ASPECTS OF CONVERGENCE FOR MIXED MULTISCALE FINITE ELEMENTS AND A NEW APPROACH TO THEIR DEFINITION

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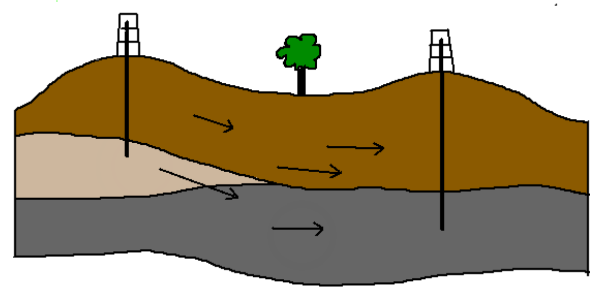
Incompressible, single phase flow in a porous medium:

$$\begin{cases} \mathbf{u} = -a_\epsilon \nabla p & \text{in } \Omega \quad \text{(Darcy's law)} \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega \quad \text{(Conservation)} \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \quad \text{(BC for simplicity)} \end{cases}$$



A mixed variational formulation:

Find $p \in W = L^2/\mathbb{R}$ and $\mathbf{u} \in \mathbf{V} = H_0(\text{div})$ such that

$$(a_\epsilon^{-1}\mathbf{u}, \mathbf{v}) = -(\nabla p, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V} \quad \text{(Darcy's law)}$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w) \quad \forall\, w \in W \quad \text{(Conservation)}$$

Mixed Finite Element Approximation

Find $p \in W_h \subset W$ and $\mathbf{u} \in \mathbf{V}_h \subset \mathbf{V}$ such that

$$(a_\epsilon^{-1}\mathbf{u}_h, \mathbf{v}) = (p_h, \nabla \cdot \mathbf{v}) \quad \forall\ \mathbf{v} \in \mathbf{V}_h$$

$$(\nabla \cdot \mathbf{u}_h, w) = (f, w) \qquad \forall\ w \in W_h$$

Problem of scale: The coefficient $a_\epsilon(x)$ varies on a fine scale $\epsilon \ll 1$. To resolve the solution, we need a mesh $\mathcal{T}_h$ of maximal spacing $h < \epsilon$. This is often not computationally feasible.

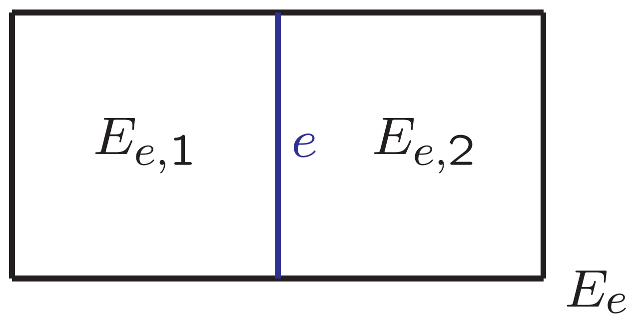
Solution: We define $\mathbf{V}_h \times W_h$ to respect the scales:

- Multiscale finite elements (Babuška, Caloz & Osborn 1994; Hou & Wu 1997; Chen & Hou 2003)
- Variational multiscale method (Hughes 1995, A., Minkoff & Keenan 1998, A. & Boyd 2006)

Mixed Multiscale Finite Elements

For this talk,

- In all cases, $W_h =$ piecewise discontinuous constants
- $\mathcal{T}_h$ is a quasiuniform rectangular grid
- $\mathcal{E}_h$ are the mesh "edges"
- For $e \in \mathcal{E}_h$, let E_e be the two elements $E_{e,1}$, $E_{e,2} \in \mathcal{T}_h$ bordering e



We consider multiscale finite elements defined either:

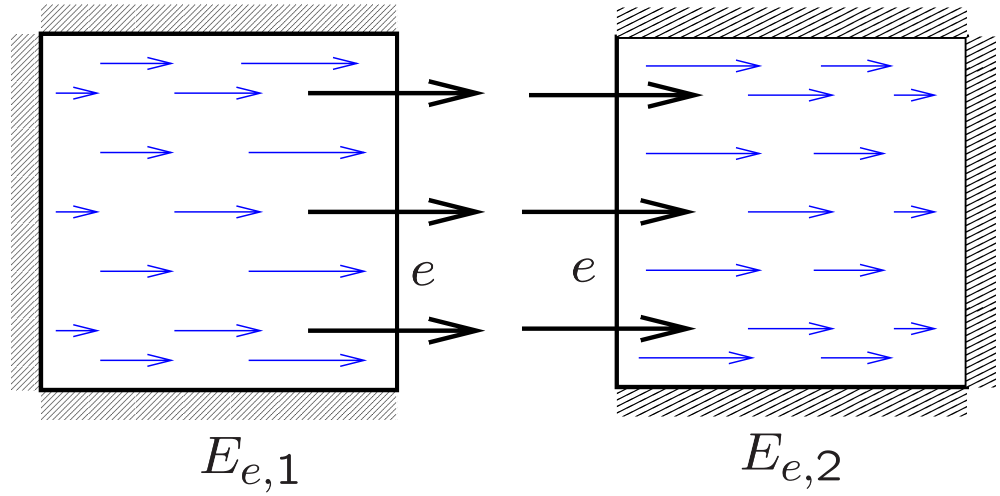
- Elementwise on $E \in \mathcal{T}_h$
- On dual-support domain E_e for $e \in \mathcal{E}_h$.

We define $\mathbf{v}_e^{\mathsf{RT}} \in V_h^{\mathsf{RT}}$ for each coarse element edge $e \in \mathcal{E}_h$.

Element definition:

For each edge $e \subset \partial E$, solve

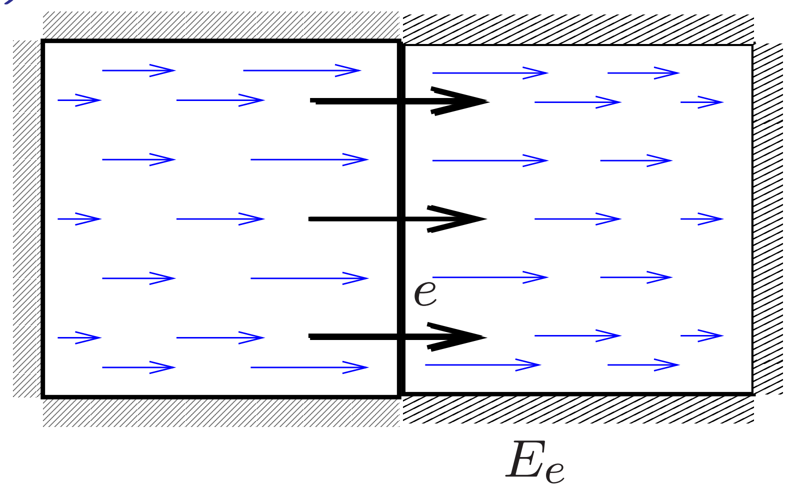
$$\begin{cases} \mathbf{v}_e^{\mathsf{RT}} = -\nabla \phi_e^{\mathsf{RT}} & \text{in } E, \\ \nabla \cdot \mathbf{v}_e^{\mathsf{RT}} = \pm |e|/|E| & \text{in } E, \\ \mathbf{v}_e^{\mathsf{RT}} \cdot \nu = \begin{cases} 0 & \text{on } \partial E \setminus e, \\ 1 & \text{on } e, \end{cases} \end{cases}$$

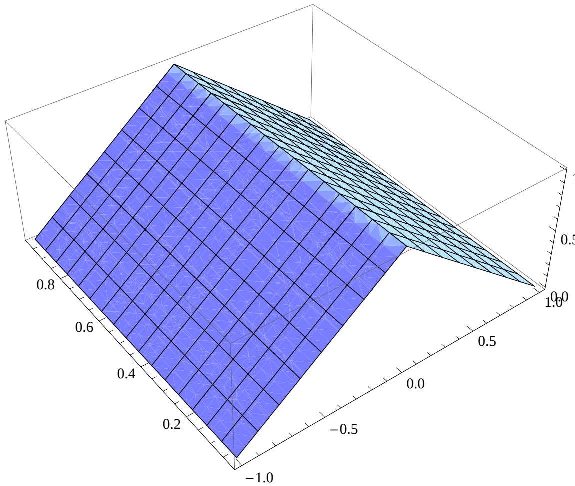


Dual-support definition (rectangular case):

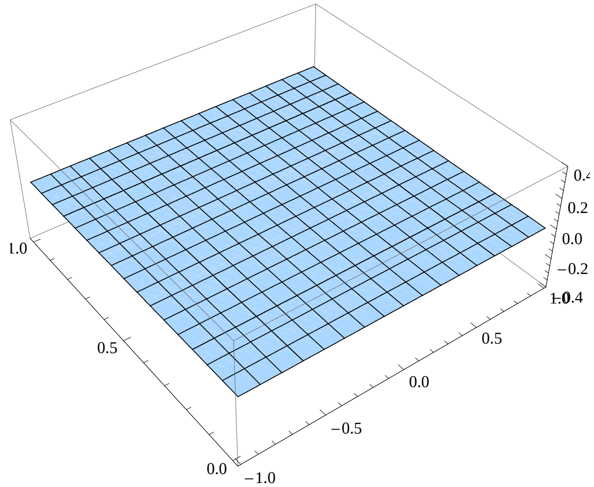
For each edge $e \in \mathcal{E}_h$, solve

$$\begin{cases} \mathbf{v}_e^{\mathsf{RT}} = -\nabla \phi_e^{\mathsf{RT}} & \text{in } E_e, \\ \nabla \cdot \mathbf{v}_e^{\mathsf{RT}} = \pm |e|/|E_{e,i}| & \text{in } E_{e,i}, \ i = 1, 2, \\ \mathbf{v}_e^{\mathsf{RT}} \cdot \nu = 0 & \text{on } \partial E_e. \end{cases}$$

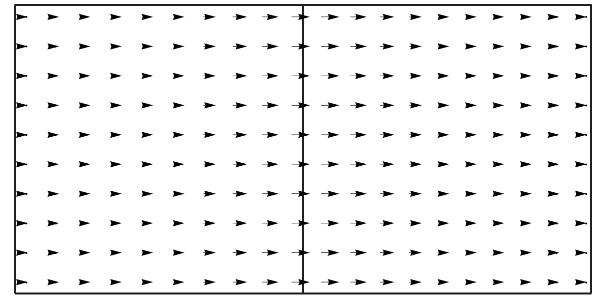




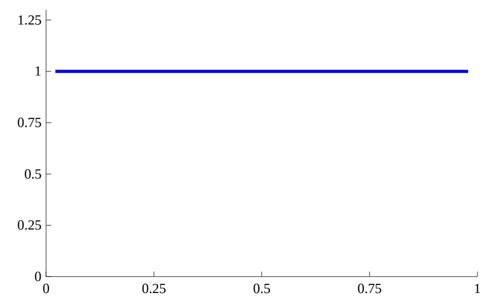
x-velocity



y-velocity



velocity



normal trace

Theorem: (Raviart & Thomas, 1977)

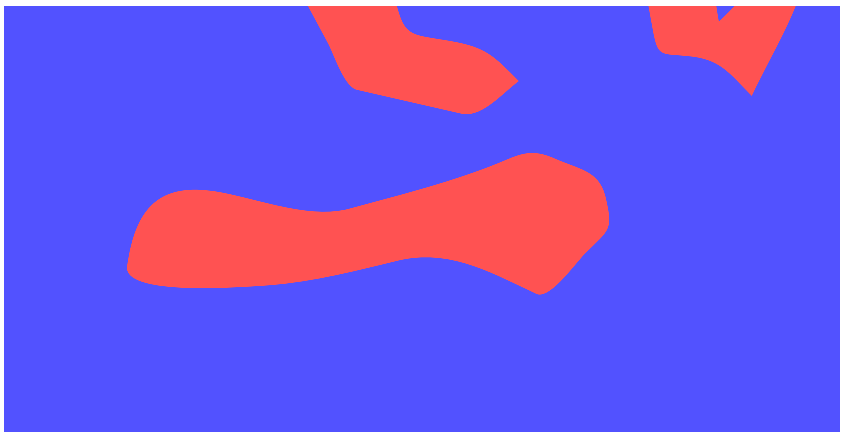
$$\|\mathbf{u} - \mathbf{u}_h^{\mathsf{RT}}\|_0 \le C\|\mathbf{u}\|_1 h = \mathcal{O}\left(\frac{h}{\epsilon}\right)$$

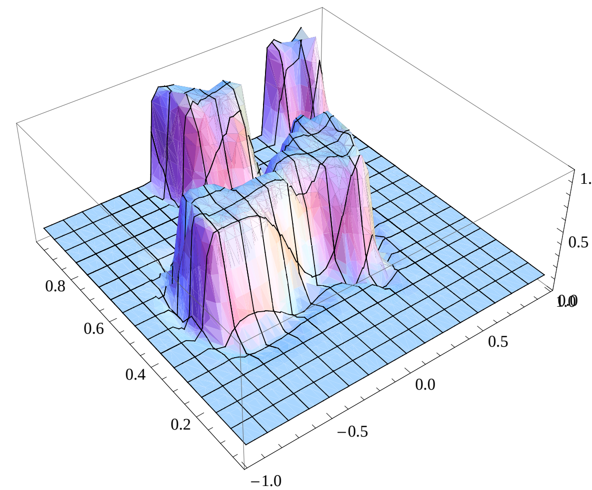
Remark: These elements have no dependence on the scale ϵ. They are accurate only when $h < \epsilon$, i.e., h resolves the fine-scale heterogeneity.

Elements Based on the Heterogeneity

Main idea of multiscale finite elements: In the boundary value problems used to define $\mathbf{v}_e^{\mathrm{RT}} \in \mathbf{V}_h^{\mathrm{RT}}$, insert the coefficient a_ϵ!

Example: An permeability coefficient a_ϵ





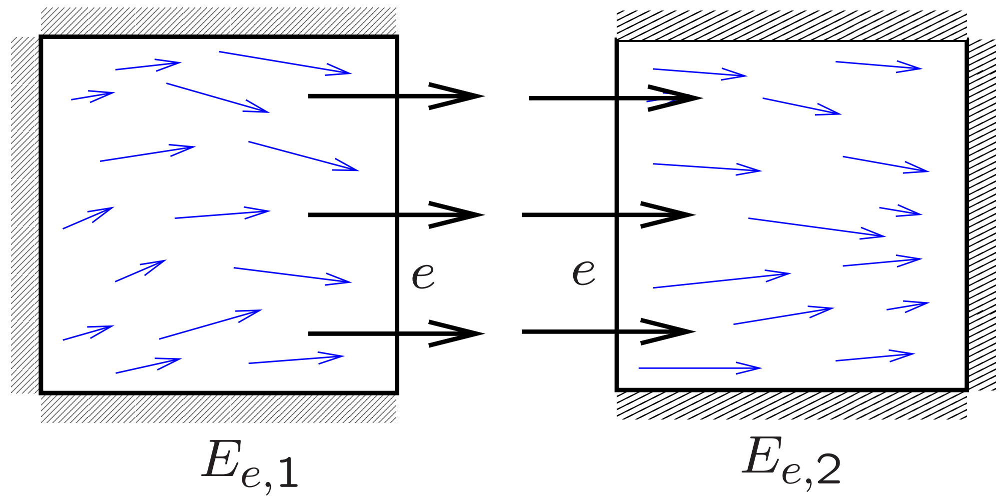
Variational Multiscale Element (ME) Based on RT—1

We define $\mathbf{v}_e^{\text{ME}} \in V_h^{\text{ME}}$ for each coarse element edge $e \in \mathcal{E}_h$.

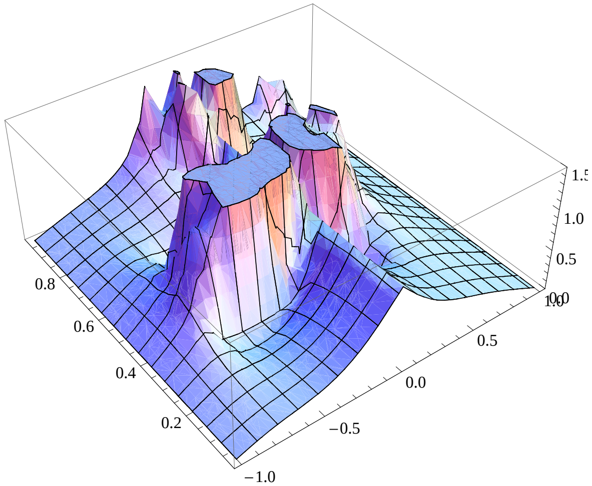
Element definition:

For each edge $e \subset \partial E$, solve

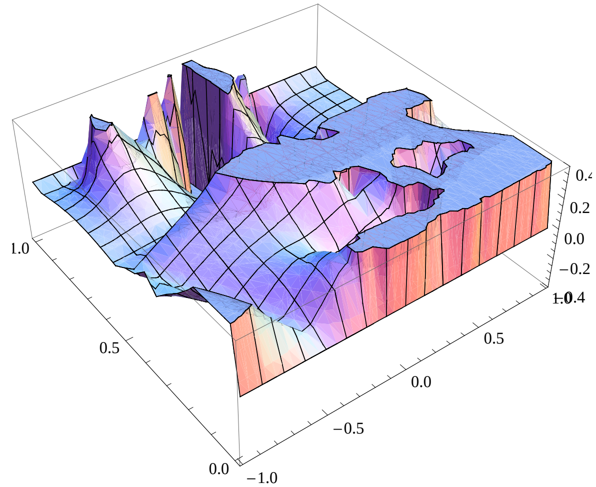
$$\begin{cases} \mathbf{v}_e^{\text{ME}} = -a_\epsilon \nabla \phi_e^{\text{ME}} & \text{in } E, \\ \nabla \cdot \mathbf{v}_e^{\text{ME}} = \pm |e|/|E| & \text{in } E, \\ \mathbf{v}_e^{\text{ME}} \cdot \nu = \begin{cases} 0 & \text{on } \partial E \setminus e, \\ 1 & \text{on } e, \end{cases} \end{cases}$$



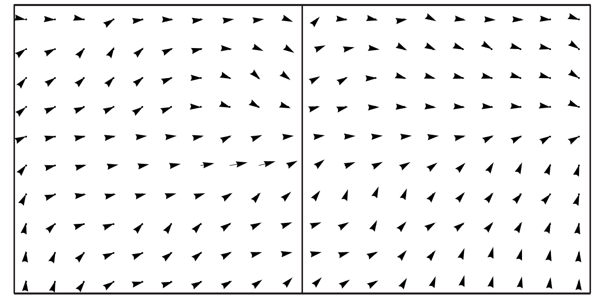
Variational Multiscale Element (ME) Based on RT—2



x-velocity



y-velocity



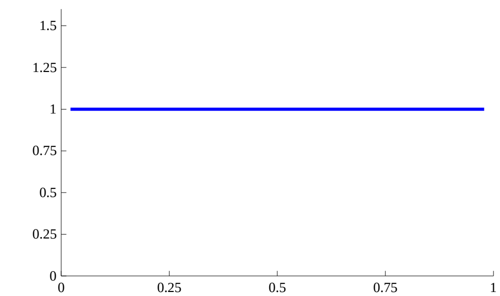
velocity

Theorem: (A. '04; Chen & Hou '03; A. & Boyd '06)

$$\|\mathbf{u}-\mathbf{u}_h^{\mathsf{ME}}\|_0 \le C\|\mathbf{u}\|_1 h,$$

$$\|\mathbf{u}-\mathbf{u}_h^{\mathsf{ME}}\|_0 \le C\Big\{h\|\mathbf{u}_0\|_1+\epsilon\|\mathbf{u}_0\|_0+\sqrt{\epsilon/h}\,\|\mathbf{u}_0\|_{0,\infty}\Big\},$$

where $\mathbf{u}_0$ is a smooth function independent of ϵ.



normal trace

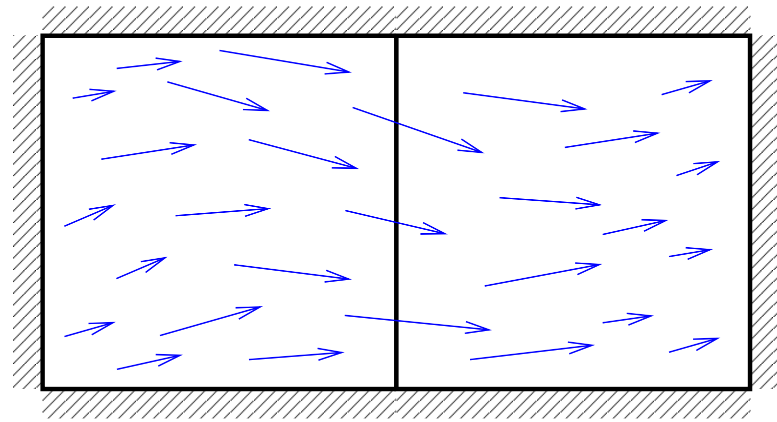
$$\|\mathbf{u}-\mathbf{u}_h^{\mathsf{ME}}\|_0 = \mathcal{O}\left(\min\left\{\frac{h}{\epsilon}\,,\ h+\epsilon+\sqrt{\frac{\epsilon}{h}}\right\}\right)$$

Multiscale Dual-Support (MD) Elements—1

(Aarnes, 2004; Aarnes, Krogstad, Lie, 2006)

We define $\mathbf{v}_e^{\mathsf{MD}} \in V_h^{\mathsf{MD}}$ for each coarse element edge $e \in \mathcal{E}_h$.

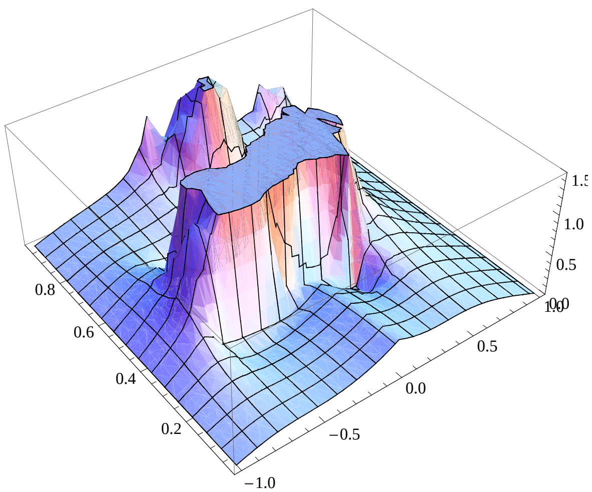
Dual support definition (rectangular case):



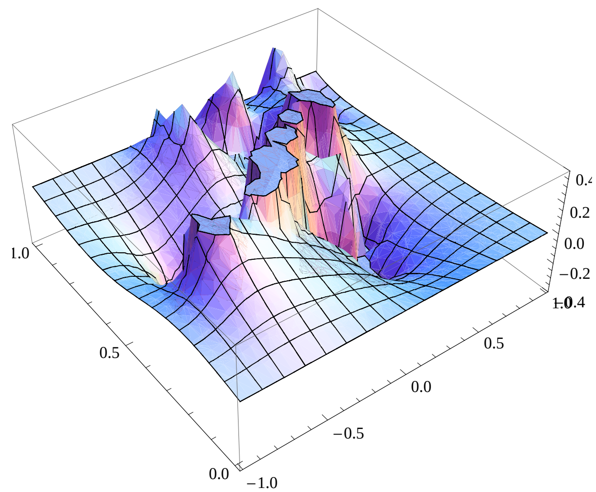
For each edge $e \in \mathcal{E}_h$, solve

$$\begin{cases} \mathbf{v}_e^{\mathsf{MD}} = -a_\epsilon \nabla \phi_e^{\mathsf{MD}} & \text{in } E_e, \\ \nabla \cdot \mathbf{v}_e^{\mathsf{MD}} = \pm|e|/|E_{e,i}| & \text{in } E_{e,i},\ i = 1, 2, \\ \mathbf{v}_e^{\mathsf{MD}} \cdot \nu = 0 & \text{on } \partial E_e. \end{cases}$$

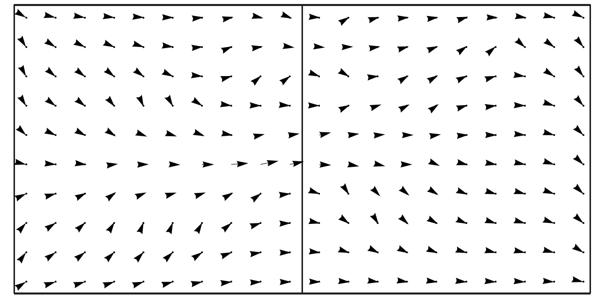
Multiscale Dual-Support (MD) Elements—2



x-velocity

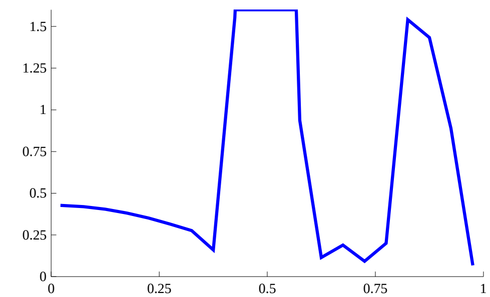


y-velocity



velocity

It is not known if this method converges, either for $\epsilon < h$ or $\epsilon > h$.



normal trace

Claim: The method cannot converge in any reasonable sense!

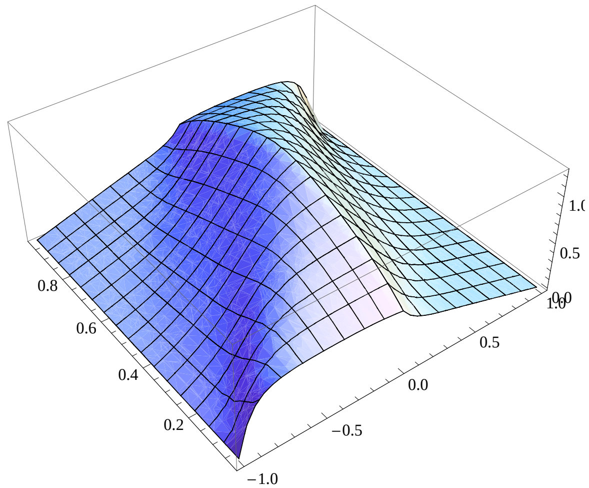
Influence of Anisotropy

Counterexample to Convergence of MD

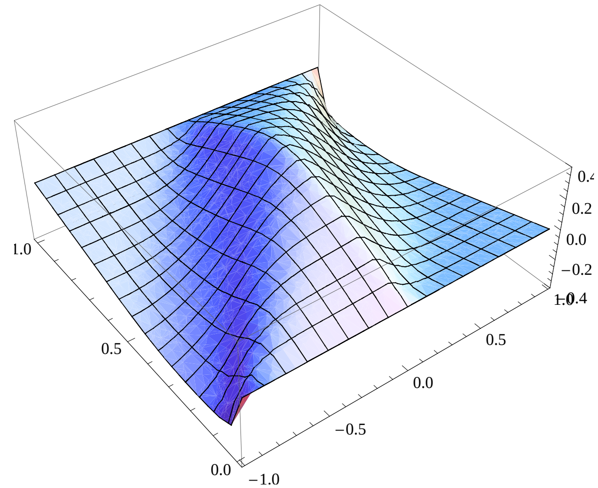
Take a constant

$$a_\epsilon(x) = a = Q\Lambda Q^T \qquad \text{with } \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \qquad \lambda = 100$$

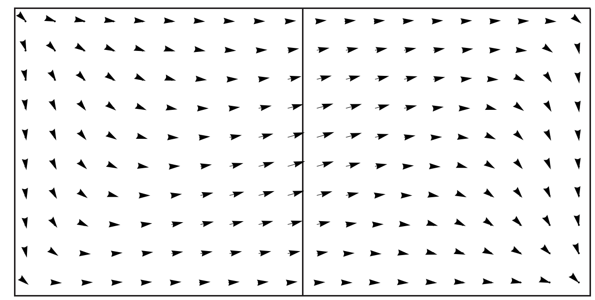
and Q a rotation (30°). We have a genuine anisotropy.



x-velocity



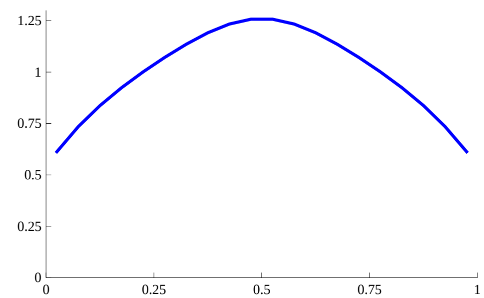
y-velocity



velocity

The space $\mathbf{V}_h^{\mathrm{MD}}$ cannot reproduce constants, so the method cannot converge in any reasonable sense.

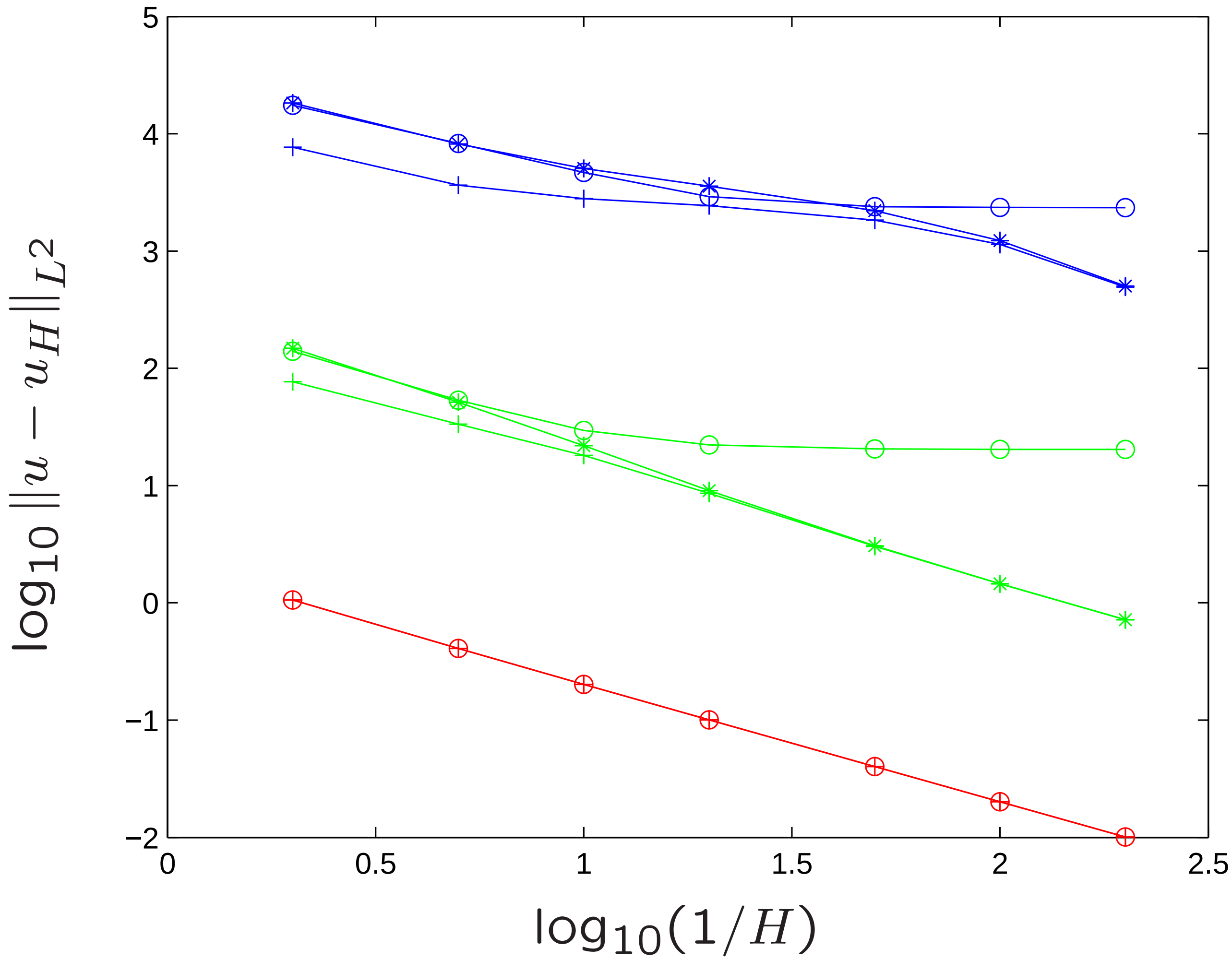
Question: Are dual-support elements infeasible?



normal trace

Numerical Convergence Study

Anisotropy at $\theta = 30°$ with ratio λ, true $p = \sin(\pi x)\sin(\pi y)$



+ RT ★ ME ○ MD

$\lambda = 1$ $\lambda = 100$ $\lambda = 10,000$

Microscale Structure
from Homogenization Theory

Suppose that a_ϵ is locally periodic of period ϵ. Then

$$a_\epsilon(x) = a(x, x/\epsilon)$$

where $a(x,y)$ is periodic in y of period 1 on the unit cube Y.

Let a_0 be the homogenized permeability matrix, defined by

$$a_{0,ij}(x) = \int_Y a(x,y) \left(\delta_{ij} + \frac{\partial \omega_j(x,y)}{\partial y_i} \right) dy$$

where, for fixed x, $\omega_j(x,y)$ is the Y-periodic solution of

$$-\nabla_y \cdot (a \nabla_y \omega_j) = \frac{\partial a}{\partial y_j}$$

Homogenized solution: Let $(\mathbf{u}_0, p_0)$ solve

$$\begin{cases} \mathbf{u}_0 = -a_0 \nabla p_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_0 = f & \text{in } \Omega \\ \mathbf{u}_0 \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

Then $(\mathbf{u}_0, p_0)$ is a smooth "approximation" of $(\mathbf{u}, p)$.

Theorem: Assume that $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Let $\alpha_0 = a_0^{-1}$ and define the fixed tensor independent of ϵ and the domain Ω

$$\mathcal{A}_{ij}(x,y) = \sum_{k,\ell} a_{ik}(x,y)\left(\delta_{k\ell} + \frac{\partial \omega_\ell(x,y)}{\partial y_k}\right)\alpha_{0,\ell j}$$

$$\mathcal{A} = a\,(I + D\omega)\alpha_0$$

Let

$$\mathcal{A}_\epsilon(x) = \mathcal{A}(x, x/\epsilon)$$

Then

$$(1) \quad \mathbf{u}_\epsilon(x) = \mathcal{A}_\epsilon(x)\,\mathbf{u}_0(x) + \theta_\epsilon^\Omega(x)$$

where

$$\|\theta_\epsilon^\Omega\|_0 \le C\left\{\epsilon\|\mathbf{u}_0\|_1 + \sqrt{\epsilon|\partial\Omega|}\|\mathbf{u}_0\|_{0,\infty}\right\} = \mathcal{O}\left(\epsilon + \sqrt{\epsilon}\right)$$

$$\mathbf{u}_\epsilon \approx \mathcal{A}_\epsilon \mathbf{u}_0 \quad \Longrightarrow \quad \mathbf{V}_h \sim \{\mathcal{A}_\epsilon \mathbf{v} : \mathbf{v} \text{ is some nice smooth function}\}.$$

However, these finite elements lie outside $H(\text{div};\Omega)$.

Definition: Let $\mathbf{v}_e^{\text{HD}} \in V_h^{\text{HD}}$ for each $e \in \mathcal{E}_h$ solve on E_e

$$\begin{cases} \mathbf{v}_e^{\text{HD}} = -\mathcal{A}_\epsilon \nabla \phi_e^{\text{HD}} & \text{in } E_e, \\ \nabla \cdot \mathbf{v}_e^{\text{HD}} = \pm |e|/|E_{e,i}| & \text{in } E_{e,i},\ i = 1, 2, \\ \mathbf{v}_e^{\text{HD}} \cdot \nu = 0 & \text{on } \partial E_e. \end{cases}$$

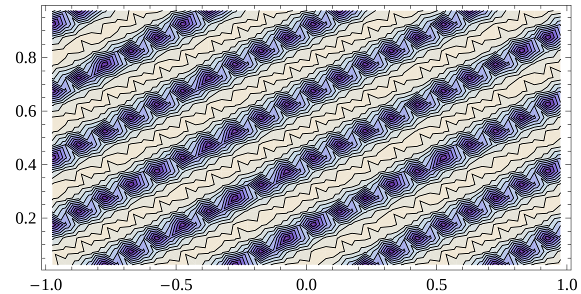
Remarks:

- This is a dual-support element.
- We have a scaling that respects the anisotropy:

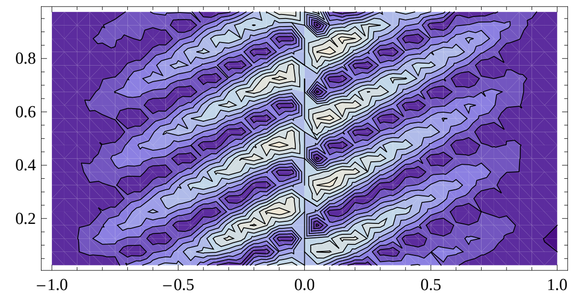
$$\int_Y \mathcal{A}\, dy = \int_Y a(I + D\omega)\, \alpha_0\, dy = I$$

Sample Basis Shapes

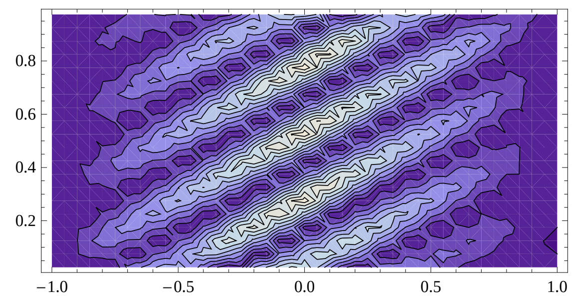
a



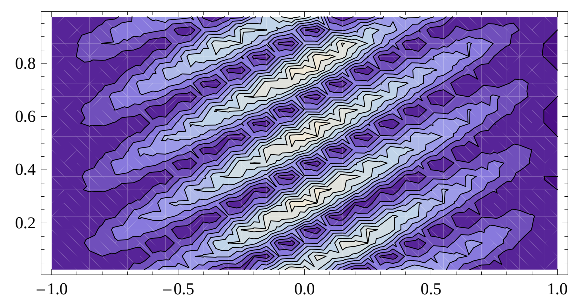
v_x^{ME}



v_x^{MD}



v_x^{HD}



Multiscale Convergence Results

Remarks

This is a multiscale error analysis

- We quantify the error in terms of h and ϵ.
- The proofs are based on comparison to the homogenized solution.
- The style of proof is due to Hou, Wu, and Cai 1999. See also
  - Efendiev, Hou, and Wu 2000
  - Chen and Hou 2003 (mixed case)
  - A. and Boyd 2006 (mixed case)

We present a new, simplified proof involving

- certain projection operators
- four key results (we saw (1))
- a one line proof

Quasi-Optimality

Assume $a_\epsilon(x)$ is smooth and

$$a_*|\xi|^2 \le \xi \cdot \alpha_\epsilon(x)\,\xi \le a^*|\xi|^2 \quad \forall x \in \Omega.$$

Let $\mathcal{P}_{W_h}$ denote L^2-projection into W_h.

Lemma: (Quasi-optimality) If $\nabla \cdot \mathbf{V}_h \subset W_h$, then

$$(2) \qquad \|\mathbf{u}_\epsilon - \mathbf{u}_h\|_0 \le \sqrt{\frac{a^*}{a_*}}\|\mathbf{u}_\epsilon - \mathbf{v}\|_0$$

for any $\mathbf{v} \in \mathbf{V}_h$ such that $\nabla \cdot \mathbf{v} = \mathcal{P}_{W_h}\nabla \cdot \mathbf{u}_\epsilon$.

Goal: Find any $\mathbf{v}_\epsilon \approx \mathbf{u}_\epsilon$ in $\mathbf{V}_h^{\mathsf{M}}$ with $\nabla \cdot \mathbf{v}_\epsilon = \mathcal{P}_{W_h}\nabla \cdot \mathbf{u}_\epsilon$.

Key idea: To deal with the ϵ scale of our finite elements, define corresponding homogenized finite elements.

Replace the true coefficient in the definitions of the finite elements with the corresponding homogenized one.

$$\begin{aligned} \text{RT}: &\quad I \longmapsto I(\text{unchanged}) \\ \text{ME and MD}: &\quad a_\epsilon \longmapsto a_0 \\ \text{HD}: &\quad \mathcal{A}_\epsilon \longmapsto \mathcal{A}_0 = I \end{aligned}$$

$$\mathbf{V}^{\mathsf{M}}_{0,h} = \operatorname*{span}_{e\in\mathcal{E}_h}\{\mathbf{v}^{\mathsf{M}}_{0,e}\}, \quad \mathsf{M} = \mathsf{ME}, \mathsf{MD}, \mathsf{HD}$$

Lemma: If $\mathcal{T}_h$ is rectangular, then HD elements are RT elements:

$$v^{\mathsf{HD}}_{0,e} = \mathbf{v}^{\mathsf{RT}}_e \quad \text{and} \quad \mathbf{V}^{\mathsf{HD}}_{0,h} = \mathbf{V}^{\mathsf{RT}}_h$$

Since our finite elements are defined by boundary value problems, the homogenization theorem applies.

Lemma: For each $e \in \mathcal{E}_h$ and method $\mathsf{M} = \mathsf{ME}$, MD, and HD,

$$\boxed{\mathbf{v}_e^{\mathsf{M}} = \mathcal{A}_\epsilon \mathbf{v}_{0,e}^{\mathsf{M}} + \theta_\epsilon^{E_e,\mathsf{M}}}$$

where

$$\begin{aligned} \|\theta_\epsilon^{E_e,\mathsf{M}}\|_{0,E_e} &\le C\Big\{\epsilon\|\mathbf{v}_{0,e}^{\mathsf{M}}\|_{1,E_e} + \sqrt{\epsilon|\partial E_e|}\|\mathbf{v}_{0,e}^{\mathsf{M}}\|_{0,\infty,E_e}\Big\} \\ &= \mathcal{O}\bigg(\Big\{\frac{\epsilon}{h} + \sqrt{\frac{\epsilon}{h}}\Big\}h^{d/2}\bigg) \end{aligned}$$

Flux-Based Projection Operators

The average normal flux across $e \in \mathcal{E}_h$ is

$$\gamma_e = \frac{1}{|e|} \int_e \mathbf{v} \cdot \nu_e \, ds$$

The Raviart-Thomas projection is

$$\pi^{\mathsf{RT}} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \gamma_e \mathbf{v}_e^{\mathsf{RT}} \in \mathbf{V}_h^{\mathsf{RT}}$$

Similarly, define

$$\pi_\epsilon^{\mathsf{M}} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \gamma_e \mathbf{v}_e^{\mathsf{M}} \in \mathbf{V}_h^{\mathsf{M}} \quad \text{and} \quad \pi_0^{\mathsf{M}} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \gamma_e \mathbf{v}_{0,e}^{\mathsf{M}} \in \mathbf{V}_{0,h}^{\mathsf{M}}$$

Lemma: For M = ME, MD, or HD,

$$\nabla \cdot \pi_\epsilon^{\mathsf{M}} \mathbf{v} = \nabla \cdot \pi_0^{\mathsf{M}} \mathbf{v} = \nabla \cdot \pi^{\mathsf{RT}} \mathbf{v} = \mathcal{P}_{W_h} \nabla \cdot \mathbf{v}$$

Multiscale Projection Approximation

Lemma: For M = ME, MD, or HD,

$$(3) \qquad \|\pi_\epsilon^{\mathsf{M}}\mathbf{v} - \mathcal{A}_\epsilon \pi_0^{\mathsf{M}}\mathbf{v}\|_0 \le C\|\mathbf{v}\|_1\Big(\epsilon/h + \sqrt{\epsilon/h}\Big)$$

Proof:

$$\pi_\epsilon^{\mathsf{M}}\mathbf{v} - \mathcal{A}_\epsilon \pi_0^{\mathsf{M}}\mathbf{v} = \sum_{e\in\mathcal{E}_h} \gamma_e(\mathbf{v}_e^{\mathsf{M}} - \mathcal{A}_\epsilon \mathbf{v}_{0,e}^{\mathsf{M}}) = \sum_{e\in\mathcal{E}_h} \gamma_e \theta_e^{E_e,\mathsf{M}}$$

$\Longrightarrow$

$$\begin{aligned}
\|\pi_\epsilon^{\mathsf{M}}\mathbf{v} - \mathcal{A}_\epsilon \pi_0^{\mathsf{M}}\mathbf{v}\|_{0,E} &\le \sum_{e\subset\partial E} |\gamma_e|\,\|\theta_e^{E_e,\mathsf{M}}\|_{0,E} \\
&\le C \sum_{e\subset\partial E} \Big(h^{-d/2}\|\mathbf{v}\|_{1,E_e}\Big)\Big(\Big\{\frac{\epsilon}{h} + \sqrt{\frac{\epsilon}{h}}\Big\}h^{d/2}\Big) \\
&= C \sum_{e\subset\partial E} \|\mathbf{v}\|_{1,E_e}\Big(\frac{\epsilon}{h} + \sqrt{\frac{\epsilon}{h}}\Big) \qquad \square
\end{aligned}$$

Lemma: If $\mathcal{T}_h$ is rectangular, then

$$(4a) \qquad \|\mathbf{v} - \pi_0^{\mathsf{HD}}\mathbf{v}\|_0 = \|\mathbf{v} - \pi_0^{\mathsf{RT}}\mathbf{v}\|_0 \le C\|\mathbf{v}\|_1 h$$

If $\mathbf{v}_0 = -a_0\nabla\phi_0$, then

$$(4b) \qquad \|\mathbf{v}_0 - \pi_0^{\mathsf{ME}}\mathbf{v}_0\|_0 \le C\|\mathbf{v}_0\|_1 h$$

The counterexamples show that similar results cannot hold for MD.

Proof: (for ME)

$$\psi = \mathbf{v} - \pi_0^{\mathsf{ME}}\mathbf{v} = -a_0\nabla\Big(\phi_0 - \sum_{e\subset\partial E}\gamma_e\phi_{0,e}^{\mathsf{ME}}\Big) \quad \text{in } E$$

is a potential field satisfying the Neumann problem

$$\begin{aligned} \nabla\cdot\psi &= \nabla\cdot\mathbf{v}_0 - \mathcal{P}_{W_h}\nabla\cdot\mathbf{v}_0 && \text{in } E \\ \psi\cdot\nu_e &= \mathbf{v}_0\cdot\nu_e - \gamma_e && \text{on } e\subset\partial E \end{aligned}$$

The standard energy estimate gives the result. □

Corollary: If Ω has elliptic regularity, and M = ME or both M = HD and $\mathcal{T}_h$ is rectangular, then there is some $\beta > 0$, independent of ϵ, such that

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h^{\mathsf{M}}} \frac{(w_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_0 + \|\nabla \cdot \mathbf{v}_h\|_0} \geq \beta \|w_h\|_0 \qquad \forall w_h \in W_h$$

Proof: Solve

$$\begin{cases} \nabla \cdot \mathbf{v}_0 = w_h & \text{in } \Omega \\ \mathbf{v}_0 = -a_0 \nabla \phi_0 & \text{in } \Omega \\ \mathbf{v}_0 \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \qquad \Longrightarrow \qquad \|\mathbf{v}_0\|_1 \leq C \|w_h\|_0$$

Take

$$\mathbf{v}_h = \pi_\epsilon^{\mathsf{M}} \mathbf{v}_0 \in \mathbf{V}_h^{\mathsf{M}} \qquad \Longrightarrow \qquad \nabla \cdot \mathbf{v}_h = \mathcal{P}_{W_h} \nabla \cdot \mathbf{v}_0 = w_h$$

Then

$$\begin{aligned} \|\mathbf{v}_h\|_0 &\leq \underbrace{\|\pi_\epsilon^{\mathsf{M}} \mathbf{v}_0 - \mathcal{A}_\epsilon \pi_0^{\mathsf{M}} \mathbf{v}_0\|_0}_{(3)} + \underbrace{\|\mathcal{A}_\epsilon (\pi_0^{\mathsf{M}} \mathbf{v}_0 - \mathbf{v}_0)\|_0}_{(4)} + \|\mathcal{A}_\epsilon \mathbf{v}_0\|_0 \\ &\leq C \|\mathbf{v}_0\|_1 \leq C \|w_h\|_0 \qquad \square \end{aligned}$$

Convergence Theorem

Theorem: If Ω has elliptic regularity and $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, then for M = ME or for M = HD and $\mathcal{T}_h$ rectangular,

$$\|\mathbf{u}_\epsilon - \mathbf{u}_h^{\mathsf{M}}\|_0 + \|\mathcal{P}_{W_h} p_\epsilon - p_h\|_0 \leq C\Big\{\Big(\epsilon + \epsilon/h + \sqrt{\epsilon/h} + h\Big)\|\mathbf{u}_0\|_1 + \sqrt{\epsilon}\,\|\mathbf{u}_0\|_{0,\infty}\Big\}$$

$$\nabla \cdot \mathbf{u}_h^{\mathsf{M}} = \mathcal{P}_{W_h} f \quad \text{and} \quad \|\nabla \cdot (\mathbf{u}_\epsilon - \mathbf{u}_h^{\mathsf{M}})\|_0 \leq C\|f\|_1 h$$

Proof: $\boxed{\mathbf{u}_\epsilon \approx \pi_\epsilon^{\mathsf{M}} \mathbf{u}_0 \in \mathbf{V}_h^{\mathsf{M}}}$ and $\nabla \cdot \pi_\epsilon^{\mathsf{M}} \mathbf{u}_0 = \mathcal{P}_{W_h} \nabla \cdot \mathbf{u}_0 = \mathcal{P}_{W_h} \mathbf{u}_\epsilon$

$$\|\mathbf{u}_\epsilon - \mathbf{u}_h^{\mathsf{M}}\|_0 \leq C\|\mathbf{u}_\epsilon - \pi_\epsilon^{\mathsf{M}} \mathbf{u}_0\|_0$$

(2) Quasi-optimality

$$\leq C\Big\{\|\mathbf{u}_\epsilon - \mathcal{A}_\epsilon \mathbf{u}_0\|_0 + \|\mathcal{A}_\epsilon(\mathbf{u}_0 - \pi_0^{\mathsf{M}} \mathbf{u}_0)\|_0 + \|\mathcal{A}_\epsilon \pi_0^{\mathsf{M}} \mathbf{u}_0 - \pi_\epsilon^{\mathsf{M}} \mathbf{u}_0\|_0\Big\}$$

(1) Homogenization (4) Smooth Proj. (3) Multiscale Proj.

Divergence result follows trivially from the definitions.
Pressure result follows from the inf-sup condition. □

Conclusions

Conclusions

1. Dual-support elements must be defined and used with care.
   - MD elements do *not* converge in any reasonable sense in the presence of anisotropy.
   - Anisotropy almost always arises from the microstructure in heterogeneous problems.
   - However, experience suggests that MD elements work well in a practically reasonable range of parameters ϵ and h.
2. A new approach was given for defining HD dual-support elements.
   - Based on the microscale structure from homogenization theory.
   - They use an anisotropy scaling factor.
3. Multiscale convergence results were given.
   - A simplified proof was presented.
   - Multiscale convergence for standard ME elements and new HD dual-support elements.

Happy Birthday Alain!