EXPLICIT STABLE METHODS FOR SECOND ORDER PARABOLIC SYSTEMS*

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Abstract. We show that it is possible to construct stable, explicit finite difference approximations for the classical solution of the initial value problem for the parabolic systems of the form $\partial_t = A(t,\mathbf{x}) + f$ on $\mathbb{R}^d$, where $A(t,\mathbf{x}) = \sum_{ij} \partial_i a_{ij}(t,\mathbf{x})\partial_j + \sum_i \partial_i b_i(t,\mathbf{x}) + c(t,\mathbf{x})$. Smoothness assumptions on the coefficients of the elliptic part are formulated so as to ensure the convergence in the appropriate Hölder norms. The numerical scheme relies on approximation of the diffusion tensor on an equidistant mesh by operators that possess Markov jump process structure. Rotations at the knots are performed in order to construct such approximations; in $\mathbb{R}^2$ scaling of second difference operators is enough. Numerical experiments illustrate the theory.

Key words. Parabolic systems, finite difference schemes, Markov chains

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1. Introduction.

We derive and analyze a class of finite-difference methods for the L_∞- and L_1- theory of 2nd order parabolic system. We consider a discretization scheme for the elliptic part of the operator $A(\mathbf{x})$ on an equidistant grid so that finite-difference approximations A_n posses certain properties. In case of L_∞-theory A_n must be generators of Markov jump processes (MJP), whereas in case of L_1-theory they must have compartmental structure. We approximate the parabolic system by systems of ODE, for which we find explicit and stable finite-difference schemes.

In the course of discretization of $A(\mathbf{x})$ we encounter difficulties caused by too large values of $|a_{ij}|, i \neq j$, compared to the values of $|a_{ii}|, |a_{jj}|$ as pointed out in [KD1]. As suggested in the mentioned reference, rotations at grid-knots might help, but they must have certain properties in order that each approximating MJP has a single class of states. The uniform continuity of coefficients enables the existence of a finite number of rotations for which MJP-s have desired properties. Section 2 contains the results for L_∞-theory, and Section 3 the corresponding results for and for L_1-theory.

In Section 4 we present and discuss a class of explicit and stable methods for ODE, and analyze the convergence and efficiency with respect to the computation time. Monte Carlo methods for a prediction of solutions fit naturally to the considered class of methods. We discuss some aspects of such methods in Section 6.

2. Numerical scheme for elliptic operators possessing MJP-structure.

The linear space of continuous and uniformly bounded functions on $\mathbb{R}^d$ is denoted by $C(\mathbb{R}^d)$. The linear space of functions having partial derivatives up to the order m, and each partial derivative continuous and uniformly bounded, is denoted by $C^m(\mathbb{R}^d)$. By convention $C^0(\mathbb{R}^d) = C(\mathbb{R}^d)$. The linear subspace of $C^m(\mathbb{R}^d)$, for which the m^{th} partial derivatives are uniformly α-Hölder continuous, $\alpha \in (0,1)$, is denoted by $C^{m+\alpha}(\mathbb{R}^d)$. All defined spaces are Banach spaces if supplied with the norm $\|\cdot\|_{k+\alpha}$ (see [LU1]). The Banach space $\dot{C}^{m+\alpha}(\mathbb{R}^d)$ is the closed subspace of $C^{m+\alpha}(\mathbb{R}^d)$, spanned by elements of $C^{m+\alpha}(\mathbb{R}^d)$ with compact supports.

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Let the functions $a_{ij} = a_{ji}$, b_i $(i,j = 1,2,\dots,d)$ and c satisfy the following conditions:

$$\text{(2.1a)} \qquad |b_i|, |c| \quad \leq M, \quad c \leq 0, \quad a_{ij}, b_i, c \quad \text{uniformly continuous on } \mathbb{R}^d,$$

$$\text{(2.1b)} \qquad M\,|z|_2^2 \geq \sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) z_i \bar{z}_j \geq \mu |z|_2^2, \quad \mu > 0, \quad \mathbf{x} \in \mathbb{R}^d,$$

where z_i are complex numbers, and $|z|_2$ the corresponding l_2-norm. The associated elliptic differential operator

$$A(\mathbf{x}) = \sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) \partial_i \partial_j + \sum_{i=1}^{d} b_i(\mathbf{x}) \partial_i + c(\mathbf{x}), \tag{2.2}$$

maps $\dot{C}^{2+\alpha}(\mathbb{R}^d)$ to $\dot{C}(\mathbb{R}^d)$.

The orthogonal coordinate system in $\mathbb{R}^d$ is determined by unit vectors $\mathbf{e}_i$ of the canonical basis. For each $n \in \mathbb{N}$, points $\mathbf{x} = h \sum_{l=1}^{d} k_l \mathbf{e}_l, h = 2^{-n}$, $k_l \in \mathbb{Z}$, define a numerical grid G_n on $\mathbb{R}^d$. Let $E_2 = \dot{C}^{2+\alpha}(\mathbb{R}^d)$, and $E_2(n)$ be the closed subspace of all elements in E_2 vanishing at G_n. If the space $F_2 = E_2/E_2(n)$ is endowed with a quotient norm, it becomes isomorphic to a subspace of l_∞. Analogously, $E_0 = \dot{C}(\mathbb{R}^d), E_0(n)$ and F_0 are defined. Because of the mentioned isomorphism, we identify F_2 with a subspace of F_0. The natural embedding from E_k into $F_k, k = 2,0$, is denoted by $\phi_k(n)$. The image F_k of $\phi(n)$ is sometimes denoted by $F_k(G_n)$ to avoid ambiguity.

We shall say that an element $u \in E_k$ is approximated *with the order* $\beta > 0$ by elements $u_n \in F_k$ if there exists an n-independent positive number κ such that $\| \phi_k(n)u - u_n \|_k < \kappa h^\beta$. In this case, u_n *factor converges* to u with the order β. An operator $A \in \mathbf{L}(E_2, E_0)$ is *approximated by the operators* $A_n \in \mathbf{L}(F_2, F_0)$ with the order of approximation β, if there exists an n-independent positive number κ such that

$$\| \phi_0(n)A - A_n \phi_2(n) \|_{\mathbf{L}(E_2,F_0)} \leq \kappa h^\beta. \tag{2.3}$$

The approximation is called *stable* if there exists an n-independent $\rho > 0$ such that $\| (A_n)^{-1} \|_{\mathbf{L}(F_0,F_2)} \leq \rho$. Because of $F_2 \subset F_0 \subset l_\infty$ the operator A_n is considered as an infinite order matrix with matrix elements $(A_n)_{ij}$. Each row and/or column of this matrix is associated with a pair of grid-knots of G_n, and consequently, each matrix element $(A_n)_{ij}$ is associated with certain grid-knots $\mathbf{x}_i, \mathbf{x}_j \in G_n$.

For $\operatorname{Re}\lambda > 0$, solutions u and u_n of respective equations,

$$\begin{gathered} (\lambda I - A)u = f, \\ (\lambda I_n - A_n)u_n = f_n, n \in \mathbb{N}, \end{gathered} \tag{2.4}$$

will be compared.

For an elliptic operator A satisfying (2.1), an approximation $A_n \in \mathbf{L}(F_2, F_0)$ is said to posses MJP-structure if A_n are generators of (regular or irregular) Markov jump processes. The property is assumed to be valid for sufficiently large natural n, for which the basic results of the following Theorem holds:

THEOREM 2.1. *Let A be defined by (2.1) and (2.2). Then there exist approximations $A_n \in \mathbf{L}(F_2, F_0)$ such that:*

(i) *each A_n possesses MJP-structure and approximates A with the order α;*

(ii) *there exists an n-independent* $\theta_A > 0$ *such that* $(A_n)_{ij} = 0$ *if the associated pair of grid-knots,* $\mathbf{x}_i, \mathbf{x}_j$*, fulfill the condition* $|\mathbf{x}_i - \mathbf{x}_j| > \theta_A h$*;*

(iii) *equations* $(\lambda I - A_n)u_n = f_n$, $\mathrm{Re}\,\lambda > 0$ *have unique solutions in* F_0 *satisfying* $\| u_n \|_\infty < \| f_n \|_\infty / \mathrm{Re}\lambda$*;*

(iv) *if* f_n *factor converges to* f *with the order* α*, then the sequence of solutions* u_n *of (2.4) also factor converges to* u *with the order* α*;*

(v) *if locally at* $\mathbf{x} \in G_n$ p *is a second degree polynomial, then* $(\phi_0(n)A - A_n\phi_2(n))p(\mathbf{x}) = 0$.

Proof. The statement (iii) is a consequence of the MJP-structure of A_n, thus ensuring the stability of operators $\lambda I - A_n$. The statement (iv) follows from (i), (iii) and the following result [KR1]: Let $A \in \mathbf{L}(E_2, E_0)$, and $A_n \in \mathbf{L}(F_2, F_0)$ be approximations of A with the order α. Let $f \in E_0$, and let f_n be approximations of f with the order α. Then, if u and u_n are the unique solutions of (2.4) and $\lambda I - A_n$ are stable, it follows that u_n factor converges to u with the order α.

To prove (i), let the approximation A_n of A have the property described in (v). Then, by using standard methods of numerical analysis for PDE [RM], the expression (2.3) can easily be derived with $\beta = \alpha$. To finish the proof we have therefore to construct a sequence A_n possessing the mentioned property and the MJP-structure. Our construction of the numerical approximations of operator A at a grid-knot $\mathbf{x}$ depends on the value $\omega(\mathbf{e}, \mathbf{x})$:

$$\omega(\mathbf{e}, \mathbf{x}) = \max_{i \neq j} \quad \frac{|a_{ij}(\mathbf{x})|}{\min\{a_{ii}(\mathbf{x}),\, a_{jj}(\mathbf{x})\}}, \tag{2.5}$$

where the symbol $\mathbf{e}$ reminds of the fact that $\omega(\mathbf{e}, \mathbf{x})$ is calculated with respect to the original coordinate system $\{\mathbf{e_i}\}_1^n$. The construction is simple in case where $\rho \in (0, 1)$ exists, such that

$$\omega(\mathbf{e}, \mathbf{x}) \leq 1 - \rho. \tag{2.6}$$

The partial differential operators of the first and second orders are approximated in the usual way:

$$\partial_i f(\mathbf{x}) \to \Box_i f(\mathbf{x}) = \frac{1}{2h}\left[f(\mathbf{x} + \mathbf{e}_i h) - f(\mathbf{x} - \mathbf{e}_i h)\right], \tag{2.7a}$$

$$\partial_i^2 f(\mathbf{x}) \to \Box_i^2 f(\mathbf{x}) = \frac{1}{h^2}\left[f(\mathbf{x} + \mathbf{e}_i h) - 2f(\mathbf{x}) + f(\mathbf{x} - \mathbf{e}_i h)\right]. \tag{2.7b}$$

The mixed partial differential operator can be approximated by one of the following four possibilities:

$$\partial_i \partial_j f(\mathbf{x}) \to \Box_i \Box_j f(\mathbf{x}) = \frac{1}{h^2}\left\{ \begin{array}{l} f(\mathbf{x} \pm \mathbf{e}_i h \pm \mathbf{e}_j h) - f(\mathbf{x} \pm \mathbf{e}_i h) - f(\mathbf{x} \pm \mathbf{e}_j h) + f(\mathbf{x}), \\ -f(\mathbf{x} \pm \mathbf{e}_i h \mp \mathbf{x}_j h) + f(\mathbf{x} \pm \mathbf{e}_i h) + f(\mathbf{x} \mp \mathbf{e}_j h) - f(\mathbf{x}). \end{array} \right.$$

In this way, the quadratic operator $a_{ii}(\partial_i)^2 + a_{jj}(\partial_j)^2$ is approximated by the matrix $a_{ii}(\Box_i)^2 + a_{jj}(\Box_j)^2$ with negative diagonal elements and non-negative off-diagonal elements. If $a_{ij} \geq 0$, then $a_{ij}\partial_i\partial_j$ is approximated by the half sum of the first two possibilities, otherwise by the half sum of the second two possibilities. Because of (2.6), the obtained approximation $\sum_{ij} a_{ij}\Box_i\Box_j$, is the matrix with MJP-structure. The approximation of $\sum b_i\partial_i$ by the difference operator $\sum b_i\Box_i$ can violate the MJP-structure only for larger values of h. Hence, if (2.6) is valid for all $\mathbf{x} \in \mathbb{R}^d$, the approximations A_n have MJP-structure for large values of n, the statement (ii) is obviously fulfilled with $\theta_A = \sqrt{d}$, and (v) is valid because, essentially, this is the way how the difference scheme is constructed.

In case that (2.6) is not valid at a grid-knot $\mathbf{x}$, the previous construction of numerical scheme does not yield an A_n with the MJP-structure. Still, the previous construction can be utilized upon applying rotations to grid-knots in which (2.6) is violated. Auxiliary results are given in Lemma 2.2–Lemma 2.5 from which the statement (i) of Theorem 2.1 obviously follows. $\square$

The original orthogonal coordinate system in $\mathbb{R}^d$ defined by the unit vectors $\mathbf{e}_i$ of the canonical basis shall be henceforth referred to as $\mathbf{e}$-system; two such systems, defined by two sets of unit vectors, shall be assumed to have the same origin at $\mathbf{x} = \mathbf{0}$. We shall translate the origin to the considered grid-knot $\mathbf{x}$ and perform the transformation in translated systems, thus simplifying the expressions calculated at the considered grid-knot. The $\mathbf{e}$-system is used to define grids G_n. To each $\mathbf{x} \in \mathbb{R}^d$ there corresponds an orthogonal transformation $O(\mathbf{x})$ transforming $\mathbf{e}$-system into $\mathbf{g}$-system in which the diffusion tensor at $\mathbf{x}$ has a diagonal form. Diagonal elements have values in the interval $[\mu, M]$. The new axis are defined by elements $s(\mathbf{x}) = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_d\} \in (\partial B)^d$, where ∂B is the unite spherical surface in $\mathbb{R}^d$ centered at $\mathbf{x}$. If $\mathbf{f}_i$, i = 1,2,...,d, are linearly independent, and $|\mathbf{f}_i|_2 = 1$, where $|\cdot|_2$ is the l_2-norm in $\mathbb{R}^d$, then the elements $\mathbf{f}_i$ define a coordinate system called $\mathbf{f}$-system.

The following statement is implied by the fact that the (central projection) image of G_n onto ∂B is a dense set in ∂B.

LEMMA 2.2. *Let the diffusion tensor at* $\mathbf{x}$ *be diagonal in the orthogonal* $\mathbf{g}$-*system, and let there exist at least one axis which does not cross grid-knots of* G_n. *Then for each* $\varepsilon_T > 0$ *there exists a* $m(\varepsilon_T) \in \mathbb{N}$, *and*

(i) *a coordinate system (not necessarily orthogonal) with unit vectors* $\mathbf{f}_i$*;*

(ii) *a cube* C *of the edge* $2m(\varepsilon_T)$, *centered at* $\mathbf{x}$, *in the* $\mathbf{e}$-*system,*

such that the axis of $\mathbf{f}$-*system intersect grid-knots in* C, *and satisfy the following inequalities:*

$$\mathbf{f}_i \cdot \mathbf{g}_i \geq 1 - \varepsilon_T, \quad i = 1, 2, \ldots, d. \tag{2.8}$$

The coefficients $a_{ij}(\cdot), b_i(\cdot)$ are defined in the $\mathbf{e}$-system. In $\mathbf{g}$-system they have different expressions and, consequently, they are denoted by $a_{ij}(\mathbf{g}, \cdot)$ and $b_i(\mathbf{g}, \cdot)$, respectively. Thus, by definition, $a_{ij}(\mathbf{x}) = a_{ij}(\mathbf{e}, \mathbf{x})$. The matrix $a(\mathbf{x}) = \{a_{ij}(\mathbf{x})\}_{11}^{dd}$ is considered as a linear operator on a d-dimensional unitary space of columns. Its l_2-norm is denoted by $|a(\mathbf{x})|_2$. We have $|a(\mathbf{x})|_2 = |a(\mathbf{g}, \mathbf{x})|_2 \leq Md$. Let $\varepsilon_a > 0$ be given and $\delta > 0$ such that $|a(\mathbf{x}) - a(\mathbf{y})|_2 < \varepsilon_a$ if $|\mathbf{x} - \mathbf{y}| < \delta$. Then

$$|\omega(\mathbf{e}, \mathbf{x}) - \omega(\mathbf{e}, \mathbf{y})| < \frac{2M}{\mu^2}\varepsilon_a,$$

holds whenever $|\mathbf{x} - \mathbf{y}| < \delta$. The index a in ε_a means diffusion tensor $a(\mathbf{x})$.

The value $\omega(\mathbf{e}, \mathbf{x})$ refers to the original coordinate system, i.e. the $\mathbf{e}$-system; in g-system it is denoted by $\omega(\mathbf{g}, \mathbf{x})$.

LEMMA 2.3.

(i) *Let an orthogonal system be defined by the unit vectors* $\mathbf{g}_i$, *i = 1,2,...,d; we can then find for each* ε_a *a* $\mathbf{g}$-*independent* $\delta > 0$, *such that the inequality*

$$|\omega(\mathbf{g}, \mathbf{x}) - \omega(\mathbf{g}, \mathbf{y})| \quad < \quad \frac{2M}{\mu^2}\varepsilon_a, \tag{2.9}$$

holds for $|\mathbf{x} - \mathbf{y}| < \delta$.

(ii) *Let* $\varepsilon_T > 0$ *and* $\mathbf{f}_i$ *be unit vectors of another orthogonal coordinate system so that (2.8) is valid. Then*

$$|\omega(\mathbf{g},\mathbf{x}) - \omega(\mathbf{f},\mathbf{x})| \quad \leq \quad \frac{4M^2d^2}{\mu^2}\varepsilon_T. \tag{2.10}$$

(iii) *Let* $\varepsilon_T > 0$ *and* $\mathbf{f}_i$ *be unit vectors of another (not necessarily orthogonal) coordinate system, such that (2.8) is valid. Then, for sufficiently small* ε_T*, the following inequality is valid:*

$$\begin{aligned} |\omega(\mathbf{g},\mathbf{x}) - \omega(\mathbf{f},\mathbf{x})| \quad &\leq \quad \frac{4M_f^2d^2}{\mu_f^2}\varepsilon_T, \quad \textit{where} \\ M_f = M(1 + 3\sqrt{2d\varepsilon_T}, \quad &\mu_f = \mu(1 - 3\sqrt{2d\varepsilon_T}. \end{aligned} \tag{2.11}$$

Proof. (Part (iii).) The transformations matrices from the original system to the $\mathbf{g}$-system and $\mathbf{f}$-system are denoted by O and T respectively; O is orthogonal. Matrix T has elements $T_{kl} = O_{kl} + \tau_{kl}$, where $\tau_{kl} = \mathbf{e}_k \cdot (\mathbf{f}_l - \mathbf{g}_l)$. It follows that $\sum_{kl} \tau_{kl}^2 \leq 2d\varepsilon_T$. An estimate of the matrix τ in l_2-norm is simple, $|\tau|_2 \leq (2d\varepsilon_T)^{1/2}$. The matrix $a(\mathbf{g},\mathbf{x}) = Oa(\mathbf{x})O^{-1}$ satisfies the ellipticity condition with the same constants as in (2.1). For sufficiently small ε_T, the matrix $a(\mathbf{f},\mathbf{x}) = Ta(\mathbf{x})T^{-1}$ satisfies another ellipticity inequality,

$$M_f|z|_2^2 \quad \geq \quad \sum_{i,j=1}^{d} a_{ij}(\mathbf{f},\mathbf{x})z_i\bar{z}_j \quad \geq \quad \mu_f|z|_2^2.$$

The property (2.11) then follows from this inequality in the same way as (2.10) follows from (2.1). ◻

It is convenient to denote $\omega(\mathbf{g}(m),\mathbf{x})$ by $\omega_m(\mathbf{x})$, and use the symbol T_m for the orthogonal system defined by unit vectors $\mathbf{g}_i(m)$.

LEMMA 2.4. *Let* $\rho \in (0, 1/4)$ *be given. There exist* $n \in \mathbb{N}$, $h = 2^{-n}$*, and a finite number of disjoint sets* $D_m, m = 0, 1, \ldots, M_a$, $\mathbb{R}^d = D_m$*, each supplied with an orthogonal coordinate systems* T_m*, such that the function* $\omega_m(\mathbf{x})$ *on* D_m *in the* T_m*-system satisfies the inequality*

$$\sup\left\{|\omega_m(\mathbf{x})| \, : \, \mathbf{x} \in G_n \cap D_m\right\} \;\leq\; 1 - 2\rho, \tag{2.12}$$

for any $m = 0, 1, \ldots, M_a$.

Proof. (A part of proof based on the uniform continuity of coefficients). To each $\mathbf{v} \in G_n$ there corresponds a grid cube $C(h,\mathbf{v}) = \prod_1^d \times[v_j, v_j + h)$, where v_j are coordinates of $\mathbf{v}$ in the original $\mathbf{e}$-system. Cubes $C(h,\mathbf{v})$ define a decomposition of $\mathbb{R}^d$ into disjoint sets. For the chosen ρ there is an $n \in \mathbb{N}$ and a ε_a of Lemma 2.3(ii), such that the difference (2.9) is less than ρ for all $\mathbf{x},\mathbf{y} \in C(h,\mathbf{v})$, and for all $\mathbf{v} \in G_n$. Let us point out that (2.9) is valid for all orthogonal systems, i.e. $|\omega(\mathbf{g},\mathbf{x}) - \omega(\mathbf{g},\mathbf{y})| < \rho$. Let $D_0 \subset \mathbb{R}^d$ be the union of all those cubes $C(h,\mathbf{v})$ for which $|\omega(\mathbf{e},\mathbf{v})| < 1 - 4\rho$. Let $C(h,\mathbf{v})$ be a cube outside of D_0. There are orthogonal unit vectors $\mathbf{g}_i(1)$, i = 1,2,…,d, and the corresponding orthogonal transformation of coordinates, so that the diffusion tensor at $\mathbf{v}$ is diagonal in the $\mathbf{g}(1)$-system. Let $D_1 \subset \mathbb{R}^d$ be the union of all the cubes, for which $|\omega(\mathbf{g},\mathbf{v})| < 1 - 4\rho$ is valid in the $\mathbf{g}(1)$-system. By induction, there is a decomposition $\mathbb{R}^d = \cup_0^\infty D_m$, where to each D_m there corresponds an orthogonal coordinate system, T_m, called $\mathbf{g}$(m)-system, so that $|\omega_m(\mathbf{x})| < 1 - 4\rho$, $\mathbf{v} \in G_n \cap D_m$.

Let $\mathbf{x} \in C(h,\mathbf{v}) \subset D_m$ for some m. Then $|\omega_m(\mathbf{x})| \le |\omega_m(\mathbf{x}) - \omega_m(\mathbf{v})| + |\omega_m(\mathbf{v})| < 1-3\rho$ for the chosen m.

(A part of proof based on the compactness of $(\partial B)^d$). Each T_k is defined by an element $s_k \in (\partial B)^d$, where s_k is the central $\mathbf{v}$-projection of $\mathbf{g}_i(k), i = 1,2,\ldots,d$, onto $(\partial B)^d$. Elements s_k and s_l are defined by the unit vectors $\mathbf{g}_i(k)$ and $\mathbf{g}_i(l)$, i = 1,2,...,d, respectively. Let $\varepsilon_T > 0$ and $U_k \subset (\partial B)^d$ be a neighbourhood of s_k containing all the elements $s = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\} \in (\partial B)^d$, defined by unit vectors $\mathbf{f}_i$ so that (2.8) is valid. Due to the compactness of $(\partial B)^d$, there is a finite sub-cover $\{V_0, V_1, \ldots, V_M\} \subset \{U_j : j \in \mathbb{N}_0\}$ of the set $\{s_j : j \in \mathbb{N}_0\}$. Let r be chosen. The chosen element s_r, associated with D_r, is contained in some of the sets V_m, m = 0,1,...,M, and

$$\mathbf{g}_i(r) \cdot \mathbf{g}_i(m) > 1 - \varepsilon_T.$$

Let ω_r and ω_m be functions defined by (2.5) in the $\mathbf{g}$(r)-system and $\mathbf{g}$(m)-system, respectively. The inequality (2.10) can be used to get:

$$|\omega_m(\mathbf{x})| \ \le\ |\omega_m(\mathbf{x}) - \omega_m(\mathbf{v})| \ +\ |\omega_m(\mathbf{v}) - \omega_r(\mathbf{v})| \ +\ |\omega_r(\mathbf{v})| \ < \\ \rho \ +\ \frac{4M^2d^2}{\mu^2}\varepsilon_T \ +\ 1 - 4\rho.$$

By choosing ε_T sufficiently small we have $|\omega_m(\mathbf{x})| \le 1 - 2\rho$. □

LEMMA 2.5. *There exists a decomposition of* $\mathbb{R}^d$ *into* $1 + M_a$ *disjoint sets* D_m, *and there exist* $1 + M_a$ *associated coordinate systems* T_m *(not necessary orthogonal) with the following properties:*

(i) *the axis of* T_m *intersect grid-knots of* G_n*;*

(ii) *the function* ω_m *on* D_m *in the coordinate system* T_m *satisfies the inequality* $|\omega_m(\mathbf{x})| \le 1 - \rho$.

Proof. Let $\mathbf{x} \in G_n$. By Lemma 2.4 there exists $m \in \{0, 1, \cdots, M_a\}$ such that $|\omega_m(\mathbf{x})| < 1 - 2\rho$. A system T_m has at least one axis which does not intersect grid-knots of $\cup_m G_m$. Therefore, we have to rotate axis $\mathbf{g}_i(m)$, $i \in \{1, 2, \cdots, M_a\}$ slightly, and obtain a new axis $\mathbf{f}_i(m)$, satisfying (2.8). Then

$$|\omega(\mathbf{f}(m),\mathbf{x})| \ \le\ |\omega(\mathbf{f}(m),\mathbf{x}) - \omega(\mathbf{g}(m),\mathbf{x})| \ +\ |\omega(\mathbf{g}(m),\mathbf{x})| \ \le\ \frac{4M_f^2d^2}{\mu_f^2}\varepsilon_T \ +\ 1 - 2\rho.$$

The statement (ii) is attained by choosing ε_T sufficiently small. □

The proof of theorem 2.1 can now be finished:

Proof. For each $\mathbf{x} \in G_n$, the collection of all the grid-knots connected by the scheme defines the numerical neighbourhood $N(\mathbf{x})$, and there are at most $1 + M_a$ different types of neighbourhoods. All the neighbourhoods at the same $\mathbf{x} \in \cup_n G_n$ are alike with respect to the scaling by positive constants 2^m, $m \in \mathbb{Z}$. All the neighbourhoods $N(\mathbf{x})$, for $\mathbf{x} \in D_m \cap G_n, m, n$, fixed, are alike with respect to translations. Each $N(\mathbf{x})$ contains at most $1 + d(1+d)$ grid-knots. This proves (ii) of Theorem 2.1.

To the differential operator $A_0(\mathbf{x}) = \sum_{ij} a_{ij}(\mathbf{x})\partial_i\partial_j$ we apply the coordinate transformation T_m. In the new coordinates we obtain $A_0(\mathbf{x}) = \sum_{ij} a_{ij}(\mathbf{f},\mathbf{x})\hat{\partial}_i\hat{\partial}_j$, the diffusion tensor $a_{ij}(\mathbf{f}(m),\cdot)$ satisfying (2.6) on D_m. For each $\mathbf{x} \in D_m \cap G_n$, the corresponding numerical scheme is:

$$A_n u(\mathbf{x}) = \sum_{\mathbf{z} \in N(\mathbf{x})} a(\mathbf{z}, h) u(\mathbf{z}),$$

where $a(\mathbf{x}, h) < 0$ and $a(\mathbf{z}, h) > 0$ for all $\mathbf{z} \in N(\mathbf{x}) \setminus \{\mathbf{x}\}$. This completes the proof of Theorem 2.1. $\square$

Since the set D_0 of the previous construction may be empty, it is assumed that an orthogonal transformation is performed at the beginning of the construction making D_0 nonempty. The cubes $C(h, \mathbf{v})$ entering the construction of the sets $D_m, m = 0, 1, \ldots, M$, are called basic cubes, i.e. the sets D_m are unions of basic cubes. Basic cubes are denoted by $C(\mathbf{v})$ rather than $C(h, \mathbf{v})$.

An infinite-order matrix $A = \{a_{ij}\}_{\mathcal{I}\mathcal{I}}$, with the index set $\mathcal{I}$, is called *reducible* if there exists a decomposition $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$, into disjoint nonempty subsets $\mathcal{I}_1$, $\mathcal{I}_2$, such that the sub-matrix X which corresponding to indices $i \in \mathcal{I}_1, j \in \mathcal{I}_2$, is zero. An infinite-order matrix A is called *irreducible* if such a decomposition is not possible. The following criterion of irreducibility will be used: Let $\mathcal{I}_1$ be any finite index set, and $A_{\mathcal{I}_1\mathcal{I}_1}$ be the corresponding submatrix of A. If there exists a finite index set $\mathcal{I}_2$, $\mathcal{I}_1 \subset \mathcal{I}_2$, such that the submatrix $A_{\mathcal{I}_2\mathcal{I}_2}$ is irreducible, then the matrix A is irreducible, too. The matrix A_n of Theorem 2.1 can be reducible. As an example, let us consider $A = d^2/dx^2$, $\mathbb{R} = D_0 \cup D_1$, $D_1 = [0, \infty)$. Let the grid steps in D_0 and D_1 be equal to h and $3h$ respectively; it follows that A_n are reducible. Reducibility can cause undesirable features of solutions of ODE which have a reducible matrix A_n as the system matrix; a demonstration of such an undesired effect is illustrated in Example 5.2.

The well known notions of recurrent classes of the theory of Markov chains can be defined and utilized here. A grid-knot $\mathbf{y}$ can be *reached* from a grid-knot $\mathbf{x}$ if there exists a sequence of grid-knots, $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m = \mathbf{y}$, such that $\mathbf{x}_{k+1} \in N(\mathbf{x}_k)$. The sequence $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is called a *path*. The grid-knots $\mathbf{x}$ and $\mathbf{y}$ *communicate* if there exist two paths such that $\mathbf{y}$ can be reached from $\mathbf{x}$ by one path, and $\mathbf{x}$ can be reached from $\mathbf{y}$ by another one. The communication of any two grid-knots of G_n is equivalent to the irreducibility of A_n.

LEMMA 2.6. *Let $C(\mathbf{v}_1)$ and $C(\mathbf{v}_2)$ be any two basic cubes, and $\mathbf{x} \in C(\mathbf{v}_1) \cap G_n$ for some n. Then, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is a grid-knot $\mathbf{y} \in C(\mathbf{v}_2) \cap G_n$ which can be reached from $\mathbf{x}$.*

Proof. Let $C(\mathbf{v}) \subset D_k$ be a basic cube defined by the vertex $\mathbf{v} \in G_n$ and let $C(\mathbf{v}_r)$ be those 2^d surrounding basic cubes facing $C(\mathbf{v})$ by means of sides. Let $\mathbf{x} \in C(\mathbf{v})$. For sufficiently large n there exist grid-knots $\mathbf{x}_r \in C(\mathbf{v}_r) \cap G_n$ which can be reached from $\mathbf{x}$. By induction, all $3^d - 1$ basic cubes which surround $C(\mathbf{v})$ have this property, i.e. they have grid-knots in G_n reachable from $\mathbf{x}$. Since the number of types of numerical neighbourhoods is finite, there exist a sufficiently large n_0 such that the established property holds for any basic cube and $\mathbf{x} \in G_n, n \geq n_0$. $\square$

PROPOSITION 2.7. *Let A_n be the approximations on G_n of the differential operator A defined by (2.1), (2.2), and let A_n have the properties (i) - (v) of Theorem 2.1. For large values of n, the set $G_n = G_n(A)$ is either one class of grid-knots with respect to A_n, or $G_n = G_n(A) \cup H_n$, where*

(i) *(a) $G_n(A)$ are recurrent classes and $\cup_n G_n(A)$ is a dense set in $\mathbb{R}^d$;*

(ii) *(b) the sets H_n contain no recurrent class;*

(iii) *(c) if $H_n \neq \emptyset$, the matrix A_n on G_n has the following block structure:*

$$A_n = \begin{pmatrix} A_n(A) & 0 \\ R & Q \end{pmatrix},$$

where $A_n(A)$ is the irreducible part of A_n.

(iv) *(d) If $H_n \neq \emptyset$, then $A_n(A)$ is the approximation on subgrids $G_n(A)$ of the operator A, and $A_n(A)$ has the properties (i) - (v) of Theorem 2.1.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in G_n$ and $C(\mathbf{v}_0) \subset D_0$ be a basic cube. There are two grid-knots $\mathbf{y}_1, \mathbf{y}_2 \in C(\mathbf{v}_0)$ which can be reached from $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. Because all the internal grid-knots in basic cubes of D_0 communicate among themselves, we conclude that all the grid-knots in D_0 communicate among themselves. Next we define the subgrid $G_n(A) \subset G_n$ containing D_0, and all of the grid-knots in G_n that communicate among themselves. This set is the only recurrent class of G_n. Since it contains grid-knots in each basic cube, there must be an n-independent $\kappa > 0$, such that any ball of the radius κh contains at least two grid-knots of $G_n(A)$. ☐

Generally, two grids $G_n(A)$ for different n cannot be ordered by inclusion, that is, $G_n(A) \subset G_m(A)$, or vice-versa. One would like at least a subsequence of such grids, $G_{n(r)}(A), r \in \mathbb{N}$, to exist, for which the inclusion $G_{n(r)}(A) \subset G_{n(r+1)}(A)$ is valid, and in addition, $\cup_r G_{n(r)}(A)$ to be a dense set in $\mathbb{R}^d$. The mentioned ordering is not always possible, as can be shown if we take $A = d^2/dx^2, G_n = \mathbb{N}, G_n(A) = \{k \in \mathbb{Z}\} : k \leq 0\} \cup \{1 + rk : k \in N\}$, where r is a fixed natural. The mentioned ordering is possible for $r = 2^m - 1$ with some natural m, and impossible for any other choice of r.

3. A weighted polynomial scheme in $\mathbb{R}^2$ possessing MJP-structure. It is possible to construct a numerical scheme possessing MJP-structure without using rotations at the knots. The approximating operator then reproduces piecewise parabolæ i.e. (v) of Theorem 2.1 holds for piecewise quadratics p, which does not influence local convergence properties; we will show how such a scheme can be constructed in the important case of an elliptic operator $\dot{C}^{2+\alpha}(\mathbb{R}^2) \to \dot{C}^{\alpha}(\mathbb{R}^d)$.

Let $x_{2i} = 2ih$ stand for the knots of G_{n-1} in x-direction, and let us denote the knots in the refined grid G_n as $x_i = ih$. Define $w : \mathbb{R} \to \mathbb{R}$ to be a positive, piecewise constant, and right-continuous function having breakpoints at $\{x_i\}$, and denote $w_j = w \mid_{[\, x_j, x_{j+1})} := w_j,\ j = i-2, \dots i+1$. The associated Lebesgue-Stieltjes measure $d\sigma_w$ is defined by $d\sigma_w(\delta) = \displaystyle\int_\delta \frac{dx}{w(x)}$ for all Lebesgue measurable sets $\delta \subset \mathbb{R}$.

For the sake of symmetry, let $w_i = w_{i-1},\ w_{i+1} = w_{i-2}$. For a sufficiently smooth f we obtain an identity, a sort of generalized Taylor expansion:

$$f(x_{i+2}) = f(x_i) + 2h\partial_x f(x_i) + \int_{x_i}^{x_i+2h} d\eta \int_x^\eta D_w^2 f(\xi) d\sigma_w(\xi), \tag{3.1}$$

where

$$D_w^2 f :\, = w \cdot \frac{d^2}{dx^2} f. \tag{3.2}$$

Upon integration of (3.1) we have

$$f(x_{i+2}) = f(x_i) + 2h\partial_x f(x_i) + w_i f''(x_i) \left(\int_{x_i}^{x_{i+1}} d\eta \int_{x_i}^\eta \frac{d\xi}{w_i} + \int_{x_{i+1}}^{x_{i+2}} d\eta \int_{x_{i+1}}^\eta \frac{d\xi}{w_{i+1}} \right).$$

Thus, for the piecewise constant w, we have, up to the terms of the 3rd order:

$$f(x_{i+2}) \approx f(x_i) + 2h\partial_x f(x_i) + + D_w^2 f(x_i) \frac{h^2}{2} \left(\frac{1}{w_i} + \frac{3}{w_{i+1}} \right). \tag{3.3}$$

In the same way, by expanding f to the left of x_i, we obtain:

$$f(x_{i-2}) \approx f(x_i) - 2h\partial_x f(x_i) + + D_w^2 f(x_i) \frac{h^2}{2} \left(\frac{1}{w_{i-1}} + \frac{3}{w_{i-2}} \right). \tag{3.4}$$

The "weighted central difference" approximation of the second derivative follows from (3.3), (3.4), and the symmetry properties of w:

$$\left(\frac{1}{w_i} + \frac{3}{w_{i+1}}\right) D_w^2 f(x_i) \approx \frac{f(x_{i+2}) + f(x_{i-2}) - 2f(x_i)}{h^2}.$$

Definition (3.2) then leads to an approximant of the second derivative in the form of

$$\partial_{xx} f \to \Box^2_{1,w} f := \frac{f(x_{i+2}) + f(x_{i-2}) - 2f(x_i)}{h^2(1 + 3w_i/w_{i+1})}. \tag{3.5}$$

For equal weights, one obtains the common result: $\Box^2_1 f := \frac{f(x_{i+2})+f(x_{i-2})-2f(x_i)}{4h^2}$. In the same manner, we may approximate ∂_{yy} by $\Box^2_{2,r}$, whereas the mixed derivatives ∂_{xy} are approximated by the usual non-weighted differences. We assume that r is a positive piecewise constant function defining a measure with density $1/r$, playing the same role as w in the y-direction, and $r_j = r\,|_{[\,y_j, y_{j+1}]}$. Let

$$\alpha_i := 1 + 3w_i/w_{i+1}, \qquad \beta_j := 1 + 3r_j/r_{j+1}. \tag{3.6}$$

The normalization constants α_i, β_i can be determined in such a way that that the difference scheme posses MJP-structure:

THEOREM 3.1. *Let* $a_{11}\partial_{xx} + 2a_{12}\partial_{xy} + a_{22}\partial_{yy} + b_1\partial_x + b_2\partial_y + c$ *be an elliptic operator* $\dot{C}^{2+\alpha}(\mathbb{R}^2) \to \dot{C}^{\alpha}(\mathbb{R}^d)$. *Then there exists a choice of weights in (3.6) such that the difference operator*

$$a_{11}\Box^2_{1,w} + 2a_{12}\Box_1\Box_2 + a_{22}\Box^2_{2,r} + b_1\Box_1 + b_2\Box_2$$

possess MJP-structure.

Proof. The weights enter the approximations of second derivatives only, the first derivatives being approximated as in (2.7a). We may assume that $a_{11}, a_{22} > 0$, $a_{12} \le 0$ (the case $a_{12} > 0$ may be treated in the same way). Then we may consider a numerical scheme obtained by approximating derivatives by finite differences, which depends on the weights; the appropriate factors are listed in the following table (indexes for α, β are omitted):

knot	factor $\times 1/h^2$
(x_i, y_i)	$-2a_{12} - \frac{2}{\alpha}a_{11} - \frac{2}{\beta}a_{22}$
(x_{i+2}, y_i)	$a_{12} + \frac{a_{11}}{\alpha}$
$(x_{i+2}, y_{i+2}$	0
(x_i, y_{i+2})	$a_{12} + \frac{a_{22}}{\beta}$
(x_{i-2}, y_i)	$a_{12} + \frac{a_{11}}{\alpha}$
(x_{i-2}, y_{i-2})	0
(x_i, y_{i-2})	$a_{12} + \frac{a_{22}}{\beta}$
(x_{i-2}, y_{i-2})	$-a_{12}$

The MJP property requires that

$$\begin{aligned} a_{12} + \frac{1}{\alpha}a_{11} + \frac{1}{\beta}a_{22} &\ge 0, \\ a_{12} + \frac{a_{11}}{\alpha} &\ge 0, \\ a_{12} + \frac{a_{22}}{\beta} &\ge 0. \end{aligned}$$

The first equation is implied by the last two, and this can be achieved if we let

$$0 < \alpha = \beta = \min\{-\frac{a_{11}}{a_{12}}, -\frac{a_{22}}{a_{12}}\}.$$

The approximations of first derivatives do not influence MJP-strucure, since the argument used in the proof of Theorem 2.1 applies. $\square$

It is not difficult to prove that the scheme reproduces constants and linear functions exactly. It does not reproduce quadratic polynomials, but it reproduces piecewise parabolæ instead, that is parabolic splines [R], having jumps in second derivatives at the interior knots (property (v) of Theorem 2.1). This is enough to establish convergence of the 2^{nd} order, since the proof relies on 'weighted' Taylor expansion.

THEOREM 3.2. *Let the difference operator A_n be determined by the scheme as in Theorem 3.1. Then*

$$\| \phi_0(n)A - A_n\phi_2(n) \|_{\mathbf{L}(E_2,F_0)} \leq \kappa h^{\alpha}, \tag{3.7}$$

that is, (2.3) holds, and the order of approximation is $\beta = \alpha$.

Proof. We first establish a Taylor expansion, that is, we prove that for $w > 0$ and u such that wu'' is integrable with respect to $d\sigma_w$, the error term in the expansion

$$u(x+h) = u(x) + hu'(x) + D_w^2 u(x)\int_0^h (h-t)d\sigma_w(t) + R(w;x) \tag{3.8}$$

is of the form

$$R(w;x) = \int_0^h [w(t+x)u''(t+x) - w(x)u''(x)](h-t)\frac{dt}{w(t+x)}. \tag{3.9}$$

In the identity

$$u(x+h) = u(x) + hu'(x) + \int_x^{x+h} d\eta \int_x^{\eta} D_w^2 u(\xi)d\sigma_w(\xi)$$

we can rearrange the order of integration in the last term to obtain

$$\int_x^{x+h} d\eta \int_x^{\eta} D_w^2 u(\xi)d\sigma_w(\xi) = \int_x^{x+h} D_w^2 u(\xi)d\sigma_w(\xi) \int_{\xi}^{x+h} d\eta =$$
$$\int_x^{x+h} (x+h-\xi)D_w^2 u(\xi)d\sigma_w(\xi) = \int_0^h w(t+x)\partial_{tt}u(t+x)\cdot(h-t)\frac{1}{w(t+x)}dt.$$

Let $d\sigma_{w,x} := \dfrac{dt}{w(t+x)}$. By adding and subtracting the "second order term"

$$(wu''(x)\int_x^{x+h}(x+h-\xi)d\sigma_w(\xi),$$

we obtain the Taylor expansion:

$$u(x+h) = u(x) + hu'(x) + D_w^2 u(x)\int_0^h (h-t)d\sigma_{w,x} + R(w;x).$$

The error term may be written as $\int_x^{x+h}(D_w^2 u(\xi) - D_w^2 u(x))(x+h-\xi)d\sigma_w(\xi)$, which implies (3.9) upon substitution $\xi = t + x$. One verifies easily that the second term in (3.8) is in fact the second term in (3.3),(3.4).

Let $d(x) := (\partial_1^2 - \Box_{1,w})u(x)$. If we multiply and divide Taylor error (3.9) by t^α, we a obtain a simple estimate $|d(x)| \leq \mu h^\alpha \|u\|_{2+\alpha}$, where μ is independent of h and α; other differentials can be estimated in more or less the same way. We have immediately that

$$\|a_{11}\, d\|_{F_0} \leq \mathrm{c} h^\alpha \|u\|_{E_2}, \qquad \mathrm{c} \leq 2\mu \|a_{11}\|_{C(\mathbb{R}^d)},$$

whence (2.3) follows. □

4. Elliptic operators in divergence form. The elliptic operator

$$B(\mathbf{x}) = \sum_{i,j=1}^{d} \partial_i a_{ij}(\mathbf{x}) \partial_j, \tag{4.1}$$

with coefficients $a_{ij} \in C^{1+\alpha}(\mathbb{R}^d)$ satisfying the strict ellipticity (2.1 b), can be rewritten in the form

$$B(\mathbf{x}) \;=\; A(\mathbf{x}) \;+\; \sum_i b_i(\mathbf{x}) \partial_i, \tag{4.2}$$

where $A = \sum_{ij}\, a_{ij}\partial_i\partial_j$ and $b_i = \sum_j\, \partial_j a_{ij}$. Hence, the classical theory can be applied to (4.1) in order to describe the properties of solutions. We wish to define approximations B_n on grids G_n with the MJP structure. In case that $a_{ij} = 0$ for $i \neq j$, the desired approximation is described in [BPV]. At each grid-knot $\mathbf{x} \in G_n$, the function $\partial_i a_{ii} \partial_i u$ is approximated by the term

$$\begin{array}{c} -\frac{1}{h^2}\Big[\, a_{ii}(\mathbf{x}+\frac{h}{2}\mathbf{e}_i) \;\;+\;\; a_{ii}(\mathbf{x}-\frac{h}{2}\mathbf{e}_i) \,\Big]\, u(\mathbf{x}) + \\ \frac{1}{h^2}\, a_{ii}(\mathbf{x}+\frac{h}{2}\mathbf{e}_i)\, u(\mathbf{x}+h\mathbf{e_i}) \;\;+\;\; \frac{1}{h^2}\, a_{ii}(\mathbf{x}-\frac{h}{2}\mathbf{e}_i)\, u(\mathbf{x}-h\mathbf{e}_i). \end{array} \tag{4.3}$$

The expression $\sum_i \partial_i a_{ii} \partial_i u$ is approximated by the sum of the appropriate terms (4.3). In case of nontrivial term $\partial_i a_{ij} \partial_j u$ the procedure is analogous to the one described in 2. Let the condition (2.6) be satisfied. Then the term under consideration can be approximated by any one of the following four different expressions:

$$\begin{array}{l} \frac{1}{h^2}\,\Big[\, a_{ij}(\mathbf{x} \pm \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x}) \;+\; a_{ij}(\mathbf{x} \pm \mathbf{e}_i \pm \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x} \pm h\mathbf{e}_i \pm h\mathbf{e}_j) \;- \\ \qquad a_{ij}(\mathbf{x} \pm \mathbf{e}_i \pm \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x} \pm h\mathbf{e}_i) \;-\; a_{ij}(\mathbf{x} \pm \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x} \pm h\mathbf{e}_j)\,\Big], \\ -\frac{1}{h^2}\,\Big[\, a_{ij}(\mathbf{x} \mp \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x}) \;+\; a_{ij}(\mathbf{x} \pm \mathbf{e}_i \mp \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x} \pm h\mathbf{e}_i \mp h\mathbf{e}_j) \;+ \\ \qquad a_{ij}(\mathbf{x} \pm \mathbf{e}_i \mp \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x} \pm h\mathbf{e}_i) \;+\; a_{ij}(\mathbf{x} \mp \frac{h}{2}\mathbf{e}_j)\, u(\mathbf{x} \mp h\mathbf{e}_j)\,\Big]. \end{array} \tag{4.4}$$

If (2.6) is holds, to obtain the MJP structure, we choose convex combinations of those expressions in (4.4) for which the coefficients at $u(\mathbf{x} \pm h\mathbf{e}_i \pm h\mathbf{e}_j)$ are nonnegative; if (2.6) is not valid, we must apply the rotations first.

THEOREM 4.1. *Let B be defined by (4.1), and let its coefficients satisfy (2.1), (2.2) and belong to the class $C^{1+\alpha}(\mathbb{R}^d)$. If B_n are approximations of B as described previously, then statements (i) to (iv) of Theorem 2.1 hold.*

Proof. Because of (4.2), the methods of Section 2 apply with minor adaptations. In particular, to prove the convergence we have to use

$$a_{ij}(\mathbf{y}) \;=\; a_{ij}(\mathbf{x}) \;+\; \frac{h}{2}\, g_{ij}(\mathbf{x}) \;+\; \int_0^{h/2} [g_{ij}(\mathbf{x}+t\mathbf{e}) \;-\; g_{ij}(\mathbf{x}) dt, \tag{4.5}$$

where $\mathbf{e}$ is either $\mathbf{e}_i$, or $\mathbf{e}_j$, $\mathbf{y} = \mathbf{x} + \frac{h}{2}\mathbf{e}$ and g_{ij} is the related differentiation of a_{ij} at $\mathbf{x}$. Insertion of the first terms of (4.5) into (4.3) and (4.4) results in the approximation of $A(\mathbf{x})$ as constructed in Section 2. The second terms of (4.5) give rise to the approximation of $\sum_i b_i \partial_i = B - A$ of Section 2. When the third terms of (4.5) are inserted into (4.3), (4.4), we have an error behaviour like h^α for small values of h. $\square$

If there exists a nontrivial a_{ij}, $i \neq j$, the matrices B_n are not symmetric. It happens that one gets nonsymmetric matrices B_n with the MJP-structure, or symmetric approximations lacking this structure. This fact can be explained from the point of view of Markov jump processes generated by B_n.

Let $l_1(n)$ be defined by the norm $u \to \|u\|_1 = h \sum_{\mathbf{x} \in G_n} |u_{\mathbf{x}}|$, where $u_{\mathbf{x}}$ are components of u. For an element $u \in \dot{C}^\alpha(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ the images $\phi_0(n)u$ in $l_1(n)$ factor converge to u. In order to ensure the L_1-convergence of solutions u_n of (2.2) to u,