A finite volume method on general meshes for a time evolution convection-diffusion equation.

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Model problem & Numerical scheme ;

Existence of the unique discrete solution, a priori estimate on the approximate solution in the discrete norms $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$;

Estimates on the differences of the space and time translates, which implies the propriety of the relative compactness by the theorem of Fréchet–Kolmogorov ;

Strong convergence in L^2 of the approximate solution to the weak solution of the problem $(\mathcal{P})$;

Numerical tests.

We consider the convection-diffusion problem

$$(\mathcal{P}) \quad \begin{cases} \partial_t u - \nabla \cdot (\mathbf{\Lambda}(\mathbf{x})\nabla u) + \nabla \cdot (\mathbf{V}u) = f(\mathbf{x},t) & \text{in} \quad Q_T = \Omega \times (0,T), \\ u(\mathbf{x},t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{for all} \quad \mathbf{x} \in \Omega. \end{cases}$$

We suppose that the following hypotheses are satisfied

($\mathcal{H}_1$) Ω is an open bounded connected polyhedral subset of $\mathbb{R}^d$, $d \in \mathbb{N} \setminus \{0\}$;

($\mathcal{H}_2$) $\mathbf{\Lambda}$ is a measurable function from Ω to $\mathcal{M}_d(\mathbb{R})$, where $\mathcal{M}_d(\mathbb{R})$ denotes the set of $d \times d$ symmetric matrices, such that for a.e. $\mathbf{x} \in \Omega$ the set of its eigenvalues is included in $[\underline{\lambda}, \overline{\lambda}]$, where $\underline{\lambda}, \overline{\lambda} \in L^\infty(\Omega)$ are such that $0 < \alpha_0 \leqslant \underline{\lambda}(\mathbf{x}) \leqslant \overline{\lambda}(\mathbf{x})$;

($\mathcal{H}_3$) $\mathbf{V} \in L^2(0, T; H(div, \Omega)) \cap L^\infty(Q_T)$ is such that $\nabla \cdot \mathbf{V} \geqslant 0$ a.e. in Q_T;

($\mathcal{H}_4$) $u_0 \in L^\infty(\Omega)$;

($\mathcal{H}_5$) $f \in L^2(Q_T)$.

The possible finite volume schemes

The essential problem is related with the fact that $\mathbf{\Lambda}$ is a full matrix, in such case it is not possible to use the classical two point discretization.
Possible solutions among the finite volume methods :

Finite Volumes - Finite Elements
(Angot A., Dolejší V., Feistauer M., Felcman j., Vohralík M.)
Idea : Use the dual finite element grid in order approximate the diffusion term.

Multipoint Flux Approximation Methods (MFAM)
(Aavatsmark I., Eigestad G. T., Klausen, R. A.)
Idea : Use several neighbors of the control volume in order to define the diffusive flux ;

Hybride Finite Volume Method
(Eymard R., Gallouët R., Herbin R.)
Idea : Take into account the supplementary unknowns associated with the cell faces.

Discretization

A discretization of Ω, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where :

1. $\mathcal{M}$ is a family of control volumes ;
2. $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}$ is a set of edges ;
3. $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ is a family of points, such that for all $K \in \mathcal{M}$, $\mathbf{x}_K \in K$ and K is $\mathbf{x}_K$-star-shaped.

$m(K)$, the measure of $K \in \mathcal{M}$;
$m(\sigma)$, the measure of $\sigma \in \mathcal{E}$;
$\mathcal{E}_K$, the set of edges of $K \in \mathcal{M}$;
$\mathcal{M}_\sigma$, the set of control volumes containing $\sigma \in \mathcal{E}$;
$\mathbf{n}_{K,\sigma}$, the unit vector outward to K and normal to $\sigma \in \mathcal{E}_K$;
$d_{K,\sigma}$, the Euclidean distance between $\mathbf{x}_K$ and $\sigma \in \mathcal{E}_K$;
$D_{K,\sigma}$, the cone with vertex $\mathbf{x}_K$ and basis $\sigma \in \mathcal{E}_K$.

A time discretization is given by $0 = t_0 < t_1 \ldots < t_N = T$ with the constant time step $k = T/N$.

DISCRETIZATION

We associate with the mesh the following spaces of discrete unknowns

$$X_{\mathcal{D}} = \{((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}), v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\}$$

$$X_{\mathcal{D},0} = \{v \in X_{\mathcal{D}} \text{ such that } (v_\sigma)_{\sigma\in\mathcal{E}_{ext}} = 0\},$$

The space $X_{\mathcal{D}}$ is equipped with the semi-norm

$$|v|_X^2 = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} (v_\sigma - v_K)^2,$$

which is the norm in $X_{\mathcal{D},0}$.

(i) The initial condition for the scheme

$$u_K^0 = \frac{1}{m(K)} \int_K u_0(\mathbf{x}) d\mathbf{x};$$

(ii) The discrete equations

$$m(K)(u_K^n - u_K^{n-1}) + k \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u^n) + k \sum_{\sigma \in \mathcal{E}_K} V_{K,\sigma} \overline{u_{K,\sigma}^n} = m(K) f_K^n \quad \forall K \in \mathcal{M},$$

$$\text{where } f_K^n = \frac{1}{m(K)} \int_{t_{n-1}}^{t_n} \int_K f(\mathbf{x}, t) d\mathbf{x} dt;$$

(iii) The local conservation of the total flux

$$(F_{K,\sigma}(u^n) + V_{K,\sigma} \overline{u_{K,\sigma}^n}) + (F_{L,\sigma}(u^n) + V_{L,\sigma} \overline{u_{L,\sigma}^n}) = 0 \quad \text{for all} \quad \sigma \in \mathcal{E}_{int}, \quad \mathcal{M}_\sigma = \{K, L\};$$

(iv) The discrete analog of the boundary conditions

$$u_\sigma^n = 0 \quad \text{for all} \quad \sigma \in \mathcal{E}_{ext} \ .$$

We put the scheme (i)-(iv) under the equivalent form

Let u^0 is defined by

$$u_K^0 = \frac{1}{m(K)} \int_K u_0(\mathbf{x}) \quad \forall K \in \mathcal{M}.$$

For each $n \in \{1, \ldots, N\}$ find $u^n \in X_{\mathcal{D},0}$ such that for all $v \in X_{\mathcal{D},0}$,

$$\sum_{K \in \mathcal{M}} m(K) v_K (u_K^n - u_K^{n-1}) + k < v, u^n >_F + k < v, u^n >_T = \sum_{K \in \mathcal{M}} m(K) v_K f_K^n,$$

where

$$< v, u^n >_F = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma) F_{K,\sigma}(u^n),$$

and

$$< v, u^n >_T = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma) V_{K,\sigma} \overline{u_{K,\sigma}^n}.$$

Discretization of the convection term

We define

$$V_{K,\sigma} = \int_\sigma \mathbf{V}(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma};$$

let an upwind value $\overline{u^n_{K,\sigma}}$ be given by

$$\overline{u^n_{K,\sigma}} = \begin{cases} u^n_K, & \text{if } V_{K,\sigma} \geqslant 0 \\ u^n_\sigma, & \text{if } V_{K,\sigma} < 0. \end{cases}$$

We have completely defined the discrete version of the convective term. We give below the definition of the discret flux $F_{K,\sigma}(u^n)$.

We denote

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u \cdot \mathbf{n}_{K,\sigma},$$

where

$$\nabla_K u = \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} m(\sigma)(u_\sigma - u_K)\mathbf{n}_{K,\sigma}$$

and where

$$R_{K,\sigma} u = \frac{\alpha_K}{d_{K,\sigma}}(u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)),$$

with some $\alpha_K > 0$, which should be chosen properly.

Optimization of α_K has been studied by O. Angelini, C. Chavant, E. Chenier, R. Eymard.

Discretization of the diffusion term

We denote

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u \cdot \mathbf{n}_{K,\sigma},$$

where

$$\nabla_K u = \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} m(\sigma)(u_\sigma - u_K)\mathbf{n}_{K,\sigma}$$

and where

$$R_{K,\sigma} u = \frac{\sqrt{d}}{d_{K,\sigma}}(u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)).$$

The choice $\alpha_K = \sqrt{d}$ yields the two point scheme in case of meshes which satisfy $\mathbf{n}_{K,\sigma} = \frac{x_\sigma - x_K}{d_{K,\sigma}}$.

Optimization of α_K has been studied by O. Angelini, C. Chavant, E. Chenier, R. Eymard.

The discret gradient $\nabla_{\mathcal{D}} u$

We define the discret gradient $\nabla_{\mathcal{D}} u$ by

$$\nabla_{\mathcal{D}} u(\mathbf{x}) = \nabla_{K,\sigma} u \quad \mathbf{x} \in D_{K,\sigma},$$

where $D_{K,\sigma}$ is the cone with vertex $\mathbf{x}_K$ and basis $\sigma \in \mathcal{E}_K$, notice that the bilinear form

$$< v, u >_F = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma) F_{K,\sigma}(u) = \int_\Omega \nabla_{\mathcal{D}} v \cdot \mathbf{\Lambda}(\mathbf{x}) \nabla_{\mathcal{D}} u$$

is symmetric. We show in what follows that it is also continuous and coercive.

Advantages of the scheme

Very general class of meshes ;

Local conservativity ;

The discretization of the convection and diffusion flux does not involve the unknowns outside of the cell ;

One can easily eliminate the cell unknowns u_K and then solve the system of $card(\mathcal{E})$ equations.

Lemma 1 Let $\mathcal{D}$ be the discretization of Ω.

(i) There exists $C_1 > 0$ and $\alpha > 0$ such that

$$| < u, v >_F | \leqslant C_1 |u|_X |v|_X$$

and

$$< u, u >_F \geqslant \alpha |u|_X^2.$$

for all $u, v \in X_{\mathcal{D}}$.
(ii) There exists $C_2 > 0$ such that

$$| < u, v >_T | \leqslant C_2 |u|_X |v|_X$$

and

$$< u, u >_T \geqslant 0$$

for all $u, v \in X_{\mathcal{D}}$.

The Lemma 1 and the Lax-Milgram Theorem implies the following result

Theorem 1 The discrete problem (i)-(iv) possess the unique solution.

Definition of the approximate solution

Let $u^n \in X_{\mathcal{D},0}$, $n = 1 \ldots N$, be a solution of the approximate problem, with $k = T/N$. We say that the piecewise constant function $u_{h,k}$ is an approximate solution of the problem $(\mathcal{P})$ if

$$u_{h,k}(\mathbf{x}, 0) = u_K^0 \qquad \text{for all } \mathbf{x} \in K$$

$$u_{h,k}(\mathbf{x}, t) = u_K^n \qquad \text{for all } (\mathbf{x}, t) \in K \times (t_{n-1}, t_n].$$

We also define its gradient by

$$\nabla_h u_{h,k}(\mathbf{x}, t) = \nabla_{\mathcal{D}} u^n(\mathbf{x}) \qquad \text{for all } (\mathbf{x}, t) \in K \times (t_{n-1}, t_n].$$

Theorem 2 (A priori estimate) Let $u_{h,k}$ be a solution of the discrete problem, then it is such that

$$\|u_{h,k}(\cdot,t)\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \|u_0\|_{L^2(\Omega)} + 2T\|f\|_{L^2(Q_T)},$$

$$\underline{\lambda}\|\nabla_h u_{h,k}\|^2_{L^2(Q_T)} \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + (\|u_0\|_{L^2(\Omega)} + 2T\|f\|_{L^2(Q_T)})T\|f\|_{L^2(Q_T)}.$$

We could show as well the estimates on time and space translates

Theorem 3 Let $u_{h,k}$ be an approximate solution. There exists $C > 0$ and $0 < \vartheta < 1/2$, which do not depend on h and k, such that

$$\|u_{h,k}(\cdot + \mathbf{y}, \cdot + \tau) - u_{h,k}\|_{L^2(Q_T)} \leqslant C(\sqrt{\tau} + |\mathbf{y}|^\vartheta)$$

In view of the Theorem 2, the Fréchet-Kolmogorov Compactness Theorem implies that the family $\{u_{h,k}\}$ is relatively compact in $L^2(Q_T)$

Theorem 4 Let $\mathcal{F}$ be a family of discretizations of Ω and let $\{u_{h,k}\}$ be a family of approximate solutions corresponding to $\mathcal{F}$ and $k = T/N$. Then there exist a function $u \in L^2(Q_T)$ such that $u_{h,k} \to u$ strongly in $L^2(Q_T)$ as $h, k \to 0$. Moreover $u \in L^2(0, T; H_0^1(\Omega))$, $\nabla_h u_{h,k}$ weakly converge in $L^2(Q_T)^d$ to ∇u as $h, k \to 0$ and u is the weak solution of the problem $(\mathcal{P})$.

Outline of the proof

Thanks to the a priori estimate there exist some function $\mathbf{G} \in L^2(Q_T)^d$ such that $\nabla_h u_{h,k}$ weakly converge in $L^2(Q_T)^d$ to $\mathbf{G}$ as $h, k \to 0$. We show then that $\mathbf{G} = \nabla u$, more particulary

$$\int_0^T \int_{\mathbb{R}^d} \nabla_h u_{h,k}(\mathbf{x}, t) \cdot \psi(\mathbf{x}, t) d\mathbf{x} dt \to - \int_0^T \int_{\mathbb{R}^d} u(x, t) div \psi(\mathbf{x}, t) d\mathbf{x} dt$$

for some $\psi(\mathbf{x}, t)$ enough regular.

In order to show that u is a weak solution of the problem $(\mathcal{P})$ we introduce the following functional space

$$\Phi = \{\varphi \in C^{2,1}(\overline{\Omega} \times [0,T]), \quad \varphi = 0 \text{ on } \partial\Omega \times [0,T], \quad \varphi(\cdot, T) = 0\}.$$

Let $\varphi \in \Phi$, we denote $\varphi_K^n = \varphi(\mathbf{x}_K, t_n)$ and $\varphi_\sigma^n = \varphi(\mathbf{x}_\sigma, t_n)$, we show in what follows that

$$\sum_{n=1}^{N} \sum_{K \in \mathcal{M}} m(K)(u_K^n - u_K^{n-1})\varphi_K^n \to \int_\Omega u_0(\mathbf{x})\varphi(\mathbf{x}, 0),$$

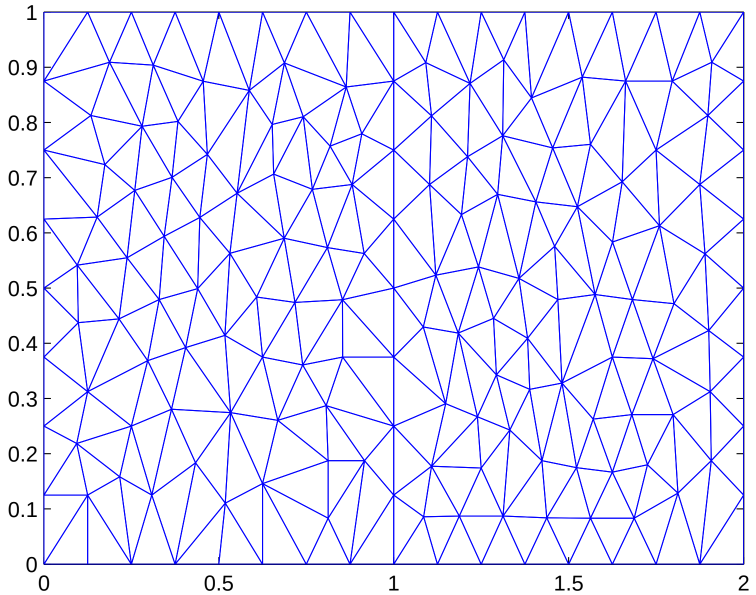
$$\sum_{n=1}^{N} k \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (\varphi_K^n - \varphi_\sigma^n) F_{K,\sigma}(u^n) \to \int_0^T \int_\Omega \nabla\varphi(\mathbf{x}, t) \cdot \mathbf{\Lambda}(\mathbf{x}) \nabla u(\mathbf{x}, t),$$

$$\sum_{n=1}^{N} k \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (\varphi_K^n - \varphi_\sigma^n) V_{K,\sigma} \overline{u_{K,\sigma}^n} \to \int_0^T \int_\Omega u(\mathbf{x}, t) \nabla(\mathbf{V}(\mathbf{x})\varphi(\mathbf{x}, t)).$$

By density of the set Φ in the set $\{\varphi \in L^2(0,T; H_0^1(\Omega)), \varphi_t \in L^\infty(Q_T), \varphi(\cdot, T) = 0\}$ u is the weak solution of the continuous problem.

Numerical test I

We consider a 2D domain $\Omega = (0,2) \times (0,1)$ and $T = 1$.



If $x_1 \leqslant 1$:

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{V} = (3, 0).$$

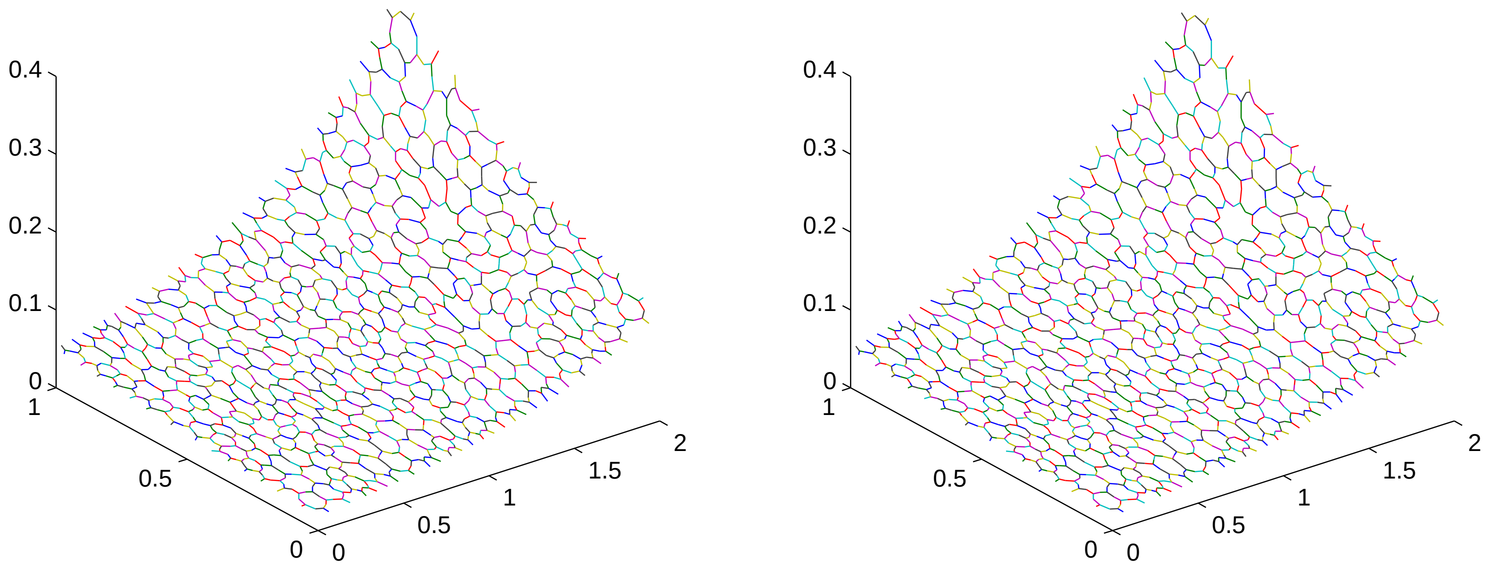
If $x_1 > 1$:

$$\mathbf{\Lambda} = \begin{pmatrix} 8 & -7 \\ -7 & 20 \end{pmatrix}, \quad \mathbf{V} = (3, 12).$$

Numerical test I

The initial data and the boundary conditions are given by the continuous exact solution

$$u(\mathbf{x}, t) = e^{x_1 + x_2 - t - 3}.$$



Approximate solution (left) and exact solution (right) on a triangular grid at $t = 1$.

Numerical test I

Number of time steps N, mesh diameter h, number of unknowns and relative error Err$= \max_{0<t\leqslant T} \frac{\|u_{h,k} - u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}$ for triangular Delaunay and structured quadrangular grids, respectively

N	h	Err	Unkn.	h	Err	Unkn.
16	0.197642	0.039027	483	0.223607	0.074821	220
32	0.103985	0.021936	1920	0.111803	0.046313	840
64	0.053386	0.014743	7560	0.055902	0.032142	3280
128	0.026758	0.012429	29802	0.027951	0.025171	12960

The problem is diffusion dominated and we observe a linear convergence of the scheme.

We consider the problem $(\mathcal{P})$ in the 2-dimensional space domain $\Omega = (0,3) \times (0,3)$ with scalar diffusion tensor given by

$$\mathbf{\Lambda}(\mathbf{x}) = \delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the constant velocity field

$$\mathbf{V}(\mathbf{x}) = (v_1, v_2).$$

The initial and boundary conditions are given by the exact solution

$$u(\mathbf{x}, t) = \frac{1}{200\delta t + 1} e^{-50 \frac{|\mathbf{x} - \mathbf{x}_0 - \mathbf{V}\cdot t|^2}{200\delta t + 1}},$$

representing a Gaussian peak centered at the point $(\mathbf{x_0})$, being transported by the convective field $\mathbf{V}$ and diffusing.

We set $\mathbf{V} = (0.8, 0.4)$ and $\mathbf{x}_0 = (0.5, 1.35)$.

Number of time steps N, mesh diameter h, number of unknowns and $L^2(\Omega)$ error for $\delta = 0.1$ at $t = 2$ for triangular Delaunay and structured quadrangular grids, respectively

N	h	Unkn.	$\|u_{h,k} - u\|_{L^2(\Omega)}$	h	Unkn.	$\|u_{h,k} - u\|_{L^2(\Omega)}$
16	0.436	477	0.00942	0.3000	220	0.0092
32	0.222	1927	0.00519	0.1500	840	0.0078
64	0.113	7686	0.00299	0.0750	3280	0.0046
128	0.057	30366	0.00160	0.0375	12960	0.0025

Again, we have a linear convergence.

Numerical test II - Convection dominated case

Number of time steps N, mesh diameter h, number of unknowns and $L^2(\Omega)$ error for $\delta = 0.001$ at $t = 2$ for triangular Delaunay and structured quadrangular grids, respectively

N	h	Unkn.	$\|u_{h,k} - u\|_{L^2(\Omega)}$	h	Unkn.	$\|u_{h,k} - u\|_{L^2(\Omega)}$
16	0.436	477	0.139	0.3000	220	0.149
32	0.222	1927	0.129	0.1500	840	0.141
64	0.113	7686	0.113	0.0750	3280	0.129
128	0.057	30366	0.091	0.0375	12960	0.112

We see that the numerical diffusion cased by the upwind scheme provides a significant error.

Numerical test II - Convection dominated case

How to reduce the numerical diffusion ?

·Use upstream weighting scheme i.e. set the amount of upstream weighting with respect to the local Péclet number in order to stabilize the scheme by adding only a necessary numerical diffusion ;

or

·Use some other, less diffusive scheme for the convection term ;

·Use the local grid refitment.

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Thank you for your attention !