Scaling Up and modeling for Transport and Flow in Porous Media

Numerical Method for Elliptic Multiscale Problems

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Position of the problem

- Motivation
  Large class of multiscale problems are described by partial differential equations with heterogeneous coefficients.
  Such coefficients represent the properties of a composite material or heterogeneity of the medium in the computation of flow in porous media
- Difficulty
  Computation of an accurate discrete solution of such problems requires a very fine discretisation.
  $\Rightarrow$ High storage and computation costs.
- Interest
  The average behaviour of the elliptic oscillatory operator on a coarse scale taking into account the small scale features of the solution.

References: Different types of multiscale methods

- Weinan/Engquist: Heterogeneous multiscales methods(2003)
- Brewster/Beylkin: Multiresolution Methods (1995)
- Babuška-Osborn: Generalised Finite Element Method: 1d (1983)
  Hou-Wu: Generalised to 2d (1997)
- Variational multiscale approach introduced by Hughes and Brezzi, Arbogast for a mixed variant.

Our approach: to provide a smoother elliptic operator which behaves like the original operator on a coarse mesh, with no smoothness or periodicity requirement.

Outline

- Multiscale approach
  - Model problem
  - Finite element framework
  - Reformulation of the problem
- Theory for periodic coefficients in 1d
- Numerical Results in 1d

Model problem

The elliptic boundary value problem on Ω, a bounded Lipschitz domain in $\mathbb{R}^d$,

$$\begin{cases} Lu = f\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \quad \text{with} \quad L = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \alpha_{ij} \frac{\partial}{\partial x_i} \qquad (1)$$

$f \in L^2(\Omega)$. The coefficients $\alpha_{ij} \in L^\infty(\Omega)$ may be **oscillatory** or **jumping**.

Let $\underline{\lambda}, \overline{\lambda} > 0$ *s.t.* the matrix function $\alpha(x) = (\alpha_{ij}(x))_{i,j=1,\dots,d}$ satisfies $0 < \underline{\lambda} \leq \lambda(\alpha(x)) \leq \overline{\lambda}$ for all eigenvalues $\lambda(\alpha(x))$ of $\alpha(x)$ and almost all $x \in \Omega$.

Difficulty: Accurate discrete solution of such problems requires a very fine discretisation.

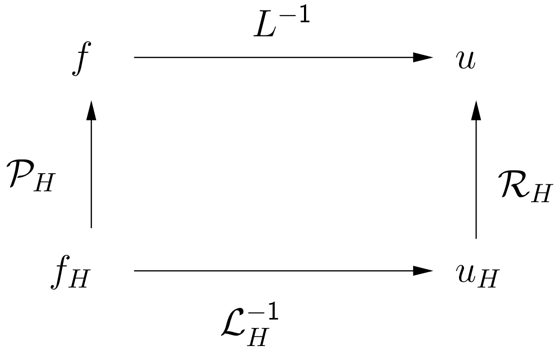
$\rightsquigarrow$ High storage and computational costs.

Notation

- $V = H_0^1(\Omega)$.
- Let $\mathcal{T}_H$ be a regular grid adapted to the coarse level.
- Let V_H be the P^1-Lagrange FE-space associated to $\mathcal{T}_H$, with $\dim V_H = m$.
- Let $\mathcal{P}_H$ be some prolongation from the macroscopic level to the continuous level.
- Let $\mathcal{R}_H$ be a restriction operator associated to $\mathcal{P}_H$.

Formulation

The use of the Green function of the operator L, allows to consider L^{-1}. We have the following diagram:



Problem 1: Let $\mathcal{L}_H \in \mathbb{R}^{m\times m}$ be defined by

$$\mathcal{L}_H := \left(\mathcal{R}_H L^{-1} \mathcal{P}_H\right)^{-1}.$$

Can $\mathcal{L}_H$ be interpreted as an approximation A_H of some local differential operator A with the step size H?

Difficulty: The Green function is not always explicitely given.
Idea Consider a very small step size h *s.t.* $h \ll H$ and the discretisation L_h of the operator L on a fine grid $\mathcal{T}_h$.

Finite element framework

1. Let V_h be the P^1-Lagrange FE space $V_h = \text{span}\{b_1^h, \dots, b_n^h\}$ and $\dim V_h = n$ *s.t.* $V_H \subset V_h$.
2. Isomorphisms P_h and its adjoint $R_h \in L(V', \mathbb{R}^n)$ defined by

$$P_h \;:\; \mathbb{R}^n \to V_h \subset V$$
$$v = (v_1, \dots, v_n) \longmapsto P_h\, v = \sum_{i=1}^{n} v_i\, b_i^h \qquad \text{and } R_h = P_h^* .$$

3. Let $M_h \in \mathbb{R}^{n\times n}$ and $M_H \in \mathbb{R}^{m\times m}$ be the mass matrices: $M_h := R_h\, P_h$ and $M_H := R_H\, P_H$.
4. The FE-stiffness matrix L_h is given by

$$L_h = R_h\, L\, P_h .$$

The inclusion $V_H \subset V_h$ ensures that the following mappings are well defined:

- The prolongation operator $P_{h \leftarrow H}$ from the **coarse** grid $\mathcal{T}_H$ to the **fine** grid $\mathcal{T}_h$ given by

$$P_{h \leftarrow H} = (P_h^{-1}\, P_H) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

- The restriction operator $R_{H \leftarrow h} = \left(P_{h \leftarrow H}\right)^*$ from the **fine** grid $\mathcal{T}_h$ to the **coarse** grid $\mathcal{T}_H$.

Normalised prolongation and restriction: $\tilde{P}_{h \leftarrow H} : \mathbb{R}^m \rightarrow \mathbb{R}^n$

$$\tilde{P}_{h \leftarrow H} := M_h\, P_{h \leftarrow H}\, M_H^{-1} \quad \text{and} \quad \tilde{R}_{H \leftarrow h} := \left(\tilde{P}_{h \leftarrow H}\right)^* .$$

Let $\|\!|\cdot|\!\|$ be the norm defined for a matrix $X \in \mathbb{R}^{m \times m}$ by

$$|\!|\!| X |\!|\!| := \| P_H\, X\, R_H \|_{L^2(\Omega) \leftarrow L^2(\Omega)} = \| M_H^{1/2} X\, M_H^{1/2} \|_2 .$$

Reformulation of the Problem

With help of $\mathcal{H}$-arithmetic the computation of the discrete operator L_h^{-1} on the fine mesh is possible. Therefore the following matrix $\mathcal{L}_{H,h}$ is available

$$\mathcal{L}_{H,h} := \left(\tilde{R}_{H \leftarrow h} L_h^{-1} \tilde{P}_{h \leftarrow H} \right)^{-1} .$$

Problem 1 with the operator $\mathcal{L}_{H,h}$ leads to Problem 2:

Problem 2: We are looking for an elliptic operator $A \in L(V, V')$ such that its discretisation A_H on the coarse grid satisfies:

$$A_H \approx \mathcal{L}_{H,h} \quad \text{for all } h \text{ small enough.} \tag{2}$$

Remarks

- Engineer's point of view: average solution for f given on a coarse level.
- The inverse L_h^{-1} is treated with Hierarchical Matrices ($\mathcal{H}$-matrices).
- Hierarchical Matrices arithmetic: low cost for arithmetic and storage.
- Storage of $\mathcal{L}_{H,h}$ and not L_h.
- Once $\mathcal{L}_{H,h}$ is computed, it can be use as much as one needs.

Solution of problem 2 in 1d

Let the elliptic operator A be defined by

$$A = -\frac{d}{dx}\left(a\frac{d}{dx}\right)$$

where the coefficient a is given on each segment $[x_j^H, x_{j+1}^H]$ by

$$a_{|[x_j^H, x_{j+1}^H]} = \frac{1}{\theta_j} \quad \text{where} \quad \theta_j = \frac{1}{x_{j+1}^H - x_j^H}\int_{x_j^H}^{x_{j+1}^H} \frac{ds}{\alpha(s)}.$$

For α T-periodic, $H > T$, H and T proportional

$$a = \frac{1}{M(\frac{1}{\alpha})} = \alpha_0 \quad \text{where} \quad M\left(\frac{1}{\alpha}\right) = \frac{1}{T}\int_0^T \frac{dx}{\alpha(x)}.$$

Homogenisation in 1d: Exact solution $u = L^{-1}f$ is approximated by the homogenised one $u_0 = L_0^{-1}f$ with a precision depending on the period T

Theory for periodic coefficient in 1d

Error estimate The following error estimate holds

$$\|\mathcal{L}_{H,h}^{-1} - L_{0,H}^{-1}\| \quad \leq \quad C\left(\varepsilon(h) + T\,M(\frac{1}{\alpha})\,(1+T) + \varepsilon_0(H)\right),$$

where $\varepsilon(h)$ (resp. $\varepsilon_0(H)$) is a bound on the FE-discretisation error of L on the fine mesh $\mathcal{T}_h$ (resp. of the homogenised L_0 on $\mathcal{T}_H$).

Idea of the proof:

1. Decomposition of the Green function G associated to L: $G(x,t) = G_0(x,t) + R_T(x,t)$,
2. Let B_h (resp. $B_{0,H}$) be Galerkin discretisation of L^{-1} on $\mathcal{T}_h$ (resp. of L_0^{-1} on $\mathcal{T}_H$) $\Rightarrow B_h = B_{0,h} + B_{T,h}$.
3. Bebendorf/Hackbusch gives

$$||L_{0,H}^{-1} - M_H^{-1}\,B_{0,H}\,M_H^{-1}||_2 \leq 2\,\|M_H^{-1}\|_2\,\varepsilon_0(H)$$
$$||L_h^{-1} - M_h^{-1}\,B_h\,M_h^{-1}||_2 \leq 2\,\|M_h^{-1}\|_2\,\varepsilon(h)\,.$$

4. $\left\|B_{T,h}\right\|_2 \leq C\,h\,T\,M\left(\frac{1}{\alpha}\right)(1+T)\,.$

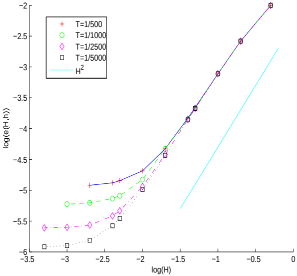
Numerical Results in 1d: $\Omega = [0,1]$

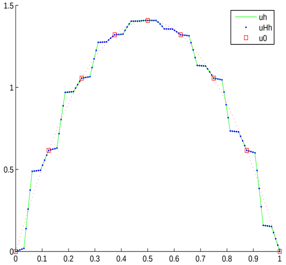
Periodic coefficient

Let $\alpha > 0$ be the T-periodic, piecewise constant defined on a period $[0, T[$ by:

$$\alpha(x) = \begin{cases} 8.1\,, & x \in [0, \frac{T}{4}[\\ 0.3\,, & x \in [\frac{T}{4}, \frac{T}{2}[\\ 20.55\,, & x \in [\frac{T}{2}, \frac{3T}{4}[\\ 1.0\,, & x \in [\frac{3T}{4}, T[\,. \end{cases}$$

- Computation of the norm $\|\mathcal{L}_{H,h}^{-1} - A_H^{-1}\|$ for $h = 10^{-5}$ and different values of H between $1/2$ and $50\,h$, and the period T is varying between $T = 10^{-3}$ and $T = 10^{-4}$.
- Computation of the discrete solution $u_h = L_h^{-1} f$, $u_0 = L_0^{-1} f$, $u_{H,h} = \mathcal{L}_{H,h}^{-1} f$ for a constant right-hand side $f = 10$ and the step sizes $h = 2^{-13}$ and $H = 2^{-7}$.





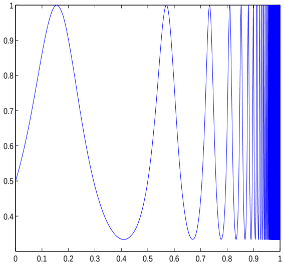
$\log\left(|||\mathcal{L}_{H,h}^{-1} - A_H^{-1}|||\right)$ represented as a function of $\log(H)$. $h = 10^{-5}$ and $T = 1/500,\ 1/1000,\ 1/2500,\ 1/5000$.

Representation of $u_h = L_h^{-1} f$, $u_0 = L_0^{-1} f$, $u_{H,h} = \mathcal{L}_{H,h}^{-1} f$, for $f = 10$, $h = 2^{-13}$, $T = 2^{-3}$, and $H = 2^{-7}$.

- Convergence of order almost 2 when H is large in comparison with T.
- $u_{H,h}$ is matching u_h. The details of u_h are well captured by $u_{H,h}$, whereas u_0 interpolates the fine solution u_h.

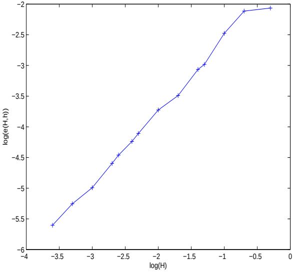
Non-periodic oscillatory coefficient

Let α be defined by $\alpha(x) = \left[2 - \sin(2\,\pi\,\tan(\frac{x\,\pi}{2}))\right]^{-1}$ on $\Omega = [0,1]$
$\longrightarrow$ contains a continuum of scales



H \ h	1/ 1000	1/ 2000	1/ 4000	1/ 8000
1/2	8.60e-3	8.58e-3	8.59e-3	8.61e-3
1/5	7.70e-3	7.68e-3	7.68e-3	7.68e-3
1/10	3.36e-3	3.35e-3	3.35e-3	3.34e-3
1/20	1.08e-3	1.06e-3	1.04e-3	1.04e-3
1/25	8.56e-4	8.57e-4	8.59e-4	8.61e-4
1/50	3.22e-4	3.22e-4	3.22e-4	3.23e-4
1/100	1.94e-4	1.91e-4	1.89e-4	1.88e-4
1/200	9.23e-5	8.29e-5	7.90e-5	7.80e-5
1/250	7.07e-5	6.21e-5	5.89e-5	5.77e-5
1/400	–	3.84e-5	3.57e-5	3.46e-5
1/500	3.42e-5	2.97e-5	2.65e-5	2.54e-5
1/1000	2.82e-15	1.30e-5	1.37e-5	1.02e-5
1/2000	–	4.55e-14	5.63e-6	5.58e-6
1/4000	–	–	6.53e-14	2.49e-6

$|||\mathcal{L}_{H,h}^{-1} - A_H^{-1})|||$, for $h = 1/1000$, $1/2000$, $1/4000$, $1/8000$



$\log\left(|||\mathcal{L}_{H,h}^{-1} - A_H^{-1})|||\right)$ is represented as a function of $\log(H)$ for $h = 1/8000$

Notwithstanding the very oscillatory behaviour of the coefficient α, Good convergence of the norm: globally order 1.

Conclusion

1. In 1d, the discrete operator $\mathcal{L}_{H,h} := \left(\tilde{R}_{H\leftarrow h} L_h^{-1} \tilde{P}_{h\leftarrow H}\right)^{-1}$ on the coarse mesh behaves like the discretisation of an elliptic operator.
2. One possible approximation is given by $A = -\frac{d}{dx}\left(a\,\frac{d}{dx}\right)$, where a is the piecewise harmonic average of α.
3. Currently: Numerical experiments in 2d.

For more details, see:
Greff/Hackbusch, Numerical methods for elliptic multiscale problems, 16 (2), J. of Numer. math, 107-138 (2008).