 = 3$ as illustrated in Figure 1 (c), to ensure the compartmental structure of the system matrix. Contour lines of the solution are shown in Figure 2.

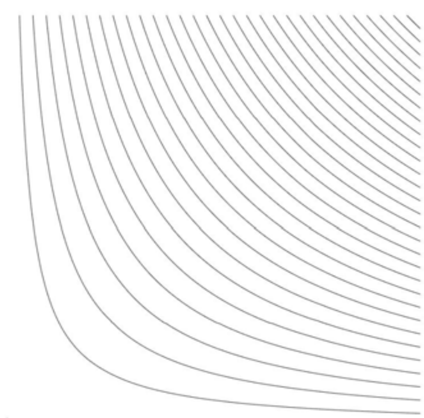


Fig. 2 Contour lines for the example.

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