Discontinuous Galerkin and Nonconforming in Time Optimized Schwarz Waveform Relaxation for Coupling Heterogeneous Problems

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Outline

1. Motivation: Application for nuclear waste disposal
2. Subdomain time stepping with nonconforming time grids
3. Numerical results
4. Conclusions

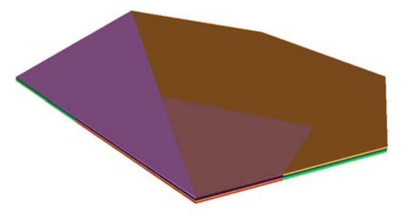
Outline

Far field 3D : The computational domain

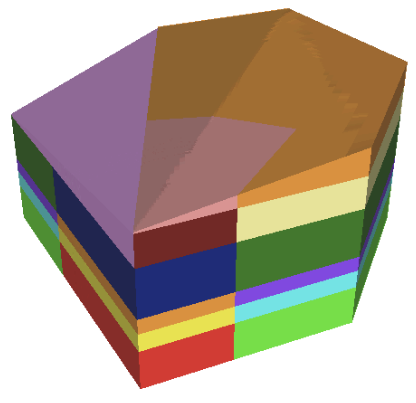
Research Group MOMAS

with Jérôme Jaffré, Michel Kern and Jean Roberts (INRIA)

Actual dimensions:
$40km \times 40km \times 500m$



A blow-up in the vertical direction (30 times)



The repository is located in the red part of the bottom layer.

Hydrogeological data

Hydrogeologic layers	Thickness [m]	Porosity [%]	Permeability [m/s]		Effective diffusion coefficient [m^2/s]	Dispersivity Coefficients [m]
			Regional	Local		
Tithonian	Variable	10	3.10^{-5}	3.10^{-5}	10^{-9}	6.0, 0.6
Kimmeridgian when it outcrops	Variable	10	3.10^{-4}	3.10^{-4}	10^{-9}	6.0, 0.6
Kimmeridgian under cover			10^{-11}	10^{-12}		
Oxfordian L2a-L2b	165	6	$2\ 10^{-7}$	10^{-9}	10^{-9}	6.0, 0.6
Oxfordian Hp1-Hp4	50	18	$6\ 10^{-7}$	$8\ 10^{-9}$	10^{-9}	1600, 30
Oxfordian C3a-C3b	60	1	10^{-10}	10^{-12}	4.10^{-12}	6.0, 0.6
Callovo-Oxfordian Cox	135	1	$K_v=10^{-14}$ $K_h=10^{-12}$		4.10^{-12}	6.0, 0.6

Nuclear waste disposal

$\Rightarrow$ **use different time and space steps, adapted to the physics**

Goal :

- decompose the time interval into windows
- in each window:
  use an Optimized Schwarz Waveform Relaxation method
  with non conforming space-time grids
  and discontinuous Galerkin method in time as subdomain solver

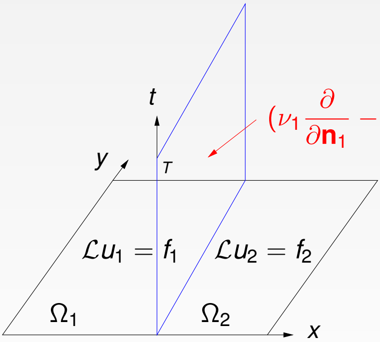
Outline

Advection-diffusion equation with discontinuous coefficients

$$\mathcal{L}u = \frac{\partial u}{\partial t} + bu + \nabla \cdot (\mathbf{a}(\mathbf{x})u - \nu(\mathbf{x})\nabla u) = f \text{ in } \Omega \times [0, T]$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T], \quad u(.,0) = u_0, \quad \text{on } \partial\Omega$$

with $\mathbf{a}(\mathbf{x})$ and $\nu(\mathbf{x})$ discontinuous, $\nu(\mathbf{x}) > 0$



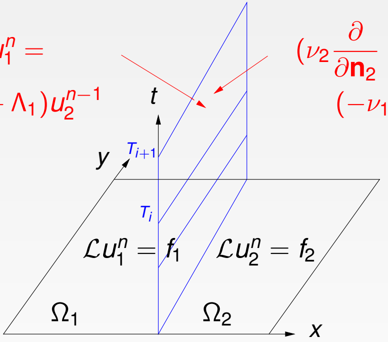
$$u_1 = u_2$$

$$(\nu_1 \frac{\partial}{\partial \mathbf{n_1}} - \mathbf{a}.\mathbf{n_1})u_1 = (-\nu_2 \frac{\partial}{\partial \mathbf{n_2}} + \mathbf{a}.\mathbf{n_2})u_2$$

Optimized Schwarz Waveform Relaxation Method

(Gander/Halpern/Nataf (DD11, 1998), Martin (2003), Bennequin/Gander/Halpern (2004), Gander/Halpern/Kern (2004), Blayo/Halpern/Japhet (2004))

$$(\nu_1 \frac{\partial}{\partial \mathbf{n}_1} - \mathbf{a}.\mathbf{n_1} + \Lambda_1) u_1^n = (-\nu_2 \frac{\partial}{\partial \mathbf{n}_2} + \mathbf{a}.\mathbf{n_2} + \Lambda_1) u_2^{n-1}$$



$$(\nu_2 \frac{\partial}{\partial \mathbf{n}_2} - \mathbf{a}.\mathbf{n_2} + \Lambda_2) u_2^n = (-\nu_1 \frac{\partial}{\partial \mathbf{n}_1} + \mathbf{a}.\mathbf{n_1} + \Lambda_2) u_1^{n-1}$$

Choose Λ_1 and Λ_2 in order to optimize the convergence rate

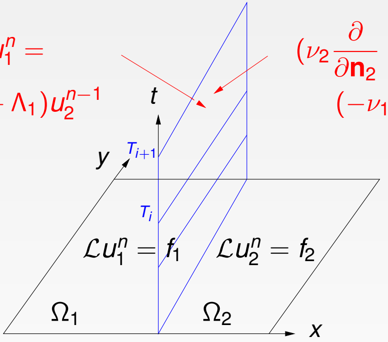
Optimized Schwarz Waveform Relaxation Method

$$\Lambda_1 = \alpha_2 + \beta_2(\partial_t + \mathbf{a}.\tau_{\mathbf{2}}\partial_{\tau_2} - \partial_{\tau_2}(\nu_2\partial_{\tau_2})), \quad \Lambda_2 = \alpha_1 + \beta_1(\partial_t + \mathbf{a}.\tau_{\mathbf{1}}\partial_{\tau_1} - \partial_{\tau_1}(\nu_1\partial_{\tau_1}))$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ optimize the convergence rate

$$(\nu_1 \frac{\partial}{\partial \mathbf{n}_1} - \mathbf{a}.\mathbf{n_1} + \Lambda_1)u_1^n = (-\nu_2 \frac{\partial}{\partial \mathbf{n}_2} + \mathbf{a}.\mathbf{n_2} + \Lambda_1)u_2^{n-1}$$

$$(\nu_2 \frac{\partial}{\partial \mathbf{n}_2} - \mathbf{a}.\mathbf{n_2} + \Lambda_2)u_2^n = (-\nu_1 \frac{\partial}{\partial \mathbf{n}_1} + \mathbf{a}.\mathbf{n_1} + \Lambda_2)u_1^{n-1}$$



How to discretize these conditions with nonmatching grids in time ?

Non conforming space-time method

Discontinuous Galerkin in time
(Eriksson-Johnson-Thomée, 1985, Halpern-Japhet, 2005)
Non conforming finite elements in space
(Gander-Japhet-Maday-Nataf, 2004)

Subdomain problem in Ω_j,
in one time window $I = (T_i, T_{i+1})$

$$\left\{ \begin{array}{rcll} \mathcal{L}u & = & f & \text{in } \Omega_j \times I, \\ u(\cdot, T_i) & = & u_0 & \text{in } \Omega_j, \\ (\nu \frac{\partial}{\partial \mathbf{n}} - \mathbf{a}.\mathbf{n} + \alpha + \beta(\frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau}(r \frac{\partial}{\partial \tau})))\, u & = & g & \text{on } \Gamma \times I \end{array} \right.$$

Weak Formulation

Let $H^s_s(\Omega) = \{v \in H^s(\Omega),\ v_{|\Gamma} \in H^s(\Gamma)\}$, equipped with

$$(u, v)_{H^s_s(\Omega)} = (u, v)_{H^s(\Omega)} + \beta(u, v)_{H^s(\Gamma)}$$

Find u such that

$$(\partial_t u, v)_{H^0_0(\Omega)} + a(u, v) = \ell(v), \quad \forall v \in H^1_1(\Omega)$$

with

$$\begin{cases} a(u, v) = \displaystyle\int_\Omega \nabla \cdot (\mathbf{a}u)v\, dx + \int_\Omega \nu \nabla u.\nabla v\, dx + \int_\Omega buv\, dx \\ \qquad\qquad + \displaystyle\int_\Gamma ((\alpha - \mathbf{a}.\mathbf{n})uv) + \beta\partial_\tau uv + \beta r \partial_\tau u \partial_\tau v)\, ds \\ \ell(v) = (f, v)_{L^2(\Omega)} + (g, v)_{L^2(\Gamma)} \end{cases}$$

Time Discontinuous Galerkin

Let $\mathcal{T}$ be a decomposition of $I = \cup_{k=1}^{K} I^k$ with $I^k = [t_k, t_{k+1}]$. We define

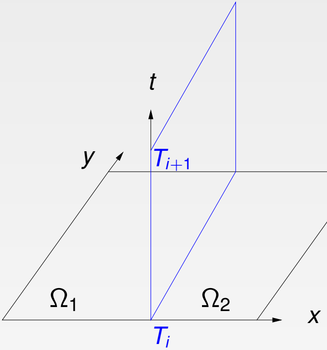
$$\begin{aligned} \mathbf{P}_q(V) &= \{\varphi : \ \varphi(t) = \sum_{i=0}^{q} \varphi_i t^i, \ \varphi_i \in V\} \\ \mathcal{P}_q(V, \mathcal{T}) &= \{\varphi : I \to V, \ \varphi_{|I_k} \in \mathbf{P}_q(V), \ 0 \leq k \leq K\}. \end{aligned}$$

Let $\varphi(t_k^{\pm}) = \lim_{t \to t_k \pm 0} \varphi(t)$

The discontinous Galerkin method defines recursively on I_k, an approximate solution U in $\mathcal{P}_q(H_1^1(\Omega), \mathcal{T})$ such that

$$\begin{cases} U(0, \cdot) = u_0, \\ \forall \varphi \in \mathcal{P}_q(H_1^1(\Omega), \mathcal{T}) : \quad \displaystyle\int_{I_k} [(\frac{dU}{dt}, \varphi)_{H_0^0(\Omega)} + a(U, \varphi)]\, dt \\ \qquad\qquad + ((U(t_k^+, \cdot) - U(t_k^-, \cdot), \varphi(t_k^+, \cdot)))_{H_0^0(\Omega)} = \displaystyle\int_{I_k} \ell(\varphi) dt \end{cases}$$

Domain decomposition



The continuous matching conditions for Ω_1 is

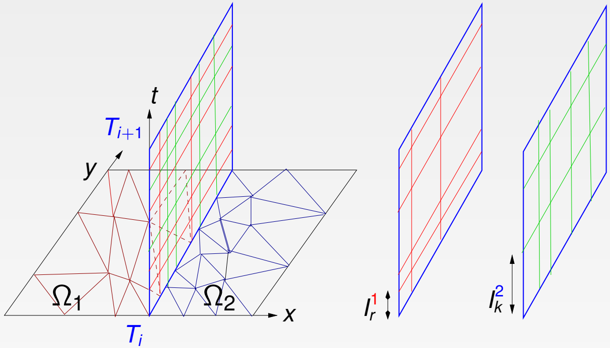
$$f_1(t) = g_2(t), \quad \forall t \in I$$

with :

$$f_1(t) = (\nu_1 \partial_{\mathbf{n}_1} - \mathbf{a}.\mathbf{n}_1 + \Lambda_1) u_1^n$$
$$g_2(t) = (-\nu_2 \partial_{\mathbf{n}_2} + \mathbf{a}.\mathbf{n}_2 + \Lambda_1) u_2^{n-1}$$

Projections between time grids

L^2 orthogonal projection on $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_1)$, restricted to $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_2)$

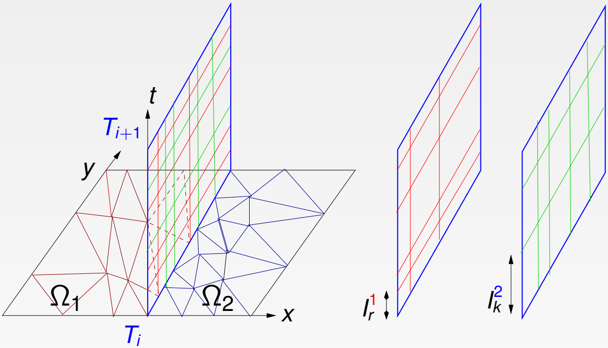


The discrete approximations F_1 of f_1 in $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_1)$, and G_2 of g_2 in $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_2)$ verify the nonconforming matching condition

$$\int_I [F_1 - G_2]\, V_1 = 0, \quad \forall V_1 \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_1)$$

An efficient way to perform the projections between time grids

Based on the method in Gander/Halpern/Nataf (2003)



Let V_k^1 (resp. V_ℓ^2) the shape functions of $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_1)$ (resp. $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_2)$)

How to compute $M_{k,\ell} = \int_I V_k^1 V_\ell^2$?

$\Rightarrow$ **Linear complexity algorithm without an additional grid**

Convergence - Error estimates

based on the theoretical results in Eriksson/Johnson/Larsson (1998), Makridakis/Akrivis (2004), Szeftel (2004)

Convergence : the continuous algorithm converges for

- $\beta_1 = \beta_2 = 0$, $\alpha_1 \neq \alpha_2,\ \nu_1 \neq \nu_2,\ \mathbf{a}_1 \neq \mathbf{a}_2$ and general decomposition
- if $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\beta_1 = \beta_2$, $\alpha_1 \neq \alpha_2,\ \nu_1 \neq \nu_2,\ \mathbf{a}_1 \neq \mathbf{a}_2$ and decomposition into strips

The coupled discret problem in time, has a unique solution and the discret Schwarz algorithm is convergent.

Error estimates : for $\beta_1 = \beta_2 = 0$

$$\sum_{i=1}^{I} \|u - U_i\|^2_{L^\infty(0,T,L^2(\Omega_i))} = \mathcal{O}(\Delta t^{q+1})$$

with $\Delta t = sup_k \Delta t_k$

Outline

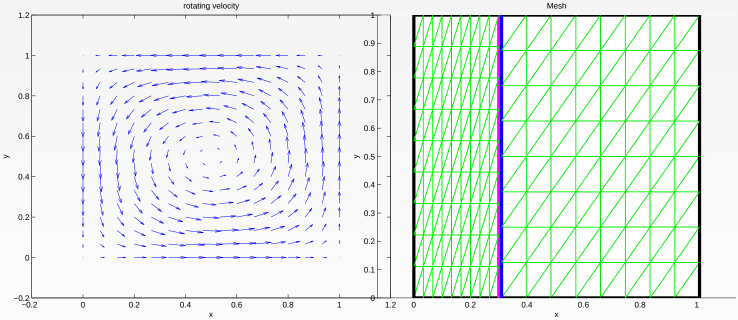
Numerical Results

Replace $H_1^1(\Omega_j)$ with V_j^h (P_1 finite element space)

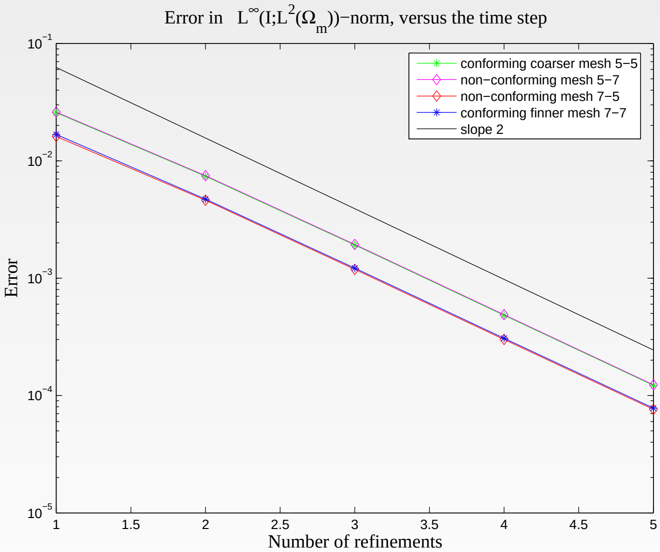
Exact solution $u(x,t) = cos(\pi x) sin(\pi y) cos(\pi t)$, in $[0,1]^3$

$\mathbf{a} = (-\sin(\pi * (y - \frac{1}{2})).*\cos(\pi * (x - \frac{1}{2})), \cos(\pi * (y - \frac{1}{2})).*\sin(\pi * (x - \frac{1}{2}))),$
$\nu_1 = \nu_2 = 1$

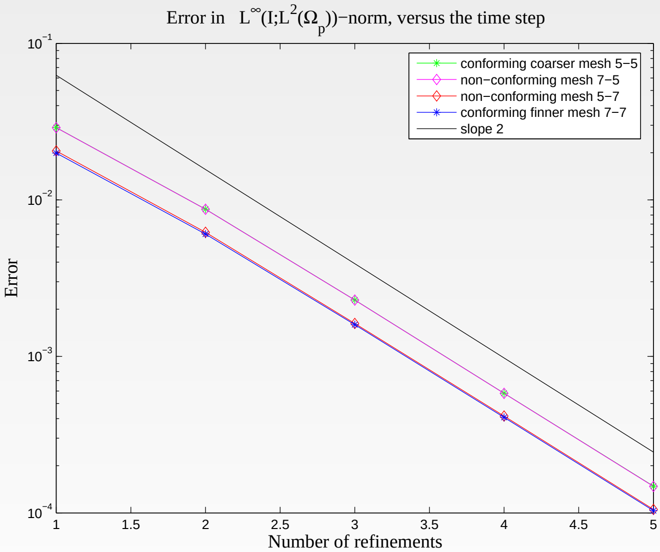
Stopping criterion : the jump of interface conditions is smaller than 10^{-6}
Space-time non conforming grids



Error in $L^\infty(I; L^2(\Omega_1))$ norm



Error in $L^\infty(I; L^2(\Omega_2))$ norm



Example with discontinuous coefficients

$$\mathcal{L}u = \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{a}(\mathbf{x})u - \nu \nabla u) = e^{-100((x-055)^2+(y-1.7)^2)} \text{ in } \Omega \times [0, T]$$

$u = 0$ on $\Gamma_0 \times [0, T]$, $\partial_{\mathbf{n}} u = 0$ on $\partial\Omega \backslash \Gamma_0 \times [0, T]$, $u(.,0) = e^{-100((x-055)^2+(y-1.7)^2)}$ on $\partial\Omega$

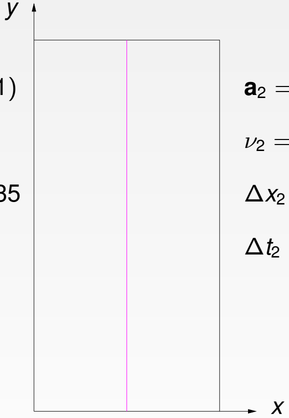
3 OSWR iterations

$\mathbf{a}_1 = (0, -1)$

$\nu_1 = 0.003$

$\Delta x_1 = 0.035$

$\Delta t_1 = 0.01$



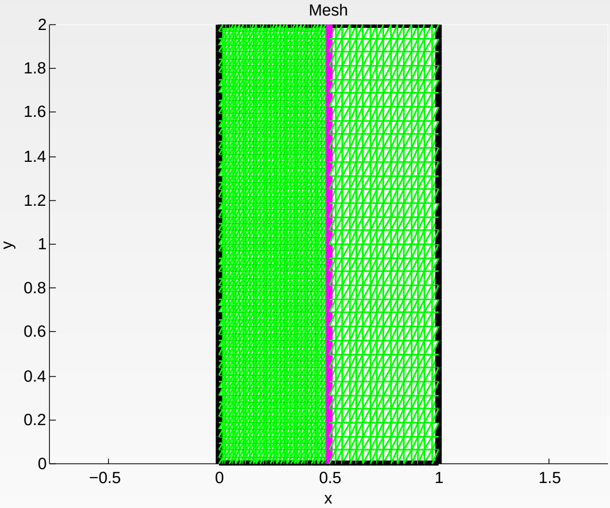
$\mathbf{a}_2 = (-1, 0)$

$\nu_2 = 0.1$

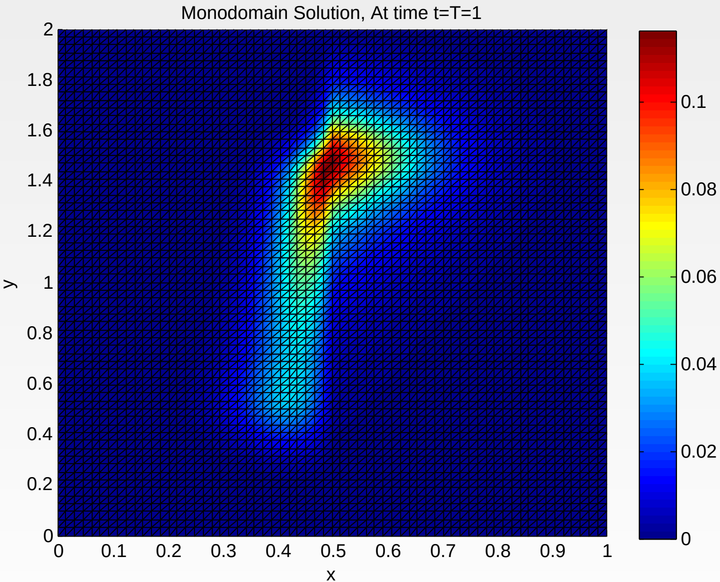
$\Delta x_2 = 0.07$

$\Delta t_2 = 0.02$

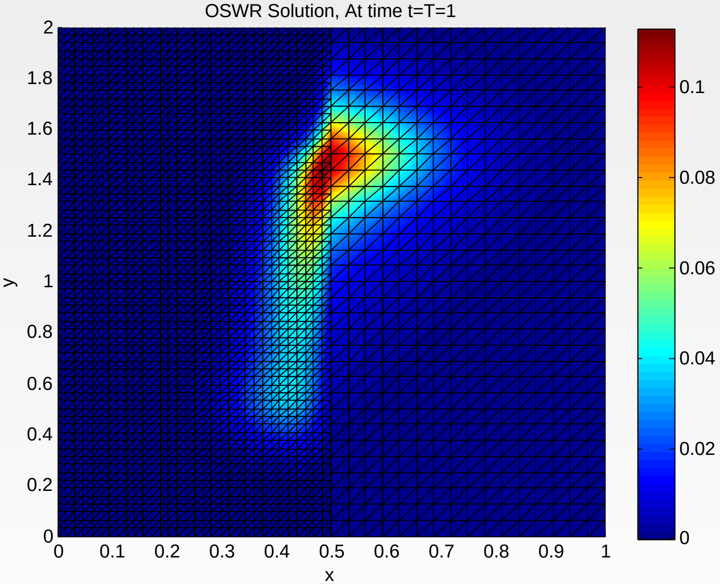
Non conforming space grid



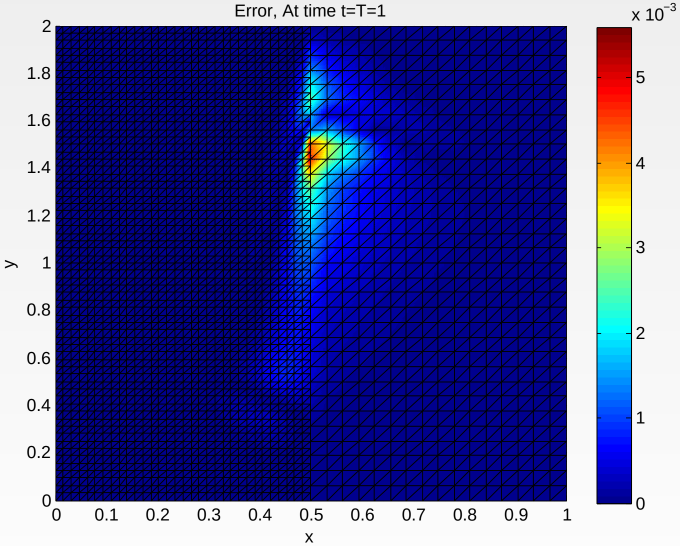
Monodomain Solution at time T=1



OSWR Solution at time T=1



Error between monodomain and multidomain solutions at time T=1



Outline

Conclusions

- Time discontinuous Galerkin method with Optimized Schwarz Waveform Relaxation

  $\Rightarrow$ lead to physical transmission conditions in very few iterations

  $\Rightarrow$ independant time steps with preservation of the scheme global order in time in the subdomains

  $\Rightarrow$ a simple and efficient algorithm to perform projection between nonmatching time grids

- Work in progress

  - numerical and mathematical analysis of the convergence rate (with M.J. Gander)
  - Extension to the MOMAS approach (Mixte Finite Element)