Leuven 2000
July 2000
pp. 1–26

Tension spline collocation methods for singularly perturbed Volterra integro-differential and Volterra integral equations *

Vilmoš Horvat †and Mladen Rogina ‡

Abstract. We consider the numerical discretization of singularly perturbed Volterra integro–differential equations (VIDE)

$$\begin{aligned} \varepsilon y'(t) &= q_1(t) - q_2(t)y(t) + \int_0^t K(t,s)y(s)ds, \quad t \in I := [0,T] \\ y(0) &= y_0 \end{aligned} \tag{1}$$

and Volterra integral equations (VIE)

$$\varepsilon y(t) = g(t) - \int_0^t K(t,s)y(s)ds \quad t \in I, \tag{2}$$

by tension spline collocation methods in certain tension spline spaces, where ε is a small parameter satisfying $0 < \varepsilon \ll 1$, and q_1, q_2, g and K are functions sufficiently smooth on their domains to ensure that the equations (1) and (2) posses a unique solution.

We give an analysis of the global convergence properties of a new tension spline collocation solution for $0 < \varepsilon \ll 1$ for singularly perturbed VIDE and VIE, thus extending the existing theory for $\varepsilon = 1$ to the singularly perturbed case.

AMS subject classification: (2000) 65R20, 45J05

Key words: Singularly perturbed Volterra integro-differential equations, Volterra integral equations, tension spline, collocation method

1. Introduction

A growing interest in Volterra integral and integro–differential equations involving small parameters motivates this research. In the area of the numerical solution of the singularly perturbed boundary value problems for ordinary differential equations, one of the very successful approaches is to use either exponentially fitted schemes or,

---

*Supported by grant 037011 of the Ministry of Science and Technology, Croatia.

†Dept. of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia, e–mail: horvat@math.hr

‡Dept. of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia, e–mail: rogina@math.hr

alternatively, tension splines. These functions are piecewisely in the space spanned by $1, x, exp(x), exp(-x)$. It is the latter approach that we consider in this paper, combining the classical collocation method with some new results in the field of tension spline spaces. We show that by employing such techniques, it is possible to achieve convergence orders like those in polynomial collocation methods.

Implicit Runge–Kutta methods were considered for singularly perturbed integro-differential–algebraic equations in [7] and for singularly perturbed integro–differential systems in [8]. There were supposed that the initial value of the reduced equation lies on the smooth outer solution. Therefore, in [7, 8] the convergence of the implcit Runge–Kutta methods had been proved out of the boundary layer.

We do not have such assumptions and by our method one can find the solution in boundary layer to. For small values of the perturbation parameter such results can might not be achieved by polynomial methods, at least not on equidistant grids for the soution in the boundery layer. We do not go into details of the numerical scheme, since it involves construction of local basis in tension spline spaces and other items from the theory of Chebyshev systems, but we give suitable references instead. Many important issues have not been discussed, like fully discretized schemes, non-linear kernels and a lot of other problems. Some are only a matter of technique, while others are not so trivial. There is an open problem of finding Gauss related quadratures of higher order in tension spline spaces, and also we do not have a completely numerical way to determine tension parameters, but depend heavily on asymptotic expansions. Construction of higher order methods is still a matter of research. However, it is our belief that this is a very powerfull and competitive method, based on simple ideas, which can be easily implemented.

In the section on tension splines we describe briefly the finite dimensional spaces used in the collocation method. It is sufficiently self-contained as far as Volterra equations are concerned. Next, in the section on quadrature formulæ we describe what we know of the nodes and weights, which will be important for determination of collocation parameters. The central part is the proof of the collocation method by tension splines in the section on the collocation method; only exact collocation is described, *i.e.* we assume that the integrals involved are known exactly. We end by showing how the method works on some examples, and discuss a choice of tension parameters.

2. Tension splines

We shall concentrate on the collocation method for problems like (1) and (2). Though there are other possible approaches (see [9]), it is our aim to show that, for a suitable choice of the underlying spline spaces it is possible to employ collocation for singularly perturbed problems in the same way as polynomial collocation for regular problems. We begin by characterizing such spaces, which will subsequently be referred to as spaces of tension splines.

For $N \in \mathbf{N}$ let $\Pi_N := \{t_j^{(N)} : 0 = t_0^{(N)} < t_1^{(N)} < \ldots < t_N^{(N)} = T\}$ be a partition of the interval $[0,T]$, $\sigma_0 := [t_0, t_1]$, $\sigma_n := [t_{n-1}, t_n], n = 1, \ldots, N-1$ subintervals of the partition, and $h_n = t_{n+1} - t_n$. The set of the associated interior points of the partition Π_N is denoted as $Z_N = \{t_n : n = 1, \ldots, N-1\}$. For such a partition Π_N and some positive real numbers $p_0, \ldots, p_{N-1}$ we can define differential operators

$$\mathcal{M}_{\Pi,\mathbf{P}} u(t) := \frac{d^2}{dt^2} u(t) - \frac{p_n^2}{h_n^2} u(t) \quad \text{for} \quad t \in \sigma_n, \quad n = 0, 1, \ldots, N-1 \quad \text{and}$$

$$\widetilde{\mathcal{M}}_{\Pi,\mathbf{P}} u(t) := \frac{h_n^2}{p_n^2} \frac{d^2}{dt^2} u(t) - u(t) \quad \text{for} \quad t \in \sigma_n, \quad n = 0, 1, \ldots, N-1.$$

Next, for $k \in \mathbf{N}$, let $\mathcal{M}_{\Pi,\mathbf{P}}^{(k)} u := \mathcal{M}_{\Pi,\mathbf{P}} u^{(k)}$; the definition of the operators $\mathcal{M}_{\Pi,\mathbf{P}}$ also holds for $p_i = 0$, $i = 0, \ldots, N-1$.

Definition 1. *Positive real numbers p_i are called* **tension parameters**, *and* $\mathbf{P} := (p_0, \ldots, p_{N-1})^\tau$ *is the* **tension vector**.

We would like to define the space of tension splines as a possible generalization of the space of polynomial splines. To this end, we consider a generalization of power functions defined by certain integral representations. We start by defining the **weight functions** on each σ_n, for $l \geq 4$, $l \in \mathbf{N}$, by

$$\begin{aligned} w_{1,n}(t) &= 1, \quad \ldots, \quad w_{l-2,n}(t) = 1, \\ w_{l-1,n}(t) &= \frac{\cosh(\frac{p_n}{h_n}(t - t_n))}{\cosh p_n}, \\ w_{l,n}(t) &= \frac{1}{\cosh p_n \cosh^2(\frac{p_n}{h_n}(t - t_n))}, \text{ for } t \in \sigma_n \text{ and } p_n \in \mathbf{R},\ p_n \geq 0. \end{aligned} \tag{3}$$

The above weights can be used to build the integral representation for a special **Extended Complete Chebyshev** (ECC)–system [19] on each σ_n, which we henceforth denote as $U_{l,n}$:

$$\begin{aligned} u_{1,n}(t) &= w_{1,n}(t), \\ u_{2,n}(t) &= w_{1,n}(t) \int_a^t w_{2,n}(s_2) ds_2, \\ &\vdots \\ u_{l,n}(t) &= w_{1,n}(t) \int_a^t w_{2,n}(s_2) \int_a^{s_2} \ldots \int_a^{s_{l-1}} w_{l,n}(s_l) ds_m \ldots ds_2. \end{aligned} \tag{4}$$

Since each $t \in \sigma_n$ can be written as $t = t_n + \xi h_n$ for some $\xi \in [0,1]$, upon integration in (4) it follows, that the elements of $U_{l,n}$ are, for $l = 2k$:

$$u_{1,n}(t_n + \xi h_n) = 1,$$

$$
\begin{aligned}
u_{2,n}(t_n+\xi h_n) &= h_n\xi, \\
&\vdots \\
u_{l-2,n}(t_n+\xi h_n) &= \frac{(h_n\xi)^{l-3}}{(l-3)!}, \\
u_{l-1,n}(t_n+\xi h_n) &= \frac{h_n^{l-2}}{p_n^{l-2}\cosh p_n}\left[\cosh(p_n\xi)-\sum_{i=0}^{k-2}\frac{(p_n\xi)^{2i}}{(2i)!}\right], \\
u_{l,n}(t_n+\xi h_n) &= \frac{h_n^{l-1}}{p_n^{l-1}\cosh^2 p_n}\left[\sinh(p_n\xi)-\sum_{i=0}^{k-2}\frac{(p_n\xi)^{2i+1}}{(2i+1)!}\right],
\end{aligned} \tag{5}
$$

and, for $l = 2k+1$:

$$
\begin{aligned}
u_{1,n}(t_n+\xi h_n) &= 1, \\
u_{2,n}(t_n+\xi h_n) &= h_n\xi, \\
&\vdots \\
u_{l-2,n}(t_n+\xi h_n) &= \frac{(h_n\xi)^{l-3}}{(l-3)!}, \\
u_{l-1,n}(t_n+\xi h_n) &= \frac{h_n^{l-2}}{p_n^{l-2}\cosh p_n}\left[\sinh(p_n\xi)-\sum_{i=0}^{k-2}\frac{(p_n\xi)^{2i+1}}{(2i+1)!}\right], \\
u_{l,n}(t_n+\xi h_n) &= \frac{h_n^{l-1}}{p_n^{l-1}\cosh^2 p_n}\left[\cosh(p_n\xi)-\sum_{i=0}^{k-1}\frac{(p_n\xi)^{2i}}{(2i)!}\right].
\end{aligned} \tag{6}
$$

Definition 2. *The ECC-system of n functions defined by (5) for even, or by (6) for odd n, is called the system of* **tension powers***.*

One easily verifies that on σ_n tension powers $u_{i,n}$, $i = 1, \ldots, l$ defined by (5) or (6) belong to the null-space of the differential operator $\mathcal{M}^{(l-2)}_{\Pi,\mathbf{P}}$. Some similar systems have appeared previously in different settings of computer aided geometric design [10], though normalized in a different way. It is also true that other integral representations can be associated with the same differential operator [17, 11], but we have found this one to be particularly useful in collocation.

Analogously, for $y \in \sigma_n$ there is a $\zeta \in [0,1]$ such that $y = t_n + \zeta h_n$. We can define Green's functions for such systems of tension powers to be zero if $\xi < \zeta$, and, otherwise:

$$
\begin{aligned}
g_1(t_n+\xi h_n; t_n+\zeta h_n) &= 1, \\
g_2(t_n+\xi h_n; t_n+\zeta h_n) &= h_n(\xi-\zeta), \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
g_{l-2}(t_n+\xi h_n;t_n+\zeta h_n) &= \frac{(h_n(\xi-\zeta))^{l-3}}{(l-3)!}, \qquad (7)\\
g_{l-1}(t_n+\xi h_n;t_n+\zeta h_n) &= \frac{h_n^{l-2}}{p_n^{l-2}\cosh p_n}\Bigg[\cosh(p_n h_n\xi)\\
&\quad -\cosh(p_n h_n\zeta)\sum_{i=0}^{k-2}\frac{(p_n h_n(\xi-\zeta))^{2i}}{(2i)!}-\sinh(p_n h_n\zeta)\sum_{i=0}^{k-2}\frac{(p_n h_n(\xi-\zeta))^{2i+1}}{(2i+1)!}\Bigg],\\
g_{l}(t_n+\xi h_n;t_n+\zeta h_n) &= \frac{h_n^{l-1}}{p_n^{l-1}\cosh^2 p_n\cosh(p_n h_n\zeta)}\Bigg[\sinh(p_n h_n(\xi-\zeta))\\
&\quad -\sum_{i=0}^{k-2}\frac{(p_n h_n(\xi-\zeta))^{2i+1}}{(2i+1)!}\Bigg],
\end{aligned}
$$

for $l=2k$, and

$$
\begin{aligned}
g_{1}(t_n+\xi h_n;t_n+\zeta h_n) &= 1,\\
g_{2}(t_n+\xi h_n;t_n+\zeta h_n) &= h_n(\xi-\zeta),\\
&\vdots\\
g_{l-2}(t_n+\xi h_n;t_n+\zeta h_n) &= \frac{(h_n(\xi-\zeta))^{l-3}}{(l-3)!}, \qquad (8)\\
g_{l-1}(t_n+\xi h_n;t_n+\zeta h_n) &= \frac{h_n^{l-2}}{p_n^{l-2}\cosh p_n}\Bigg[\sinh(p_n h_n\xi)\\
&\quad -\sinh(p_n h_n\zeta)\sum_{i=0}^{k-2}\frac{(p_n h_n(\xi-\zeta))^{2i+1}}{(2i+1)!}-\cosh(p_n h_n\zeta)\sum_{i=0}^{k-1}\frac{(p_n h_n(\xi-\zeta))^{2i}}{(2i)!}\Bigg],\\
g_{l}(t_n+\xi h_n;t_n+\zeta h_n) &= \frac{h_n^{l-1}}{p_n^{l-1}\cosh^2 p_n\cosh(p_n h_n\zeta)}\Bigg[\cosh(p_n h_n(\xi-\zeta))\\
&\quad -\sum_{i=0}^{k-1}\frac{(p_n h_n(\xi-\zeta))^{2i}}{(2i)!}\Bigg],
\end{aligned}
$$

for $l=2k+1$. Chebyshev theory then tells us how to construct the generalized derivatives $L_1,\dots,L_m$ on σ_n. The result is, that for sufficiently smooth f

$$
\begin{aligned}
L_i f(t) &:= D^i f(t), \qquad i=0,1,\dots,m-2, \qquad (9)\\
L_{m-1} f(t) &:= \cosh p_n\,\frac{D^{m-1}f(t)-\frac{p_n}{h_n}\sinh(\frac{p_n}{h_n}(t-t_n))D^{m-2}f(t)}{\cosh^2(\frac{p_n}{h_n}(t-t_n))},\\
L_m f(t) &:= \cosh^2 p_n(D^2-\frac{p_n^2}{h_n^2})D^{m-2}f(t).
\end{aligned}
$$

The reader may skip details from the constructive parts of the Chebyshev theory, since for our purposes it is sufficient to state that derivatives (9) act on tension powers in such a way that $L_j u_i = w_i(t_n)\delta_{j,i-1}$ as can also be directly verified. By elementary calculus it is not difficult to show that in terms of the corresponding Green's functions (7) and (8), we obtain the **generalized Taylor expansion** in the standard way (as suggested also by the general theory [19][p.364]):

$$\begin{aligned} f(t_n+\xi h_n) &= L_0 f(t_n) u_1(t_n+\xi h_n) + L_1 f(t_n) u_2(t_n+\xi h_n) + \ldots + \\ & L_{m-1} f(t_n) u_m(t_n+\xi h_n) + \int_{t_n}^{t_n+\xi h_n} g_m(t_n+\xi h_n; y) L_m f(y) dy. \end{aligned} \tag{10}$$

Expansion (10) is an identity for functions f for which L_mf is integrable with respect to the appropriate weight functions (3).

Definition 3. *Let $d \in \mathbf{Z}$ and $m \in \mathbf{N}$ such that $-1 \le d \le m-1$. The space*

$$\begin{aligned} \mathcal{T}_m^{(d)}(Z_N) &:= \{u \mid u(t)|_{t\in\sigma_n} := u_n(t) \text{ and } \mathcal{M}_{\Pi,\mathbf{P}}^{(m-1)} u_n = 0,\ n = 0,\ldots,N-1, \\ & u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \quad \text{for} \quad j = 0,\ldots,d \quad \text{and} \quad t_n \in Z_N\} \end{aligned}$$

is the **space of tension splines of order** $m+1$ *with defect* $\delta := m-d$.

If $d = -1$, the elements of the space $\mathcal{T}_m^{(-1)}(Z_N)$ can have jump discontinuities at $t_n \in Z_N$.

Definition 4. *Let $d \in \mathbf{Z}$ and $m \in \mathbf{N}$ such that $-1 \le d \le m-1$, and let $\mathcal{P}_m$ denote the space of polynomials of order $m+1$, that is, degree at most m. Then the* **space of polynomial splines** *of order $m+1$ with defect $\delta := m-d$ is defined by*

$$S_m^{(d)} := \{u : I \to \mathbf{R}; u|_{\sigma_n} =: u_n \in \mathcal{P}_m, \quad u_{n-1}^{(\nu)}(t_n) = u_n^{(\nu)}(t_n),\ \nu = 0,...,d\}. \tag{11}$$

It follows easily that for zero tension vectors $\mathcal{T}_m^{(d)}(Z_N) \equiv S_m^{(d)}(Z_N)$, whereas it is more difficult to show that for tension parameters tending to infinity in a quite general way $\mathcal{T}_m^{(d)}(Z_N)$ approaches $S_{m-2}^{(d)}$ (see [6] for the case $m = 4$).

3. Collocation method

The general Volterra integro–differential equations can be written as:

$$\varepsilon y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s)) ds, \quad t \in I, \tag{12}$$

$$y(0) = y_0, \tag{13}$$

where ε is a known parameter; we are particularly interested in the case when $0 < \varepsilon \ll 1$. Smoothness assumptions on f and k imply existence of a unique solution of (12) for $\varepsilon > 0$. It is assumed further that there exists $\kappa_1 > 0$ such that $f_y(t, y(t)) \le -\kappa_1 < 0$ for all $t \in I$. For $\varepsilon = 0$, we obtain from (12) the **reduced equation**

$$0 = f(t, y_r(t)) + \int_0^t k(t, s, y_r(s)) ds, \tag{14}$$

which is an implicit Volterra integral equation of the second kind. Henceforth we will consider only the equations (12) and (14) which are not consistent for $t = 0$, *i.e.* $y(0) \neq y_r(0)$. The equation (12) is then called singularly perturbed Volterra integro–differential equation. The inconsistency for $t = 0$ suggests the existence of a boundary layer near $t = 0$, where the solution exhibits exponential behaviour.

Most of the following theory applies as well to the singularly perturbed Volterra integral equations

$$\varepsilon y(t) \quad = \quad g(t) - \int_0^t k(t, s, y(s))ds, \qquad t \in I. \tag{15}$$

For $\varepsilon > 0$ the equation (15) has a unique solution. If $\varepsilon = 0$, we obtain from (15) the reduced equation

$$0 = g(t) - \int_0^t k(t, s, y_r(s))ds. \tag{16}$$

Equation (16) is a Volterra integral equation of the first kind, and will possess a unique solution $y_r \in C([0, T])$ if:

$$g(0) = 0 \quad \text{and } \kappa > 0 \text{ exists such that} \quad k_y(t, t, y_r) \geq \kappa \quad \forall t \in [0, T]. \tag{17}$$

The collocation method to be described is convergent both for regular and singularly perturbed Volterra integro-differential and integral equations. The techniques are similar, but since for integro-differential equations it is easier to find a good choice of tension parameters, we will skip integral equations in the proofs, and concentrate on equations (12), (14).

A general methodology for deriving asymptotic expansion of the solution y of (12) and (15) is developed in [1, 2] in terms of ε. The existence of such asymptotic expansion of the solution in terms of ε was proved in [18]. Cited references show us that the solution of (2) can be written as

$$y(t) = \sum_{n=0}^{\infty} \varepsilon^n [v_n(t) + z_n(\tau)] \quad t \in I, \quad \tau = \frac{t}{\varepsilon}, \tag{18}$$

with the necessary assumption

$$\lim_{\tau \to \infty} z_j(\tau) = 0, \quad j = 0, 1, 2, \ldots \tag{19}$$

Since the non-linearity of the kernel does not play a significant role in our considerations, and can be dealt with by supposing some Lipshitz constraints on the kernel, we will only attempt to solve the linear case of (12) given by (1) where $q_2(t) > 0$ for all $t \in I$.

For simplicity assume that Π_N is the uniform partition of I, and that h is such that $Nh = T$. The convergence of the method can also be proved under milder assumptions, like quasi-uniformity of the partition. As $\varepsilon \to 0$ the solution of the equation (1) does not converge uniformly to the solution of the reduced equation only in the region $O(\varepsilon)$ wide, and therefore it makes sense to approximate the solution by tension splines with tension parameter $p > 0$ at σ_0, while at $[t_1, T]$ we can use tension splines with zero tension parameters, *i.e.* polynomial splines. This in fact does depend on an interplay between h, p and ε, but is not a severe restriction, since one can use tension splines in some neighbouring intervals as well, if ε is larger. However, we are interested mainly in getting ε as small as possible, without making h smaller than the machine accuracy, and the above hypothesis will significantly simplify the notation used in proof of Theorem 5.

Let us call $\{c_j(p_n)\}$ the set of **collocation parameters** if $0 \leq c_1(p_n) < \ldots < c_m(p_n) \leq 1$ for all $n = 0, \ldots, N-1$, and let $X_N := \{t_{n,j} := t_n + c_j(p_n)h, j = 1, \ldots, m; n = 0, \ldots, N-1\}$ be the set of **collocation points**. We want to find an approximation $u \in \mathcal{T}_m^{(0)}(Z_N)$ to the solution of (1) in the form

$$u_n(t) = \alpha_{n,0} + \sum_{l=1}^{m} \alpha_{n,l} \widetilde{u}_{n,l}(t), \quad t \in \sigma_n, \quad n = 0, \ldots, N-1, \tag{20}$$

where

$$\widetilde{u}_{n,l}(t) = \int_0^t u_{n,l}(s) ds, \qquad l = 1, \ldots, m, \tag{21}$$

and the functions $\{u_{n,l}\}_{l=1}^m$, $n = 0, \ldots, N-1$ are powers in tension (5). The collocation solution $u(t)|_{\sigma_n} := u_n(t) \in \mathcal{T}_m^{(0)}(Z_N)$ is the exact solution of integro–differential equation on the set of collocation points X_N:

$$\varepsilon u'(t_{n,j}) = q_{1;n,j} - q_{2;n,j} u(t_{n,j}) + \int_0^{t_{n,j}} K_{n,j}(s) u(s) ds, \quad j = 1, \ldots, m, \tag{22}$$

where we use the notation $q_{1;n,j} := q_1(t_{n,j})$, $q_{2;n,j} := q_2(t_{n,j})$ and $K_{n,j}(\cdot) := K(t_{n,j}, \cdot)$. The equation (22) can then be written in the form

$$\begin{aligned} \varepsilon u'(t_{n,j}) &= q_{1;n,j} - q_{2;n,j} u(t_{n,j}) + F_n(t_{n,j}) \\ &+ h \int_0^{c_j} K_{n,j}(t_n + h\nu) u(t_n + h\nu) d\nu, \end{aligned} \tag{23}$$

where

$$F_n(t_{n,j}) := h \sum_{i=0}^{n-1} \int_0^1 K_{n,j}(t_i + h\nu) u(t_i + h\nu) d\nu, \quad j = 1, \ldots, m, \tag{24}$$

$n = 0, \ldots, N-1$. In general, the collocation parameters $\{c_j(p_n)\}$ depend on the selection of tension parameters on each interval σ_n, and are chosen in such a way that the quadrature formulæ have order of convergence not less than m, and integrate the functions from the space $\mathcal{T}_m^{(0)}(Z_N)$ exactly. This leads to a problem that is interesting in itself, namely, to find formulæ of Gauss and Gauss-Lobatto type for tension spline spaces. Since $U_{l,n}$ are Chebyshev spaces and have an explicit integral representation (4), unique such formulæ exist [16]. This is not enough, since we need to recalculate collocation parameters when ε is changed and no tabulation is possible. Therefore, we need an efficient algorithm for calculating Gaussian points, that is, collocation parameters. Numerical experiments show that tension spline collocation methods are extremely sensitive to the choice of such parameters. We will not pursue this problem further, but only mention that for Gaussian formulæ of order up to eight we can find the four collocation parameters numerically achieving machine accuracy for all tension parameters of interest, by exploiting semianalitic techniques, translation invariance [10], and known behaviour of tension powers for $p \to 0$ and $p \to \infty$ [6]. Some special quadratures and integration error estimates are considered in the following section.

4. Quadrature formulæ

We would like to define quadrature formulæ which will be exact on spaces spanned by tension powers (5). To this end, we consider quadrature formulæ for functions $f \in C[a,b]$ of the form:

$$I(f) = \sum_{i=1}^{r} \omega_i f(t_i), \tag{25}$$

where the nodes $a \le t_1(p) < \ldots < t_r(p) \le b$, and the weights $\omega_1(p), \ldots, \omega_r(p)$ depend on the tension parameters. The error function of the quadrature formula is defined by

$$R(f) = \int_a^b f(s)ds - I(f). \tag{26}$$

Given the existence of Gauss quadrature formulæ for Chebyshev spaces in $C[a,b]$, we skip a methodology for calculating quadrature weights and nodes, keeping in mind that for higher orders of ECC–systems we haven't yet resolved this issue completely. The quadrature formula can be determined by solving the non-linear system:

$$\sum_{i=1}^{k} \omega_i(p) u_j(t_i(p)) = \int_a^b u_j(t)dt, \quad j = 1, \ldots, 2k. \tag{27}$$

Specially for the ECC–system U_{2k}, we have the system of non-linear equations

$$\begin{aligned}
\sum_{i=1}^{k} \omega_i(p) t_i^j &= \begin{cases} \frac{2}{j+1}, & \text{for } j = 0, 2, \ldots, 2k-4, \\ 0, & \text{for } j = 1, 3, \ldots, 2k-3, \end{cases} \\
\sum_{i=1}^{k} \omega_i(p) \sinh(pt_i) &= 0, \\
\sum_{i=1}^{k} \omega_i(p) \cosh(pt_i) &= \frac{2}{p} \sinh p,
\end{aligned} \tag{28}$$

possessing a unique solution [16, Theorem 5.9]. We want to bound the error of the quadrature formula, and to do this, for all $f \in C^{2k}[a,b]$ we define the Hermite interpolation problem at the nodes of the Gaussian quadrature:

$$u^{(i)}(t_j) = f^{(i)}(t_j), \quad i = 0,1, \quad j = 1,\ldots,k. \tag{29}$$

There exists a unique solution $u \in \mathcal{U}_{2k}$ in the ECC–space $\mathcal{U}_{2k}$ $(k \geq 2)$ spanned by the ECC–system U_{2k} (5) to the interpolation problem (29) (this is in fact an alternative definition of ECC–spaces, see [19, 16]). Functions in $\mathcal{U}_{2k}$ belong to the null-space of the operator

$$\mathcal{L}_m := (D^2 - p^2)D^{m-2}, \tag{30}$$

whence for $p \ll \infty$ we employ the Hermite interpolation bounds for L–splines ([19, teorem 10.24])

$$\| f - u \|_\infty \leq Ch^{2k} \| \mathcal{L}_{2k} f \|_\infty . \tag{31}$$

The same bound holds as tension parameter $p \to \infty$, [12]. The error of the Gauss quadrature formula can be bounded by using the interpolation bound (31), and thus we have

Theorem 1. *For $f \in C^{2k}[a,b]$ the error of the Gaussian quadrature (28), that integrates functions in $\mathcal{U}_{2k}$ exactly, can be estimated by:*

$$R_k(f) \leq h^{2k+1} C_{2k,0} \| \mathcal{L}_{2k} f \|_\infty . \tag{32}$$

Proof.

Since Gauss formula is exact for all $u \in \mathcal{U}_{2k}$, we have by (31)

$$\begin{aligned} R_k(f) &= \left| \int_a^b f(s)ds - I(u) \right| \leq \int_a^b | f(s) - u(s) | \, ds \\ &\leq h^{2k+1} C_{2k,0} \| \mathcal{L}_{2k} f \|_\infty . \quad \blacksquare \end{aligned} \tag{33}$$

For $f \in C^{2k-2}[a,b]$ let us consider the Hermite interpolation problem in the nodes of the quadrature formula (25) of order k, *i.e.* the formula which is exact for all $f \in \mathcal{U}_{2k-2}$ $(k \geq 3)$):

$$\begin{aligned} u(t_1) &= f(t_1), \quad u^{(i)}(t_j) = f^{(i)}(t_j), \quad i = 0,1, \quad j = 2,\ldots,k-1, \\ u(t_k) &= f(t_k), \quad \text{where} \quad t_1 = a \quad \text{i} \quad t_k = b. \end{aligned} \tag{34}$$

The interpolation problem (34) has a unique solution $u \in \mathcal{U}_{2k-2}$ by the same type of argument. The elements spanned by the ECC–space $\mathcal{U}_{2k-2}$ belong to the null–space of the operator $\mathcal{L}_{2k-2} := (D^2 - p^2)D^{2k-4}$. Hence for $p \ll \infty$ we use Hermite interpolation bounds for L–splines ([19, teorem 10.24]), or else, for $p \to \infty$ the bound from [12], thus obtaining

$$\| f - u \|_\infty \leq Ch^{2k-2} \| \mathcal{L}_{2k-2} f \|_\infty . \tag{35}$$

The error of the Gauss–Lobatto quadrature formula can be bounded by using the interpolation bound (35) which results in the following theorem that can be proved in the same way.

Theorem 2. *For $f \in C^{2k-2}[a,b]$ the error of the generalized Gauss–Lobatto quadrature formula, that integrates functions from $\mathcal{U}_{2k-2}$ exactly, can be estimated by*

$$R_k(f) \leq h^{2k-1} C_{2k-2,0} \, \| \, \mathcal{L}_{2k-2} f \|_{\infty} \, . \tag{36}$$

Properties of the Gauss–Lobatto quadratures depending on p follow easily from the asymptotics for the elements of the ECC–system U_{2k-2}. For $p \to 0$ we obtain standard Gaussian formulæ in a straightforward way, whereas for $p \to \infty$ we have the following theorems:

Theorem 3. *Let $\bar{t}_i$, $i = 2, \ldots, k-1$, $(k \geq 2)$, be zeroes of $P'_{k-1}(t)$, where P_{k-1} is $k-1^{\text{st}}$ Legendre polynomial. Then the Gaussian quadrature (25), that is exact on $\mathcal{U}_{2k}$, has the following asymptotic properties:*

$$\begin{aligned}
\lim_{p\to\infty} t_1(p) &= a, \\
\lim_{p\to\infty} t_i(p) &= \frac{b-a}{2}\bar{t}_i + \frac{b+a}{2}, \quad i = 2, \ldots, k-1, \\
\lim_{p\to\infty} t_k(p) &= b, \\
\lim_{p\to\infty} \omega_i(p) &= \frac{b-a}{k(k-1)}, \quad i \in \{1, k\}, \\
\lim_{p\to\infty} \omega_i(p) &= \frac{b-a}{k(k-1)} [P_{k-1}(\bar{t}_i)]^2, \quad i = 2, \ldots, k-1.
\end{aligned} \tag{37}$$

Proof.

As in [6], we know that when $p \to \infty$ dimension of the space spanned by tension powers reduces by two, and that $\mathcal{U}_{2k} \equiv \mathcal{P}_{2k-2}$. The non-linear system (27) for $\mathcal{U}_{2k}$ reduces thus to the first $2k-2$ linearly independent equations, which is equivalent to the non-linear system obtained in deducing the polynomial Gauss–Lobatto quadratures of order k defined on $[a,b]$. The statement (37) then follows since Gauss formula of this type is unique. ■

By modifying the proof slightly, we also can also prove the following

Theorem 4. *Let $\bar{t}_i$, $i = 2, \ldots, k-1$, $(k \geq 3)$ denote the zeros of the Legendre polynomial $P_{k-2}(t)$. Then the weights and nodes of the quadrature formula (25), exact on the space $\mathcal{U}_{2k-2}$, have the following asymptotic properties:*

$$\begin{aligned}
\lim_{p\to\infty} t_1(p) &= a, \\
\lim_{p\to\infty} t_i(p) &= \frac{b-a}{2}\bar{t}_i + \frac{b+a}{2}, \quad i = 2, \ldots, k-1, \\
\lim_{p\to\infty} t_k(p) &= b, \\
\lim_{p\to\infty} \omega_i(p) &= 0 \quad i \in \{1, k\}, \\
\lim_{p\to\infty} \omega_i(p) &= \frac{b-a}{(1-\bar{t}_i)^2} [P'_{k-2}(\bar{t}_i)]^2, \quad i = 2, \ldots, k-1.
\end{aligned} \tag{38}$$

5. Convergence of the exact collocation method

The following theorem establishes convergence rates for the tension spline collocation method. The proof follows mainly the ideas by Brunner [5], combined with tension spline techniques used to solve boundary value problems for singularly perturbed ordinary differential equations [14, 15], but it contains a lot of new elements as well.

Theorem 5. *Let be q_1, q_2 in (1) be of class $C^m(I)$, $q_2(t) > 0$, for all $t \in I$ and let $K \in C^m(S)$. Let $\tau := t/\varepsilon$, $\widetilde{\mathcal{M}}_{\Pi,\tau} z(\tau) := \frac{h^2}{p_0^2}\frac{d^2}{d\tau^2}z(\tau) - z(\tau)$, where z denotes the inner solution of (1) and suppose that the tension vector satisfies $\|\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)} z\| \leq \varepsilon e^{-p_0}$, $p_i = 0$, for $i = 1, \ldots, N-1$. Then there exists $h > 0$ for which the error $e := y - u$ of the collocation solution $u \in \mathcal{T}_m^{(0)}(Z_N)$, $m \geq 4$, of the equation (1) on $[0, T]$ satisfies:*

$$\|e\|_\infty \leq C_0 h^{m-1} \quad \text{and} \quad \|e'\|_\infty \leq C_1 h^{m-2}, \tag{39}$$

where C_0 and C_1 are positive constants independent of h. The error estimate (39) holds for every choice of collocation parameters $\{c_j(p_n)\}_{j=1}^m$ chosen is such a way that the quadrature formulæ, possessing collocation parameters as nodes, are of order not less then m for all $u \in \mathcal{T}_m^{(0)}(Z_N)$.

Proof.

The smoothness hypotheses for g and K imply that for every $\varepsilon > 0$ the equation (1) has a unique solution $y \in C^{m+1}(I)$. The solution of the equation (1) is of the form

$$y(t) = v(t) + z(\tau), \tag{40}$$

where $z(\tau)$ is exponentially decaying. We will consider only even $m \geq 4$. Let us first prove the inequality (39) on the segment σ_0. Upon substitution, (40) takes the form

$$y(\xi h) = v(\xi h) + z(\frac{\xi h}{\varepsilon}), \qquad \xi h \in \sigma_0, \qquad \xi \in [0,1]. \tag{41}$$

By the smoothness argument, it follows that y can be expanded via generalized Taylor polynomial (10) in the neighbourhood of $t_0 = 0$:

$$\begin{aligned}
v(\xi h) &= v(0) + \ldots + \frac{v^{(m-2)}(0)}{(m-2)!}(\xi h)^{m-2} + R_v(\xi h),\\
z(\frac{\xi h}{\varepsilon}) &= z(0) + \frac{1}{\varepsilon} z'(0)\xi h + \ldots + \frac{z^{(m-2)}(0)}{\varepsilon^{m-2}(m-2)!}(\xi h)^{m-2}\\
&+ \frac{1}{\varepsilon^{m-1}} z^{(m-1)}(0) \frac{h^{m-1}}{p_0^{m-1} ch p_0} \left(\sinh(p_0\xi) - \sum_{i=0}^{\frac{m}{2}-2} \frac{(p_0\xi)^{2i+1}}{(2i+1)!} \right)\\
&+ \frac{1}{\varepsilon^m} z^{(m)}(0) \frac{h^m}{p_0^m \cosh^2 p_0} \left(\cosh(p_0\xi) - \sum_{i=0}^{\frac{m}{2}-1} \frac{(p_0\xi)^{2i}}{(2i)!} \right) + R_z(\frac{\xi h}{\varepsilon}).
\end{aligned} \tag{42}$$

The remainder terms are

$$\begin{aligned}
R_v(\xi h) &= h^{m-1}\int_0^{\xi}(\xi-\nu)^{m-2}\frac{v^{(m-1)}(\nu h)}{(m-2)!}d\nu,\\
R_z(\frac{\xi h}{\varepsilon}) &= \frac{h^{m-1}}{p_0^{m-2}}\int_0^{\xi}\left(\cosh(p_0(\xi-\nu))-\sum_{i=0}^{\frac{m}{2}-1}\frac{(p_0(\xi-\nu))^{2i}}{(2i)!}\right)\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)}z(\frac{\nu h}{\varepsilon})d\nu,
\end{aligned} \tag{43}$$

where

$$\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)}z = \widetilde{\mathcal{M}}_{\Pi,\tau}(\tfrac{d^{m-1}}{d\tau^{m-1}}z). \tag{44}$$

The derivative of the solution y'

$$y'(\xi h)=v'(\xi h)+\frac{1}{\varepsilon}z'(\frac{\xi h}{\varepsilon}),\qquad \xi h\in\sigma_0,\qquad \xi\in[0,1] \tag{45}$$

is analogously expanded in the generalized Taylor series, and we get

$$\begin{aligned}
v'(\xi h) &= v'(0)+\ldots+\frac{v^{(m-2)}(0)}{(m-3)!}(\xi h)^{m-3}+R_v'(\xi h),\\
z'(\frac{\xi h}{\varepsilon}) &= z'(0)+\frac{1}{\varepsilon}z''(0)\xi h+\ldots+\frac{z^{(m-2)}(0)}{\varepsilon^{m-3}(m-3)!}(\xi h)^{m-3}\\
&+ \frac{1}{\varepsilon^{m-2}}z^{(m-1)}(0)\frac{h^{m-2}}{p_0^{m-2}\cosh p_0}\left(\cosh(p_0\xi)-\sum_{i=0}^{\frac{m}{2}-2}\frac{(p_0\xi)^{2i}}{(2i)!}\right)\\
&+ \frac{1}{\varepsilon^{m-1}}z^{(m)}(0)\frac{h^{m-1}}{p_0^{m-1}\cosh^2 p_0}\left(\sinh(p_0\xi)-\sum_{i=0}^{\frac{m}{2}-2}\frac{(p_0\xi)^{2i+1}}{(2i+1)!}\right)+R_z'(\frac{\xi h}{\varepsilon}),
\end{aligned} \tag{46}$$

where the remainder terms are:

$$\begin{aligned}
R_v'(\xi h) &= h^{m-2}\int_0^{\xi}(\xi-\nu)^{m-3}\frac{v^{(m-1)}(\nu h)}{(m-3)!}d\nu,\\
R_z'(\frac{\xi h}{\varepsilon}) &= \frac{h^{m-2}}{p_0^{m-3}}\int_0^{\xi}\left(\sinh(p_0(\xi-\nu))-\sum_{i=0}^{\frac{m}{2}-2}\frac{(p_0(\xi-\nu))^{2i+1}}{(2i+1)!}\right)\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)}z(\frac{\nu h}{\varepsilon})d\nu.
\end{aligned} \tag{47}$$

Assumption $\|\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)}z\|\le\varepsilon e^{-p_0}$ implies that $|\cosh p_0-\sum_{i=0}^{m/2-1}\frac{p_0^{2i}}{(2i)!}|\cdot\|\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)}z\|\le\varepsilon$ and $|\sinh p_0-\sum_{i=0}^{m/2-2}\frac{p_0^{2i+1}}{(2i+1)!}|\cdot\|\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-1)}z\|\le\varepsilon$. This bounds, when used in (43) and (47), imply that

$$|\,R_z(\frac{\xi h}{\varepsilon})\,| \;\le\; \frac{h^{m-1}}{p_0^{m-2}}\xi\le C_{01}h^{m-1}\qquad\text{and}$$

$$| R_z'(\frac{\xi h}{\varepsilon}) | \leq \frac{h^{m-2}}{p_0^{m-3}}\xi \leq C_{11}h^{m-2}.$$

For functions v and v' smooth enough, we can estimate the remainders of Taylor polynomials:

$$\begin{aligned} | R_v | &\leq C_{00}h^{m-1}, \\ | R_v' | &\leq C_{10}h^{m-2}. \end{aligned}$$

Let

$$R(\xi h) := R_v(\xi h) + R_z(\frac{\xi h}{\varepsilon}) \quad \text{and} \quad R'(\xi h) := R_v'(\xi h) + R_z'(\frac{\xi h}{\varepsilon}),$$

and let us denote the error and its derivative by

$$\begin{aligned} e(\xi h) &= y(\xi h) - u(\xi h), \\ e'(\xi h) &= y'(\xi h) - u'(\xi h). \end{aligned}$$

Taylor expansions of y and y' imply that:

$$e(\xi h) = h^q[\beta_{0,0} + \sum_{l=1}^{m} \beta_{0l}\underline{\tilde{u}}_{0,l}(\xi) + h^{m-1-q}R(\xi)], \tag{48}$$

$$e'(\xi h) = h^{q-1}[\sum_{l=1}^{m} \beta_{0,l}\underline{u}_{0,l}(\xi) + h^{m-1-q}R'(\xi)], \tag{49}$$

where

$$\begin{aligned} h^q\beta_{0,0} &= y(0) - u(0) = 0, \\ h^q\beta_{0,l} &= v^{(l)}(0) - \alpha_{0,l} + z_\tau^{(l)}(0), \quad l = 1, \ldots, m-2, \\ h^q\beta_{0,m-1} &= -\alpha_{0,m-1} + z_\tau^{(m-1)}(0), \\ h^q\beta_{0,m} &= -\frac{\alpha_{0,m}}{\cosh p_0} + z_\tau^{(m)}(0), \\ h^l\underline{\tilde{u}}_l(\xi) &= \tilde{u}_l(\xi h), \quad l = 1, \ldots, m, \\ h^{l-1}\underline{u}_l(\xi) &= u_l(\xi h), \quad l = 1, \ldots, m, \\ h^{m-1}R(\xi) &= R_v(\xi h) + R_z(\frac{\xi h}{\varepsilon}), \\ h^{m-2}R'(\xi) &= R_v'(\xi h) + R_z'(\frac{\xi h}{\varepsilon}). \end{aligned}$$

If we subtract the collocation equation (22) from the integro–differential equation (1), then, for $q = m-2$, from (48) and (49) we get

$$\sum_{l=1}^{m} \left[\varepsilon\underline{u}_{0,l}(c_j) + hq_{2;0,j}\underline{\tilde{u}}_{0,l}(c_j) - h^2\int_0^{c_j} K_{0,j}(h\nu)\underline{\tilde{u}}_{0,l}(\nu)\,d\nu\right]\beta_{0,l} = hr_{0,j}, \tag{50}$$

$$r_{0,j} = -\varepsilon R'(c_j) - hq_{2;0,j}R(c_j) + h^2\int_0^{c_j} K_{0,j}(h\nu)R(\nu)\,d\nu, \quad j = 1, \ldots, m.$$

In order to make the analysis of the system of equations (50) more transparent, we introduce the following matrices and vectors:

$$\begin{aligned} V &:= (\underline{u}_{0,l}(c_j)), \quad V_1 := (q_{2;0,j}\tilde{\underline{u}}_{0,l}(c_j)) \\ V_2 &:= (\int_0^{c_j} K_{0,j}\tilde{\underline{u}}_{0,l}(\nu)d\nu), \quad l,j=1,\dots,m, \\ \beta &:= (\beta_{0,1},\dots,\beta_{0,m})^\tau, \quad r := (r(c_1),\dots,r(c_m))^\tau. \end{aligned} \tag{51}$$

Equations (50) can be put to the matrix form:

$$(\varepsilon V + hV_1 - h^2 V_2)\beta = hr. \tag{52}$$

Matrices V, V_1 i V_2 are regular, because the columns of the matrices are the basis functions of an ECC–system. By the assumption $\varepsilon < h$, and taking into account the regularity of V_1, we conclude by the standard Neumann argument that there exists $h > 0$ for which the matrix $h(V_1 + \frac{\varepsilon}{h}V - hV_2)$ is regular. Hence, we have

$$||\beta||_1 \le hC||r||_1. \tag{53}$$

The equations (48) and (49), together with (53), imply that the orders of the error and its derivative are

$$||e||_\infty \le C_0 h^{m-1}, \qquad ||e'||_\infty \le C_1 h^{m-2}.$$

The tension parameters on $[t_1, T]$ are zero, and therefore the tension splines are polynomial splines belonging to $S_m^{(0)}(Z_N)$. But then it follows that the order of convergence is m (see [5]), since the solution is dominated by the smooth outer solution, or else we can reaplly the above arguments on the following intervals with different tension parameters, until effect of the boundary layer is unsignificant. This discussion is somewhat technical in nature and will be omitted here. The order of the method will be the minimum of the order on the first few intervals, and the order of the polynomial spline collocation method, as follows easily by the well known techniques. In the same way we can prove the theorem if $m > 4$ is odd. ■

If we consider linear singularly perturbed Volterra integral equations (2), then we can define its collocation solution $u \in \mathcal{T}_{m-1}^{(-1)}(Z_N)$ as a solution of the equations

$$\varepsilon u(t_{n,j}) = g(t_{n,j}) - \int_0^{t_{n,j}} K_{n,j}(s)u(s)ds, \quad j=1,\dots,m. \tag{54}$$

We will not go into details, but it is not difficult to modify the proof of Theorem 5. slightly, and prove

Theorem 6. *Let $g \in C^m(I)$, $K \in C^m(S)$. Let $\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-2)}$ be defined as in Theorem 5., and let the vector of tension parameters $\mathbf{P}$ be defined in such a way that $||\widetilde{\mathcal{M}}_{\Pi,\tau}^{(m-2)}z|| \le \varepsilon e^{-p_0}$, where z denotes the inner solution of (2) and $p_i = 0$, $i = 1,\dots,N-1$. Then* *there exists $h > 0$, for which the error $e := y - u$, of the collocation solution $u \in \mathcal{T}_{m-1}^{(-1)}(Z_N)$ $(m \geq 4)$ of the equation (54) on $[0, T]$ satisfies:*

$$\|e\|_\infty \leq C_0 h^{m-2}, \tag{55}$$

where C_0 is a positive constant independent of h. The error estimate (55) holds for every choice of collocation parameters $\{c_j(p_n)\}_{j=1}^m$ chosen is such a way that the quadrature formulæ, possessing collocation parameters as nodes, are of order not less then m for all $u \in \mathcal{T}_{m-1}^{(-1)}(Z_N)$.

Singularly perturbed Volterra integral and integro-differential equations can be fully discretized by intepolatory quadrature formulæ for ECC–spaces, but the analysis is somewhat technical in nature, and the result is similar to the one already known from the theory of polynomial collocation [4, 5].

Finally, we note that generally we have not resolved the question of choice of tension parameters, which is an open problem. For DVIDE that can be converted to differential equations (at least formally) the choice in [13] is adequate, but this is a rare case. Some possibilities are discussed in the next section.

6. Numerical examples

We begin by considering a singularly perturbed linear VIE (1). There is an interesting case described below (though not very common in practice), in which we can determine the tension parameter in Theorem 6. exactly, and that is the case when the kernel K is a polynomial of order $n-1$ in either t or s:

$$\begin{array}{rcl} K(t,s) & = & \sum_{i=0}^{n-1} \frac{a_i(t)}{i!}(t-s)^i \quad \text{or} \\ K(t,s) & = & \sum_{i=0}^{n-1} \frac{b_i(s)}{i!}(t-s)^i, \end{array} \tag{56}$$

where are a_i, $b_i \in C[0, T]$, $i = 0, \ldots, n-1$. We also suppose that the functions g and K satisfy the assumptions of the Theorem 6.

The following Lemma is easily proved by standard analytic methods for Volterra integral equations.

Lemma 1. *Suppose that the kernel of the linear VIE is a polynomial of order $n-1$ in either t or s given by (56). Let the function $f(t, s)$ be a solution of the differential equation*

$$\varepsilon \frac{d^n f}{dt^n}(t,s) + \sum_{i=1}^{n} a_{i-1}(t) \frac{d^{n-i} f}{dt^{n-i}}(t,s) = 0, \tag{57}$$

or

$$\varepsilon \frac{d^n f}{ds^n}(t,s) - \sum_{i=1}^{n} b_{i-1}(s) \frac{d^{n-i} f}{ds^{n-i}}(t,s) = 0, \tag{58}$$

respectively, satisfying the initial conditions

$$f|_{t=s} = \frac{df}{dt}|_{t=s} = \ldots = \frac{d^{n-2}f}{dt^{n-2}}|_{t=s} = 0 \quad and \quad \frac{d^{n-1}f}{dt^{n-1}}|_{t=s} = 1.$$

Then the resolvent kernel of the equation (1) is

$$R(t,s) = -\varepsilon\frac{d^n f}{dt^n}(t,s) \quad or \quad R(t,s) = \varepsilon\frac{d^n f}{ds^n}(t,s),$$

respectively.

Let Λ be the set of the real zeroes of the characteristic equation of (57) or (58). The tension parameter on σ_0 can be determined by the formula:

$$p_0 := h \mid \min_{i\in\{i:\lambda_i\in\Lambda\}} \{\lambda_i\} \mid . \tag{59}$$

We leave the proof to the reader. The singularly perturbed linear VIDE of the form (1) where the kernel K is a polynomial as in (56), q_1, q_2 and K satisfying the assumptions of Theorem 5., can be transformed to VIE, and then we can choose the tension parameter as in (59).

For general kernels one should utilize the asymptotic expansion of the solution in terms of ε. We can write both, VIE's and VIDE's, in the form

$$\varepsilon[a(\varepsilon)y'(t) + b(\varepsilon)y(t)] = g(t) + \int_0^t K(t,s)y(s)ds \tag{60}$$

and assume that an asymptotic expansion of the solution exists in the form

$$y(t) = \sum_{i=0}^{\infty}[v_i(t) + z_i(\tau)]\varepsilon^i,$$

where are $\tau = t/\varepsilon$ and $\lim_{\tau\to\infty} z_i(\tau) = 0$ for all $i = 0, 1, \ldots$. To determine the coefficient of ε^0 in the asymptotic expansion, we have

$$0 = g(t) + \int_0^t K(t,s)v_0(s)ds.$$

If $a = 0$ and $b \neq 0$, we have

$$v_0(0) + z_0(\tau) \quad = \quad \tilde{g}(\tau) + \int_0^\tau \tilde{K}(\tau,\sigma)[v_0(0) + z_0(\sigma)]d\sigma,$$

where

$$\begin{aligned} \tilde{K}(\tau,\sigma)[v_0(0) + z_0(\sigma)] &= \lim_{\varepsilon\to 0}\frac{1}{b(\varepsilon)}K(\varepsilon\tau,\varepsilon\sigma)[v_0(\varepsilon\sigma) + z_0(\sigma)]d\sigma, \\ \tilde{g}(\tau) &= \lim_{\varepsilon\to 0}\frac{g(\varepsilon\tau)}{\varepsilon b(\varepsilon)}. \end{aligned}$$

If $a \neq 0$ and $b = 0$, then

$$z_0'(\tau) \;=\; \bar{g}(\tau) + \int_0^\tau \bar{K}(\tau,\sigma)[v_0(0) + z_0(\sigma)]d\sigma,$$

with the initial conditions $z_0(0) = y_0 - v_0(0)$, where

$$\begin{aligned}\bar{K}(\tau,\sigma)[v_0(0) + z_0(\sigma)] &= \lim_{\varepsilon\to 0} \frac{\varepsilon}{a(\varepsilon)} K(\varepsilon\tau, \varepsilon\sigma)[v_0(\varepsilon\sigma) + z_0(\sigma)]d\sigma,\\ \bar{g}(\tau) &= \lim_{\varepsilon\to 0} \frac{g(\varepsilon\tau)}{a(\varepsilon)}.\end{aligned}$$

Once we have determined the functions v_0 and z_0, we can determine the coefficient of ε^1 in the asymptotic expansion. The function z_0 is exponentially decaying, and, without loss of generality, can be represented in the form $z_0(\tau) = c_0 e^{-c_1\tau}$, where $c_0 \neq 0$ and $c_1 > 0$. Therefore, the tension parameter p_0 can be approximated by

$$p_0 := \frac{hc_1}{\varepsilon}.$$

Theoretically, if the approximated tension parameter is not adequate, we can calculate the coefficient of the next term in the asymptotic expansion of the inner solution. Then, after determining the parameter of the exponential function $e^{-c\tau}$, $c > 0$, which is a good approximation to the linear combination $z_0(\tau) + \varepsilon z_1(\tau)$, the tension parameter could be $p_0 := hc/\varepsilon$. We also note that functions q_1, q_2 and g can implicitly depend on ε.

To support the convergence analysis we give examples of VIDE and VIE possessing solutions in a closed form, and demonstrate errors and convergence orders of the collocation solution.

Example 1. *The Volterra integro-differential equation*

$$\varepsilon y'(t) = (1+\varepsilon)e^{-1} - \varepsilon - y(t) + \int_0^t (1+\varepsilon)y(s)ds,$$

with the initial condition $y(0) = 1 + e^{-1}$.

The solution is $y(t) = e^{t-1} + e^{-\frac{t}{\varepsilon}(1+\varepsilon)}$, while the corresponding reduced equation is $y_r(t) = e^{-1} + \int_0^t y_r(s)ds$. The solution of the reduced equation is e^{t-1}, and $y_r(0) = e^{-1} \neq y(0)$.

Example 2. *The Volterra integral equation*

$$\varepsilon y(t) = \sin t - \int_0^t y(s)ds.$$

The solution is $y(t) = \frac{1}{1+\varepsilon^2}[\cos t + \varepsilon \sin t - e^{-\frac{t}{\varepsilon}}]$, with the corresponding reduced equation $\sin t - \int_0^t y_r(s)ds = 0$. The solution of the reduced equation is $y_r(t) = \cos t$.

The collocation solutions of the equations in Examples 1. and 2. are in the spaces $\mathcal{T}_4^{(0)}(Z_N)$ and $\mathcal{T}_3^{(-1)}(Z_N)$, respectively. The B-spline representation of the solution has been heavily exploited in the construction of collocation matrices, but we don't want to go into details here; a hint of how to calculate with basis of splines in tension with compact support efficiently, for all tension vectors, can be found in [17]. Details of the construction can be found in [10].

We have used three choices of collocation parameters, referred to as *Gauss2, Gauss4* and *Lobatto4*. The *Gauss2* collocation parameters $c_1 = 0$, $c_4 = 1$, and $c_2(p)$, $c_3(p)$ are obtained as the nodes of of the generalized Gauss quadrature formula (shifted on $[0, 1]$) of order 2. The collocation parameters *Gauss4* and *Lobatto4* are the nodes of the corresponding generalized Gauss and Gauss-Lobatto quadrature formulæ (shifted on $[0, 1]$) of order 4.

For $h > \varepsilon$ the tension parameter p_0 in the Examples 1. and 2. is

$$p_0 := \frac{h}{\varepsilon}\frac{1+\sqrt{1+4\varepsilon(1+\varepsilon)}}{2} \tag{61}$$

and

$$p_0 := \frac{h}{\varepsilon}, \tag{62}$$

respectively. For $h \leq \varepsilon$ we have set $p_0 = 0$. The special choice (61) is motivated in [15]. We have not considered superconvergence phenomena. Numerical tests show superconvergence not to be an important issue for smaller ε.

Table 1: Convergence orders for the derivative of the solution to the model integro–differential equation by collocating tension splines with *Gauss2* collocation points

2^n	$k \quad (\varepsilon = 2^{-k})$				
	6	7	8	9	10
4	3.00	2.45	2.02	1.87	2.25
8	1.92	3.06	2.52	2.09	1.94
16	2.72	1.95	3.10	2.56	2.12
32	3.17	2.73	1.96	1.88	2.57
64	3.56	3.17	2.73	1.97	1.21
128	3.77	3.56	3.17	2.73	1.97
256	3.89	3.78	3.56	3.17	2.73
512	3.94	3.89	3.78	3.56	3.17
1024	3.97	3.94	3.89	3.78	3.56

2^n	$k \quad (\varepsilon = 2^{-k})$				
	11	12	13	14	15
4	2.24	2.17	2.11	2.08	2.05
8	2.30	2.30	2.22	2.17	2.14
16	1.98	2.32	2.33	2.25	2.20
32	2.14	2.00	2.33	2.35	2.27
64	2.58	2.15	2.01	2.34	2.35
128	1.23	2.59	2.15	2.01	2.34
256	1.97	1.24	2.59	2.16	2.01
512	2.73	1.98	1.24	2.59	2.16
1024	3.17	2.73	1.98	1.24	2.59

Table 2: Convergence orders for the solution of the model integro–differential equation by collocating tension splines with *Gauss2* collocation points

2^n	$k \quad (\varepsilon = 2^{-k})$				
	6	7	8	9	10
4	3.78	3.38	3.15	3.03	3.06
8	2.84	3.85	3.43	3.22	3.10
16	3.50	2.84	3.88	3.46	3.25
32	4.15	3.51	2.85	2.50	3.48
64	4.36	4.16	3.51	2.85	2.04
128	4.16	4.36	4.16	3.51	2.85
256	4.08	4.16	4.36	4.16	3.51
512	4.04	4.08	4.16	4.36	4.16
1024	4.02	4.04	4.08	4.16	4.36

2^n	$k \quad (\varepsilon = 2^{-k})$				
	11	12	13	14	15
4	3.23	3.12	3.05	3.01	2.98
8	3.17	3.32	3.22	3.15	3.10
16	3.13	3.29	3.34	3.26	3.20
32	3.27	3.15	3.34	3.36	3.28
64	3.49	3.27	3.16	3.37	3.36
128	2.06	3.49	3.28	3.16	3.38
256	2.85	2.07	3.49	3.28	3.16
512	3.51	2.85	2.07	3.50	3.28
1024	4.16	3.51	2.85	2.07	3.50

Table 3: Convergence orders for the solution of the model integral equation by collocating tension splines with *Gauss2* collocation points

2^n	k $(\varepsilon = 2^{-k})$				
	6	7	8	9	10
4	3.12	2.58	2.15	2.01	2.34
8	3.62	3.13	2.59	2.16	2.01
16	2.73	3.52	3.13	2.59	2.16
32	3.17	2.73	1.98	3.13	2.59
64	3.56	3.17	2.73	1.98	3.13
128	3.78	3.56	3.17	2.73	1.98
256	3.89	3.78	3.56	3.17	2.73
512	3.94	3.89	3.78	3.56	3.17
1024	3.97	3.94	3.89	3.78	3.56

2^n	k $(\varepsilon = 2^{-k})$				
	11	12	13	14	15
4	2.37	2.30	2.24	2.21	2.18
8	2.34	2.36	2.29	2.23	2.20
16	2.01	2.35	2.36	2.28	2.23
32	2.16	2.01	2.35	2.36	2.28
64	2.59	2.16	2.01	2.35	2.36
128	3.13	2.59	2.16	2.01	2.35
256	1.98	3.13	2.59	2.16	2.01
512	2.73	1.98	3.13	2.59	2.16
1024	3.17	2.73	1.98	2.24	2.59

Figure 1: Collocation error for the model integral equation in the logarithmic scale, depending on the number of knots ($N = 2^n + 1$), $\varepsilon = 2^{-12}$

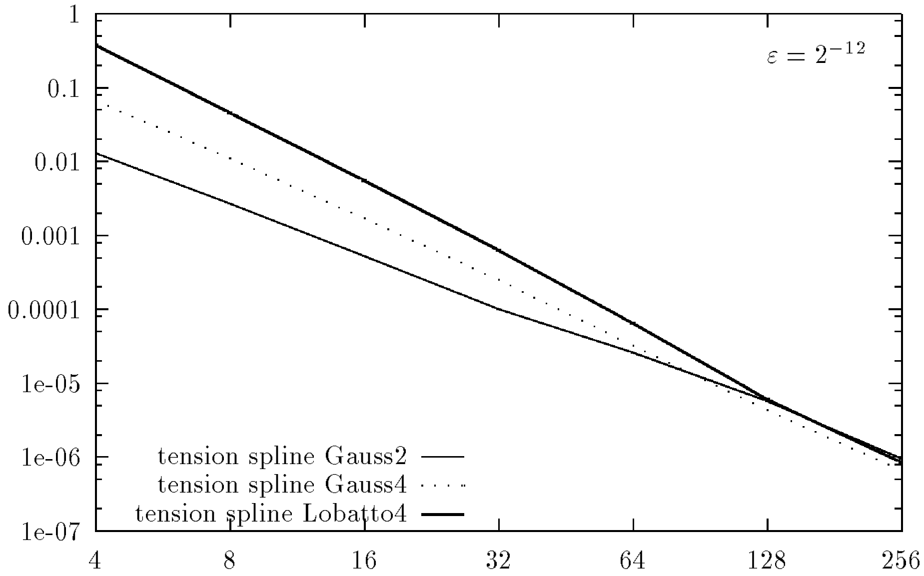


Figure 2: Collocation error for the derivative of the solution to the model integro–differential equation in the logarithmic scale, depending on the number of knots ($N = 2^n + 1$), $\varepsilon = 2^{-12}$

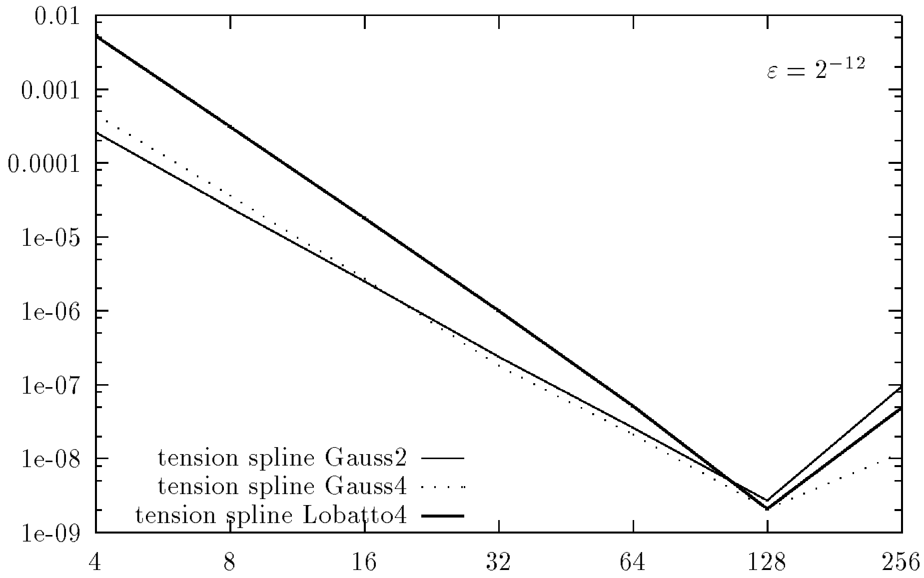
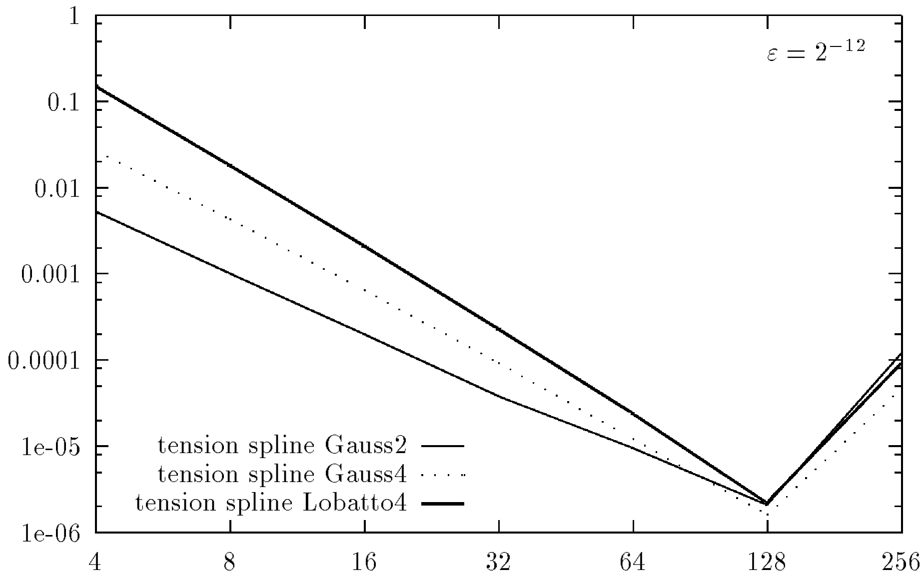


Figure 3: collocation error for the derivative of the solution to the model integro–differential equation in the logarithmic scale, depending on the number of knots ($N = 2^n + 1$), $\varepsilon = 2^{-12}$



References

[1] J.S. Angell, W.E. Olmstead, *Singularly perturbed Volterra integral equations* , SIAM J. Appl. Math. Vol. 47. No 1 (1987), 1-14.

[2] J.S. Angell, W.E. Olmstead, *Singularly perturbed Volterra integral equations II*, SIAM J. Appl. Math. Vol. 47. No 6 (1987), 1150-1162.

[3] H. Brunner, *Collocation methods for non-linear Volterra integro–differential equations with infinite delay*, Math. Comp. 53 (1989), 571–587.

[4] H. Brunner,*The use of splines in the numerical solution of differential and Volterra integral equations*, CRM Proceedings and Lecture Notes Vol. 18 (1999) 15–29.

[5] H. Brunner, P. J. van der Houwen, *The Numerical Solutions of Volterra Equations*, CWI Monograph 3, Nort-Holland, Amsterdam, 1986.

[6] C. Grandison, *Behaviour of Exponential Splines as Tension Increase without Bound*, J. Approx. Theory 21 (1997), 289–307.

[7] J. P. Kauthen, *Implicit Runge–Kutta methods for some integro differential–algebraic equations*, Appl. Numer. Math. 13 (1993) 125–134.

[8] J. P. Kauthen, *Implicit Runge–Kutta methods for singularly perturbed integro–differential systems*, Appl. Numer. Math. 18. (1995) 201–210.

[9] J. P. Kauthen, *A survey of Singularly Perturbed Volterra Equations*, to appear in Appl. Num. Math.

[10] B. Kvasov, P. Sattayatham, *GB splines of arbitrary order*, J. Comp. Appl. Math. 104 (1999), 63–88.

[11] M.-L. Mazure, *Blossoming, a geometrical approach*, Constructive Approximation 15 (1999), 33–68.

[12] M. Marušić, *On Interpolation by Hermite Tension Splines of Arbitrary Order*, in *Approximation Theory IX*, Vanderbilt Univ. press, L.L. Schumaker eds., Nashville TN 1998.

[13] M. Marušić, *A fourth/second order accurate collocation method for singularly perturbed two–point boundary value problems using tension splines*, Numer. Math. 88 (2001), 135-158

[14] M. Marušić, M. Rogina, *Sharp Error Bounds for Interpolating Splines in Tension*, J. Comp. Appl. Math. 61 (1995), 205–223 .

[15] M. Marušić, M. Rogina, *A collocation error for singularly perturbed two–point boundary value problems with splines in tension*, Adv. Comput. Math. 6, (1996), 65–75.

[16] G. Nürnberger, *Approximation by Spline Functions*, Springer–Verlag, Berlin Heidelberg New York, 1989.

[17] M. Rogina, T. Bosner, *On calculating with lower order Chebyshev splines*, in *Curves and Surface III*, 343–352, Vanderbilt Univ. press, J.P.Laurent, P. Sablonière, L.L.Schumaker eds., Nashville TN 2000.

[18] L. A. Skinner, *Asymptotic solution to a class of singularly perturbed Volterra integral equations*, Methods Appl. Anal. 2 (1995) 212–221.

[19] L. L. Schummaker, *Spline Functions:Basic Theory*, John Wiley and Sons, New York, 1981.