A posteriori error estimators for a model for flow in a porous medium with fractures

Zoubida MGHAZLI

(LIRNE, EIMA, Université Ibn Tofail, Kénitra, Morocco)

Jean E. ROBERTS

(INRIA-Roquencourt, France)

Ali SAADA

(LAMSIN-ENIT,Tunisia)

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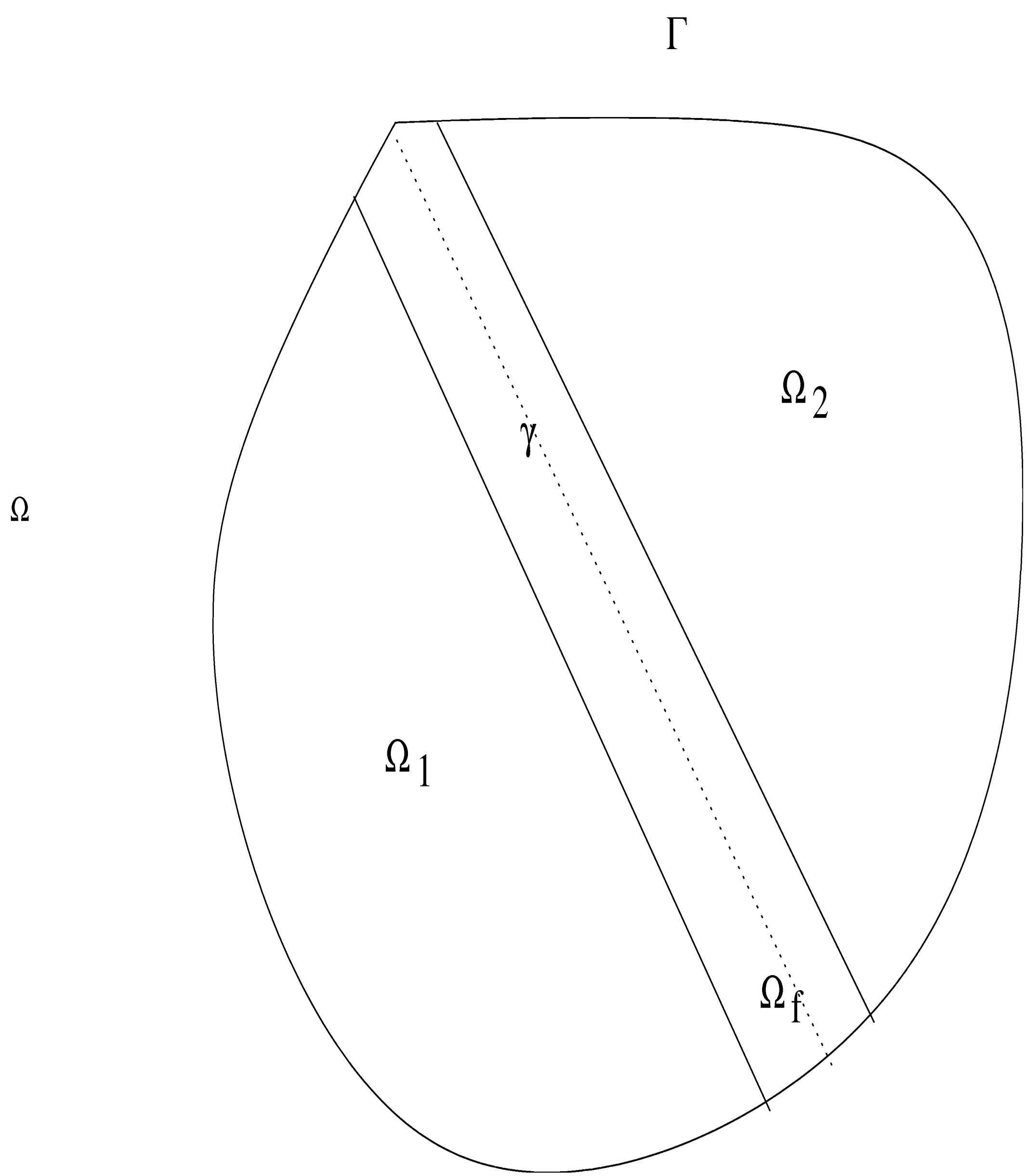
I) Introdution

$\Omega \subset \mathbb{R}^n$, $n = 2, 3$, $\bar{\Omega}_f$ = fracture; $\Omega \backslash \bar{\Omega}_f = \Omega_1 \cup \Omega_2$, $\quad \Omega_1 \cap \Omega_2 = \emptyset$;
$\Omega_f \equiv \gamma$: hyperplane; $\Gamma = \partial\Omega$, $\Gamma_i := \partial\Omega \cap \Omega_i$, $i = 1, 2$; $Z_{[2]}$ is so that 1+1=2 and 2+1=1, (cf V.Martin, J. Jaffré, J. Roberts, 2005)

$$
\left\{
\begin{array}{rll}
u_i & = -K_i \nabla p_i & \text{in } \Omega_i, \quad i = 1, 2 \\
\text{div } u_i & = q_i & \text{in } \Omega_i, \quad i = 1, 2 \\
u_f & = -K_f \nabla_\tau p_f & \text{on } \gamma, \\
\text{div}_\tau u_f & = q_f + \sum_{i=1}^{2} u_i \cdot n_i & \text{on } \gamma, \\
p_i & = p_f + \dfrac{d}{2K_f}[\xi u_i \cdot n_i & \quad (\xi \in]1/2, 1]) \\
 & \quad -(1-\xi) u_{i+1} \cdot n_{i+1}] & \text{on } \gamma, \quad i \in \mathbb{Z}_{[2]} \\
p_i & = \bar{p}_i & \text{on } \Gamma_i, \quad i = 1, 2 \\
p_f & = \bar{p}_f & \text{on } \partial\gamma.
\end{array}
\right.
$$


Γ
Ω
Ω2
γ
Ω1
Ωf

Weak Formulation

$$\begin{cases} \quad u \in W, \quad p \in M \\ a_\xi(u,v) - \beta(v,p) = -L_d(v), \quad \forall v \in W \\ \beta(u,r) = L_q(r), \quad \forall r \in M. \end{cases}$$

$$\begin{aligned}
W &= \{v = (v_1, v_2, v_f) \in \Pi_{i=1}^2 \mathcal{H}(div; \Omega_i) \times \mathcal{H}(div; \gamma) : v_i \cdot n_i \in L^2(\gamma), i = 1, 2\} \\
M &= \{r = (r_1, r_2, r_\gamma) \in L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\gamma)\} \\
a_\xi(u,v) &= \sum_{i=1}^{2} (K_i^{-1} u_i, v_i)_{\Omega_i} + (K_f^{-1} u_f, v_f)_\gamma \\
&\quad + \sum_{i=1}^{2} \left(\frac{d}{2K_f} \left[\xi u_i \cdot n_i - (1-\xi) u_{i+1} \cdot n_{i+1} \right], v_i \cdot n_i \right)_\gamma, \\
\beta(u,r) &= \sum_{i=1}^{2} (\text{div } u_i, r_i)_{\Omega_i} + (\text{div}_\tau u_f, r_f)_\gamma - \left(\sum_{i=1}^{2} u_i \cdot n_i, r_f \right)_\gamma, \\
L_q(r) &= \sum_{i=1}^{2} (q_i, r_i)_{\Omega_i} + (q_f, r_f)_\gamma, \quad L_d(v) = \sum_{i=1}^{n} (v_i \cdot n_i, \bar{p}_i)_{\Gamma_i} + (v_f \cdot n_f, \bar{p}_f)_{\partial\gamma}.
\end{aligned}$$

Norms in M and W

$$\|r\|_M^2 = \sum_{i=1}^{2} \|r_i\|_{0,\Omega_i}^2 + \|r_f\|_{0,f}^2$$

$$\|v\|_W^2 = \sum_{i=1}^{2} \left(\|v_i\|_{0,\Omega_i}^2 + \|div\, v_i\|_{0,\Omega_i}^2 \right) + \|v_f\|_{0,\gamma}^2 + \|div_\tau\, v_f\|_{0,\gamma}^2 + \sum_{i=1}^{2} \|v_i \cdot n_i\|_{0,\gamma}^2.$$

Unique Solution : $a_\xi(\cdot,\cdot)$ is $\tilde{W}-$elliptic:

$$\exists C_\alpha > 0, \qquad \inf_{v\in\tilde{W}} \frac{a_\xi(v,v)}{\|v\|_W^2} \geq C_\alpha$$

$$\tilde{W} = \{v \in W : \beta(v,r) = 0 \ \forall r \in M\}$$

$\beta(\cdot,\cdot)$ satisfies the inf-sup condition:

$$\exists C_\beta > 0, \qquad \inf_{r\in M} \sup_{v\in W} \frac{\beta(v,r)}{\|v\|_W \|r\|_M} \geq C_\beta.$$

(cf. V.Martin, J. Jaffré, J. Roberts, 2005)

Discretization with RT_0

$$\mathbf{Z}_i = \boldsymbol{H}(\boldsymbol{div};\Omega_i), \quad i = 1,2, \quad \mathbb{Z} = \bigoplus_{\mathrm{i=1,2}} \mathbf{Z}_\mathrm{i}$$

$$\boldsymbol{N}_i = \boldsymbol{L}^2(\Omega_i), \quad i = 1,2, \quad \mathbb{N} = \bigoplus_{i=1,2} \boldsymbol{N}_i = \boldsymbol{L}^2(\Omega).$$

$$\mathbf{Z}_{h,i} \times \boldsymbol{N}_{h,i} \subset \mathbf{Z}_i \times \boldsymbol{N}_i, \quad i = 1,2$$

$$\boldsymbol{W}_{h,\gamma} \subset \boldsymbol{H}(\boldsymbol{div}_\tau;\gamma), \qquad \Lambda_h = \boldsymbol{P}_0(\gamma)$$

$$\boldsymbol{W}_h = \bigoplus_{i=1,2} \mathbf{Z}_{h,i} \oplus \boldsymbol{W}_{h,\gamma}, \qquad \boldsymbol{M}_h = \bigoplus_{i=1,2} \boldsymbol{N}_{h,i} \oplus \Lambda_h.$$

$$\begin{cases} \boldsymbol{u}_h \in \boldsymbol{W}_h, \quad \boldsymbol{p}_h \in \boldsymbol{M}_h \\ \boldsymbol{a}_\xi(\boldsymbol{u}_h, \boldsymbol{v}) - \boldsymbol{\beta}(\boldsymbol{v}, \boldsymbol{p}_h) = -\boldsymbol{L}_d(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{W}_h \\ \boldsymbol{\beta}(\boldsymbol{u}_h, \boldsymbol{r}) = \boldsymbol{L}_q(\boldsymbol{r}), \quad \forall \boldsymbol{r} \in \boldsymbol{M}_h. \end{cases}$$

$\Pi_h : \boldsymbol{W} \longrightarrow \boldsymbol{W}_h$ interpolation operateur in $\boldsymbol{RT}_0$ and $\boldsymbol{P}_h : \boldsymbol{M} \longrightarrow \boldsymbol{M}_h$ is the $\boldsymbol{L}^2$-projection.

II) A posteriori error estimate

Introduction

→ *a priori* error estimates $\| \boldsymbol{u} - \boldsymbol{u}_h \| \leq \boldsymbol{F}(\boldsymbol{h}, \boldsymbol{u}, \boldsymbol{f})$

→ only yield information on the asymptotic error behaviour

→ require regularity assumptions about $\boldsymbol{u}$ which is not satisfied in the presence of singularity (as sharp fronts, wells,...)

→ overall accuracy of the numerical approximation is deteriored by local singularity

Obvious remedy:

→to refine the discretization near the critical regions = to place more grid point where the solution is less regular.

→how to identify those regions,

→ how to obtain a good balance between the refined and unrefined regions such that the overall accuracy is "optimal".

Obtain reliable estimates of the accuracy of the computed numerical solution

The need: error estimator which can, *a posteriori*, be extracted from the computed numerical solution and the given data of the problem

Reasonable error estimator should at least satisfy three minimal requirements

Reliability = estimator yields upper bounds on the error

$$\|u - u_h\|_V \leq G(u_h, f_h, f)$$

u_h is the computed solution.

Efficiency = estimator yields lower bounds on the error

$$G(u_h, f_h, f) \leq C\|u - u_h\|_V$$

Locality= information on the local distribution of the error.

$$G(u_h, f_h, f) = (\sum_{T \in \mathcal{T}_h} \eta_T^2)^{1/2}, \qquad \eta_T \leq C\|u - u_h\|_{W(T)}$$

η_T : Local error estimator

Principal tools

$$e_h = u - u_h, \qquad \varepsilon_h = p - p_h.$$

The residual equations

$$\left\{ \begin{aligned} a_\xi(e_h, v) - \beta(v, \varepsilon_h) &= -L_d(v) - a_\xi(u_h, v) + \beta(v, p_h), && \forall v \in W \\ \beta(e_h, r) &= \ \ L_q(r) - \beta(u_h, r), && \forall r \in M. \end{aligned} \right.$$

The orthogonality property

$$\left\{ \begin{aligned} a_\xi(e_h, v_h) - \beta(v_h, \varepsilon_h) &= 0, \ \forall v_h \in W_h \\ \beta(e_h, r_h) &= 0, \ \forall r_h \in M_h. \end{aligned} \right.$$

Local interpolation error bounds on elements and faces

Residual error estimators

Notations: $\mathcal{T}_{ih}$: for i=1,2, regular families of triangulations of Ω_i by closed triangles ($n = 2$) or tetrahedra ($n = 3$) and $\mathcal{T}_{fh}$: regular families of triangulations of the hyperplane γ by closed segments ($n = 2$) or triangles ($n = 3$). $\mathcal{E}_{ih}$: the set of the edges or faces E of $\mathcal{T}_{ih}$,and $\mathcal{E}_{ih}^0 := \mathcal{E}_{ih} \setminus (\Gamma_i \cup \gamma)$, $\mathcal{E}_{ih}^\Gamma := \mathcal{E}_{ih} \cap \Gamma_i$, $\mathcal{E}_{ih}^\gamma := \mathcal{E}_{ih} \cap \gamma$, $\mathcal{E}_{fh}$ the set of the edges of $\mathcal{T}_{fh}$, $\mathcal{E}_{fh}^0 := \mathcal{E}_{fh} \setminus \partial\gamma$.

For $w_h \in W_h$ the tangential component of w_h on a face E is defined by

$$(w_h)_{\tau,E} := \begin{cases} w_h \cdot t_E & \text{in 2D} \\ w_h \times n_E & \text{in 3D} \end{cases}$$

and, if $[\cdot]_E$ denote the jump across E, we define the jump of $K^{-1}v_h$ in the tangential direction across E by

$$J_{t,E}(u_h) := \begin{cases} \frac{1}{2}[(K^{-1}u_h)_{\tau,E}]_E & \text{if } E \cap \Gamma_i = \emptyset \\ \left(K^{-1}u_h + \nabla\left(P_h \bar{p}_i\right)\right)_{\tau,E} & \text{if } E \cap \Gamma_i \neq \emptyset, \end{cases}$$

For any $T \in \mathcal{T}_{i,h}$, for $i = 1, 2$, and for any $t \in \mathcal{T}_{f,h}$,

$$\eta_T^{(i)} := h_T \| K_i^{-1} u_{ih} + \nabla p_{ih} \|_{0,T} \qquad \eta_t := h_t \| K_f^{-1} u_{fh} + \nabla_\tau p_{fh} \|_{0,t}$$

$$\zeta_T^{(i)} := h_T \| curl(K_i^{-1} u_{h,i}) \|_{0,T} + \sum_{E \in \partial T} h_E^{1/2} \| J_{t,E}(u_{h,i}) \|_{0,E}$$

$$\zeta_t := h_t \| curl(K_f^{-1} u_{h,f}) \|_{0,t} + \sum_{e \in \partial t} h_e^{1/2} \| J_{t,e}(u_{h,f}) \|_{0,e}$$

$$\omega_T^{(i)} := \| (q_i - P_{hi} q_i) \|_{0,T} \qquad \omega_t := \| (q_f - P_{hf} q_f) \|_{0,t}$$

$$\overline{\omega}_T^{(i)} := h_T^{1/2} \| P_h \bar{p}_i - \bar{p}_i \|_{0,\partial T \cap \Gamma_i} \qquad \overline{\omega}_t := h_t^{1/2} \| P_h \bar{p}_f - \bar{p}_f \|_{0,\partial t \cap \partial \gamma}$$

$$\delta_t^{(i)} := h_t^{1/2} \| p_{ih} - p_{fh} - \frac{d}{2K_f} \left(\xi u_{ih} \cdot n_i - (1 - \xi) u_{i+1,h} \cdot n_{i+1} \right) \|_{0,t}$$

Upper bound on the velocity

Proposition 1 : *The error e_u is bounded from above by the indicators*

$$\| e_u \|_W \preceq \sum_{i=1}^{2} \sum_{T \in \mathcal{T}_{ih}} \left(\zeta_T^{(i)} + \omega_T^{(i)} + \overline{\omega}_T^{(i)} \right) + \sum_{t \in \mathcal{T}_{fh}} (\zeta_t + \omega_t + \overline{\omega}_t) \qquad (1)$$

Proof:

for i=1,2,f (as in Hoppe and Wohlmuth (1999))

$$\boldsymbol{H}(div;\Omega_i) = \boldsymbol{H}^0(div;\Omega_i) \oplus \boldsymbol{H}^1(div;\Omega_i)$$

where

$$\begin{aligned}\boldsymbol{H}^0(div;\Omega_i) &:= \{\boldsymbol{v}_i \in \boldsymbol{H}(div;\Omega_i), \text{s.t.} \\ &\qquad \text{div } \boldsymbol{v}_i = \boldsymbol{0}, \text{ and } (\boldsymbol{v}_i \cdot \boldsymbol{n}_i)|_\gamma = \boldsymbol{0}\,\} \\ \boldsymbol{H}^1(div;\Omega_i) &:= \{\boldsymbol{v}_i \in \boldsymbol{H}(div;\Omega_i), \text{s.t. };\\ &\qquad \int_{\Omega_i} \boldsymbol{K}_i^{-1}\boldsymbol{w}_i^0\boldsymbol{v}_i d\boldsymbol{x} = \boldsymbol{0}, \forall \boldsymbol{w}_i^0 \in \boldsymbol{H}^0(div;\Omega_i)\}.\end{aligned}$$

For n=2 we have $\boldsymbol{H}^0(div;\gamma) = \{\boldsymbol{0}\}$ and so $\boldsymbol{H}(div;\gamma) = \boldsymbol{H}^1(div;\gamma) = \boldsymbol{H}^1(\gamma)$.

Notations

$$\boldsymbol{W}^0 = \boldsymbol{H}^0(\boldsymbol{div}; \Omega_1) \times \boldsymbol{H}^0(\boldsymbol{div}; \Omega_2) \times \boldsymbol{H}^0(\boldsymbol{div}_{\boldsymbol{\tau}}; \gamma)$$

$$\boldsymbol{W}^1 = \boldsymbol{H}^1(\boldsymbol{div}; \Omega_1) \times \boldsymbol{H}^1(\boldsymbol{div}; \Omega_2) \times \boldsymbol{H}^1(\boldsymbol{div}_{\boldsymbol{\tau}}; \gamma)$$

$$\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^1$$

$$\boldsymbol{u}^0 \in \boldsymbol{W}^0, \quad a_{\boldsymbol{\xi}}(\boldsymbol{u}^0, \boldsymbol{v}^0) = -L_{\boldsymbol{d}}(\boldsymbol{v}^0), \qquad \forall \boldsymbol{v}^0 \in \boldsymbol{W}^0, \tag{2}$$

$$\boldsymbol{u}^1 \in \boldsymbol{W}^1, \quad \beta(\boldsymbol{u}^1, r) = L_q(r), \qquad \forall r \in M. \tag{3}$$

The problems (2) and (3) are independent, and we check successively bounds of $\boldsymbol{e}_{\boldsymbol{u}}^0$ and $\boldsymbol{e}_{\boldsymbol{u}}^1$.

- $v^0 = \text{curl } \phi \in W^0$, $\phi_h := P_h^C \phi$, and $v_h^0 = curl \phi_h$

$$
\begin{aligned}
a_\xi(e_u, v^0) &= -L_d(v^0 - v_h^0) - a_\xi(u_h, v^0 - v_h^0) \\
&= \sum_{i=1}^{2} \left\{ \sum_{T \in \mathcal{T}_{ih}} \left(\int_T curl(K_i^{-1} u_{h,i})(\phi_i - \phi_{ih}) + \sum_{E \subset \partial T} \int_E J_{t,E}(u_{h,i})(\phi_i - \phi_{ih}) \right) \right\} \\
&\quad + \sum_{t \in \mathcal{T}_{fh}} \left(\int_t curl(K_f^{-1} u_{h,f})(\phi_f - \phi_{fh}) + \sum_{e \in \partial t} \int_e J_{t,e}(u_{h,f})(\phi_f - \phi_{fh}) \right) \\
&\quad + \sum_{i=1}^{2} (curl(\phi_i - \phi_{ih}) \cdot n_i, (\bar{p}_i - \bar{p}_{ih})_{\Gamma_i} \\
&\quad + (curl(\phi_f - \phi_{fh}) \cdot n_f, \bar{p}_f - p_{fh})_{\partial\gamma}.
\end{aligned}
$$

$v^0 = e_u^0$ gives

$$
\|e_u^0\|_0 = \|e_u^0\|_{div} \preceq \sum_{i=1}^{2} \sum_{T \in \mathcal{T}_{ih}} \left(\zeta_T^{(i)} + \bar{\omega}_T^{(i)} \right) + \sum_{t \in \mathcal{T}_{fh}} \left(\zeta_t + \bar{\omega}_t \right).
$$

- From the approximate problem we have

$$\boldsymbol{div u_{ih} = P_{ih} q_i} \text{ for } \boldsymbol{i = 1, 2}$$

$$\boldsymbol{div u_{fh} = P_{fh} \left(\sum_{i=1}^{2} u_{ih} \cdot n_i + q_f \right)}$$

$$\boldsymbol{\left(\sum_{i=1}^{2} \| div e^1_{u_i} \|^2_{0,\Omega_i} + \| div e^1_{u_f} \|^2_{0,\Omega_f} + \| \sum_{i=1}^{2} e^1_{u_i} \cdot n_i \|^2_{0,\gamma} \right)^{1/2} \preceq \sum_{T \in \mathcal{T}_{ih}} \omega^{(i)}_T + \sum_{t \in \mathcal{T}_{fh}} \omega_t}$$

Upper bound on the pressure

Proposition 2 *The error ε_p is bounded from above by the indicators*

$$\|\varepsilon_p\|_W \preceq \sum_{i=1}^{2} \left\{ \sum_{T\in\mathcal{T}_{ih}} \left(\eta_T^{(i)} + \omega_T^{(i)} + \overline{\omega}_T^{(i)}\right) + \sum_{t\in\mathcal{T}_{fh}} \delta_t^{(i)} \right\}$$

$$\sum_{t\in\mathcal{T}_{fh}} (\eta_t + \omega_t + \overline{\omega}_t)$$

Proof:

Dual problem (for the pressure)

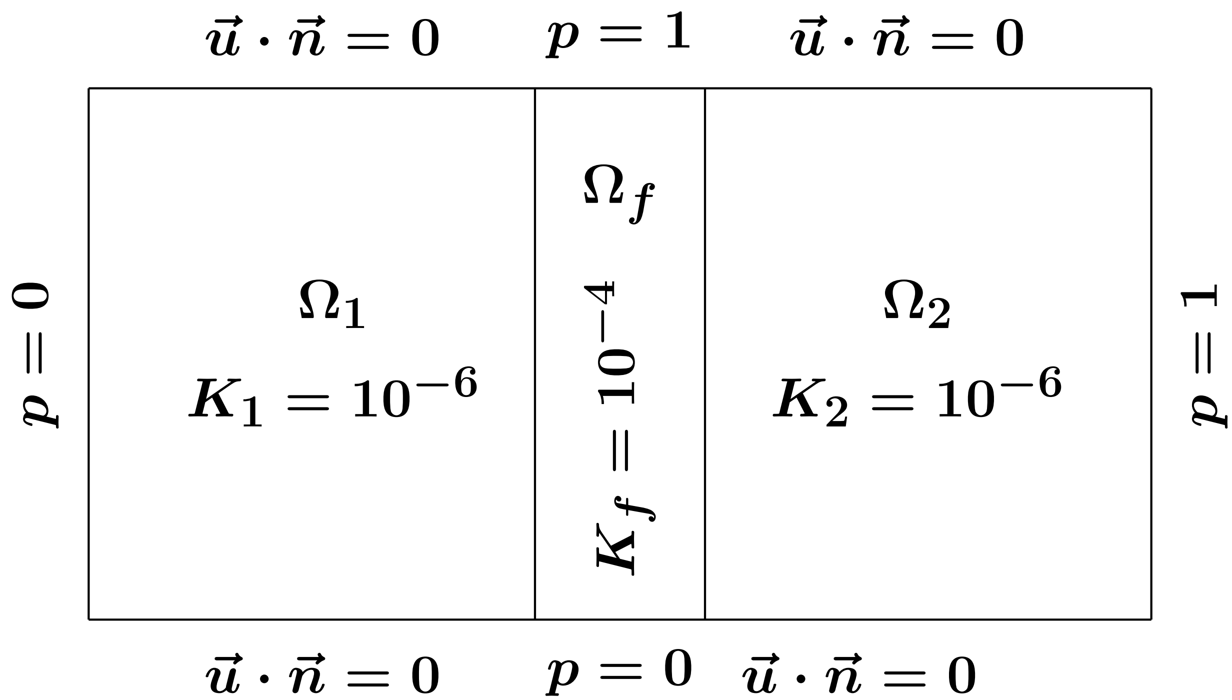
$$\left\{ \begin{array}{rll} \text{Find} & \bar{v} \in W, & \bar{r} \in M \text{ solution of} \\ a_\xi(\bar{v}, v) - \beta(v, \bar{r}) & = 0, & \forall v \in W \\ \beta(\bar{v}, r) & = -(\varepsilon_p, r), & \forall r \in M. \end{array} \right.$$

Regularity result

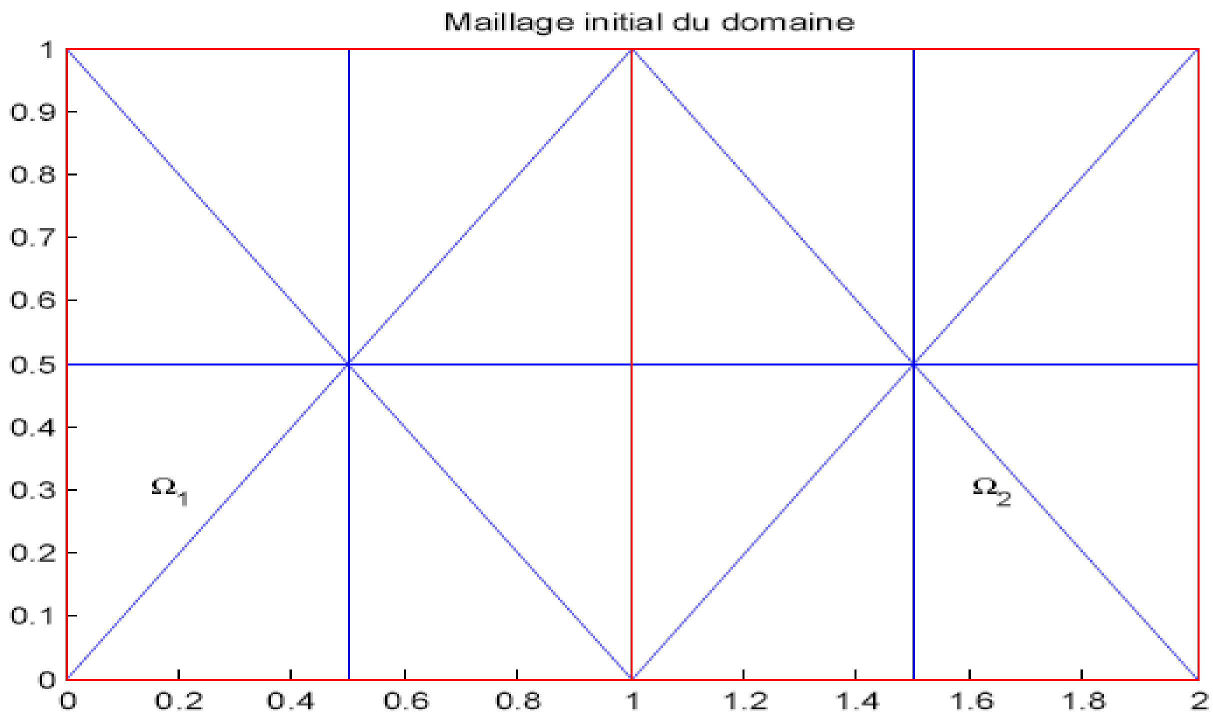
$$\exists C_s > 0, \qquad \|\bar{v}\|_1 + \|\bar{r}\|_1 \leq C_s \|\varepsilon_h\|_M.$$

Lower bound on the error by standards arguments

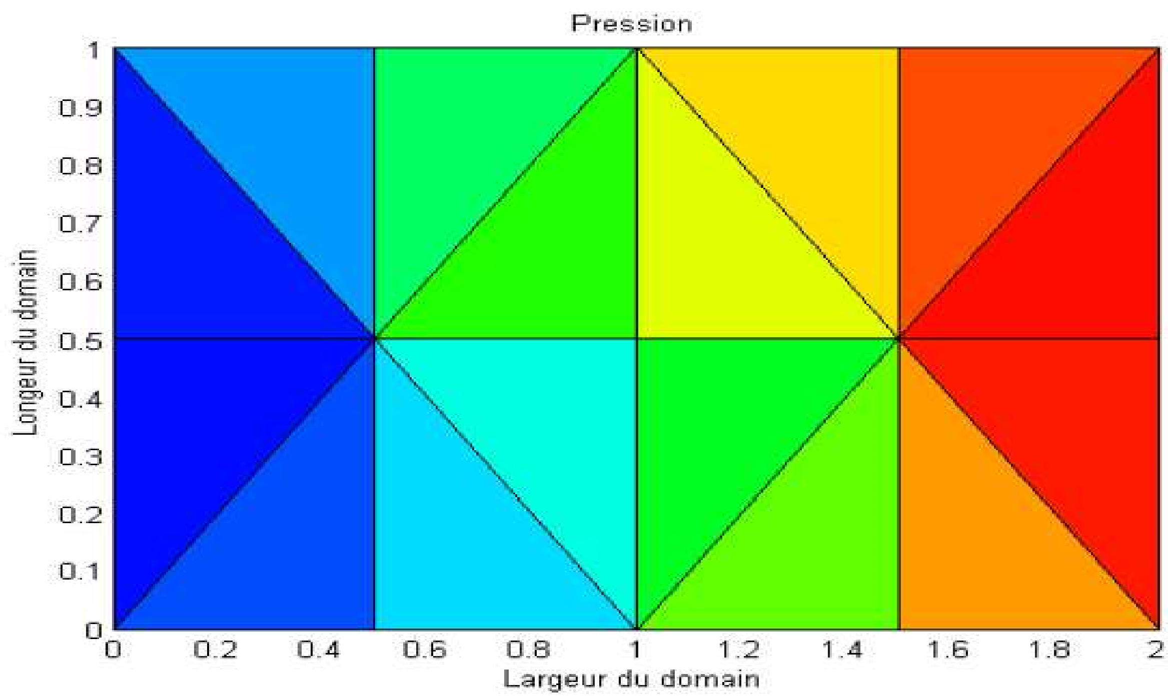
III) A Numerical test in 2D



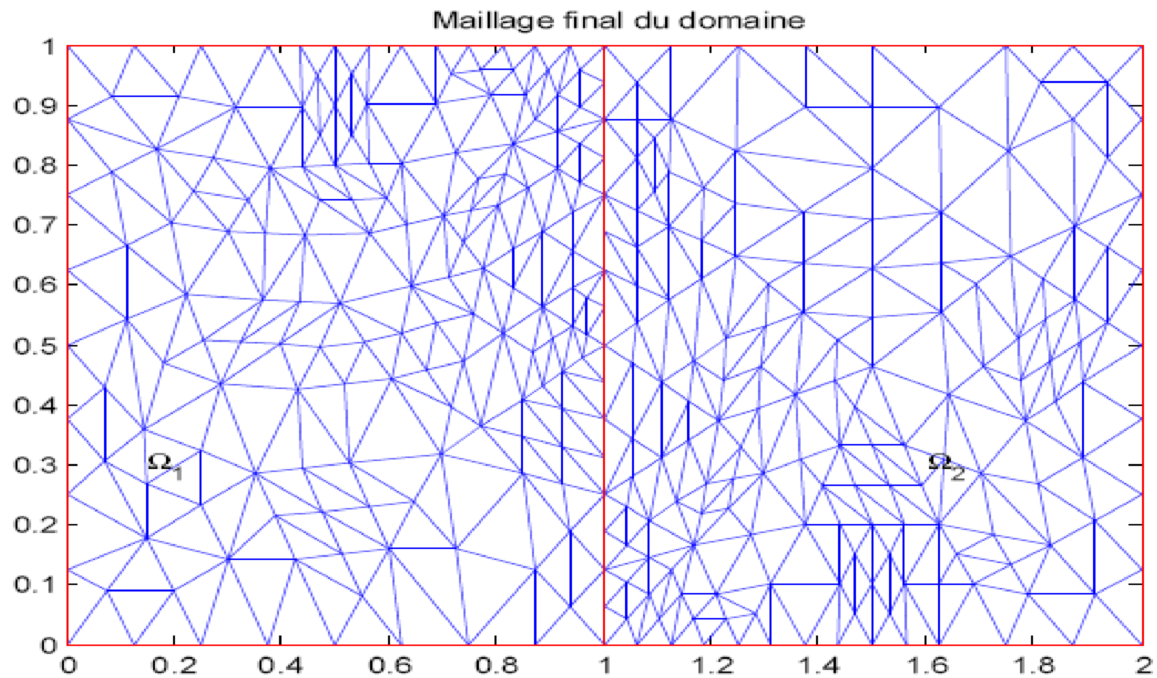
d=0,01.



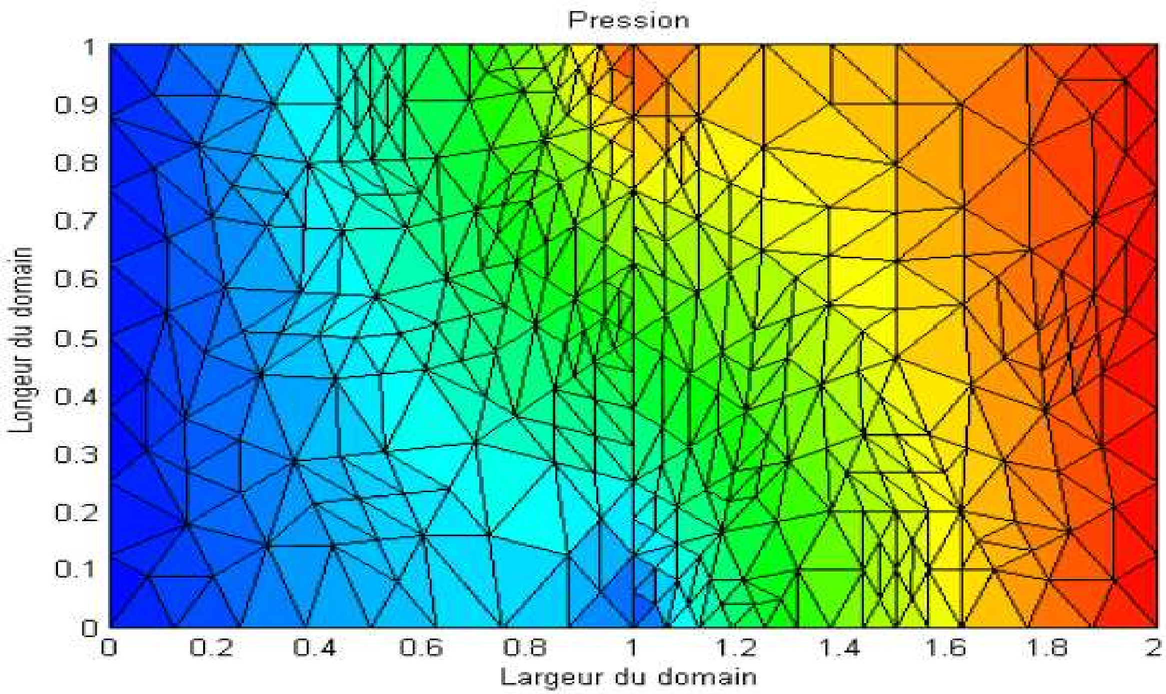
Initial mesh



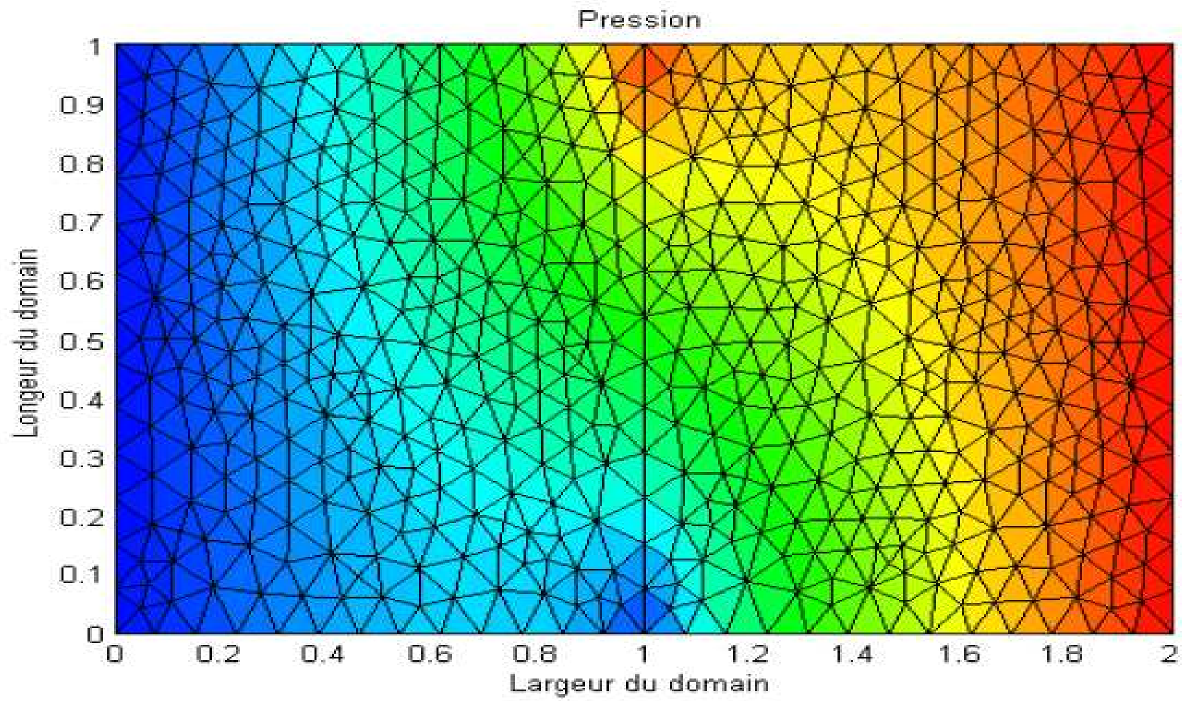
Initial pressure



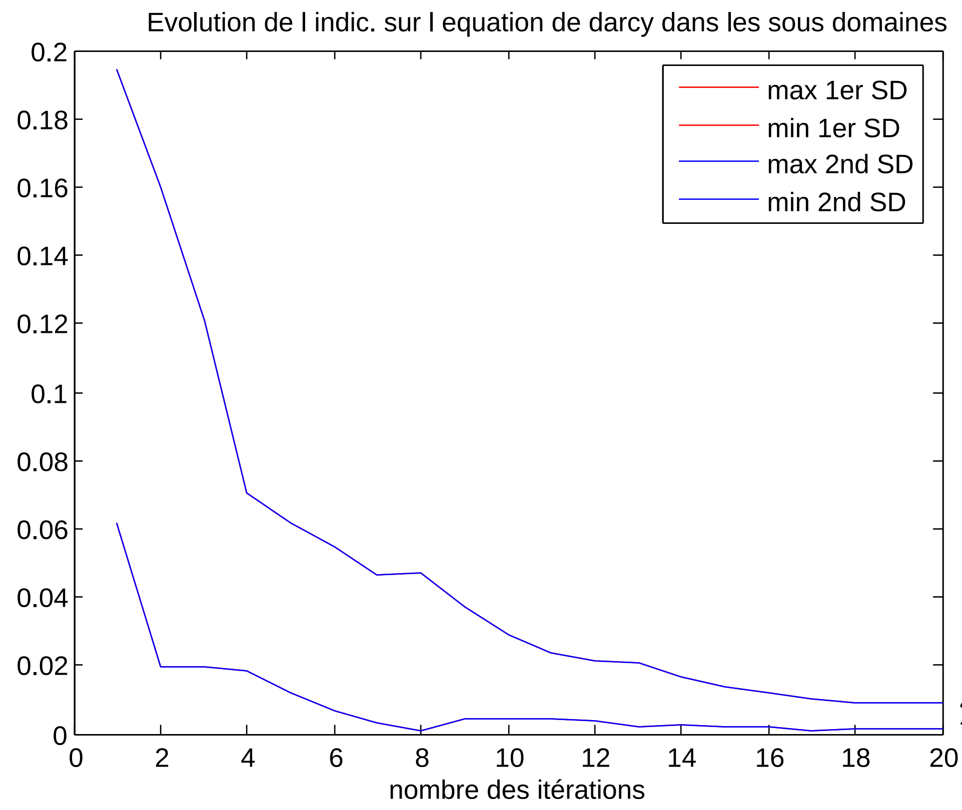
Final mesh: 320 elements 500 edges, in each Ω_i

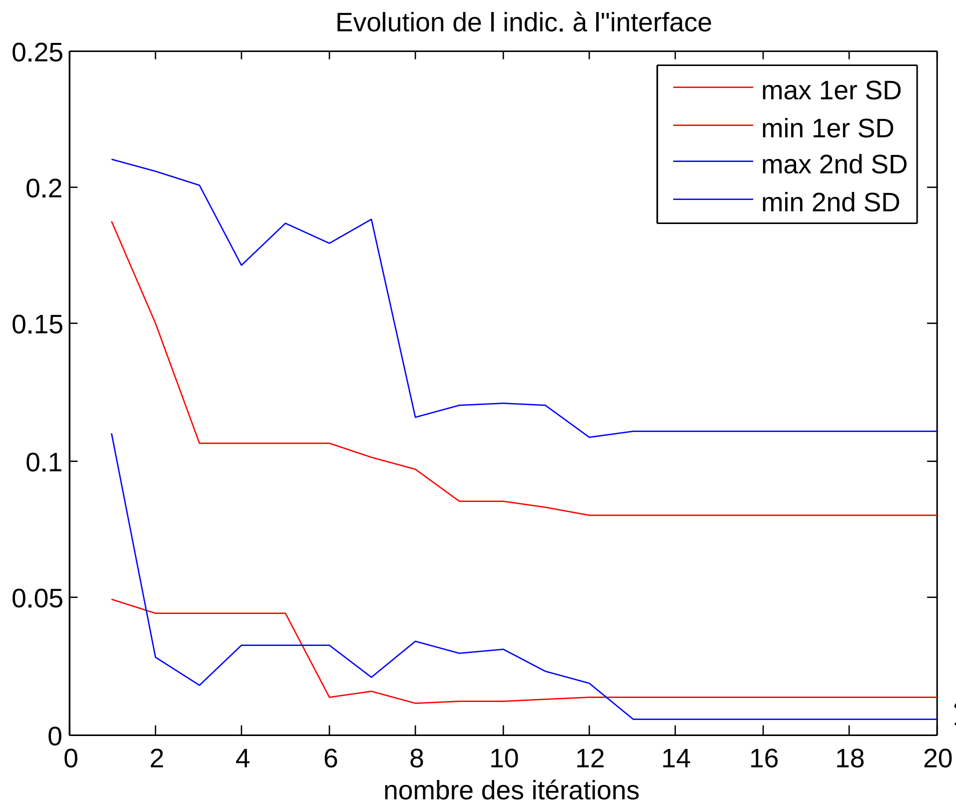


Final pressure

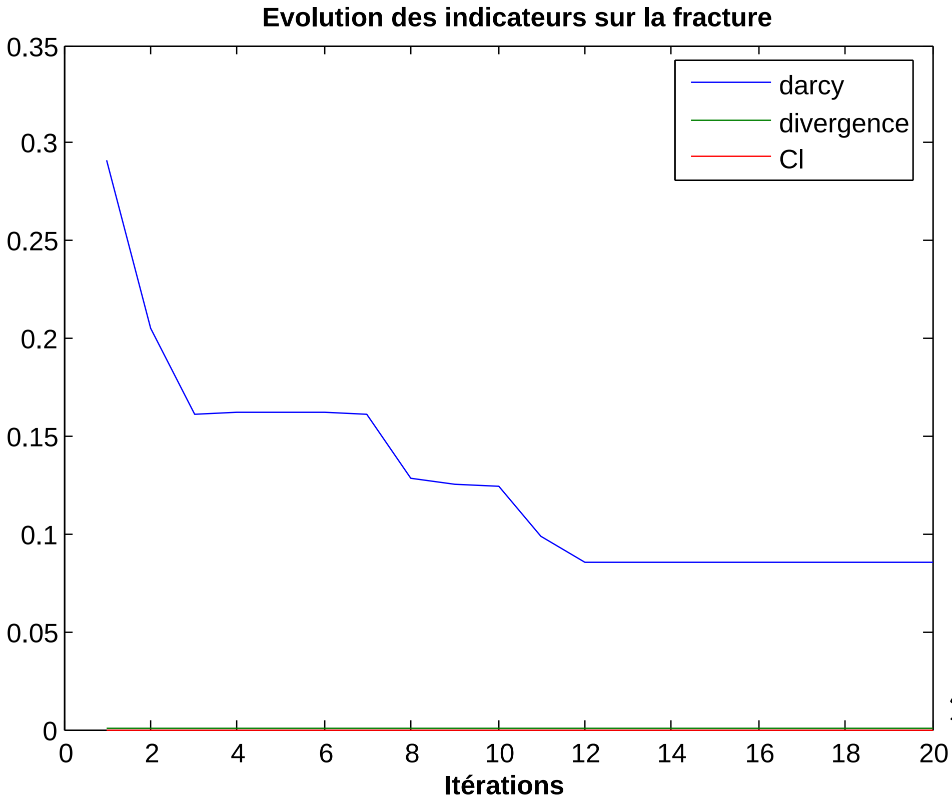


Pressure with uniform mesh: 520 elements, 800 edges in each Ω_i


Evolution de l indic. sur l equation de darcy dans les sous domaines
max 1er SD
min 1er SD
max 2nd SD
min 2nd SD
nombre des itérations
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0
2
4
6
8
10
12
14
16
18
20



Evolution de l indic. à l''interface
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
12
14
16
18
20
nombre des itérations
max 1er SD
min 1er SD
max 2nd SD
min 2nd SD


Evolution des indicateurs sur la fracture
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
12
14
16
18
20
Itérations
darcy
divergence
Cl