Small-amplitude homogenisation of parabolic equations

Nenad Antonić

Department of Mathematics
Faculty of Science
University of Zagreb

Dubrovnik, 13th October, 2008

Joint work with Marko Vrdoljak

H-convergence and G-convergence

Homogenisation:

in the sense of G-convergence (S. Spagnolo) and
H-convergence (F. Murat & L. Tartar)

H-convergence and G-convergence

Homogenisation:

in the sense of G-convergence (S. Spagnolo) and
H-convergence (F. Murat & L. Tartar)

Recall small-amplitude homogenisation for

$$-\mathrm{div}\,(\mathbf{A}\nabla u) = f\ .$$

Small-amplitude homogenisation

Consider

$$-\text{div}\,(\mathbf{A}_\gamma^n \nabla u_n) = f\ ,$$

where $\mathbf{A}_\gamma^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(\Omega; \mathrm{M}_{d\times d})$, which is bounded from below; for small γ function $\mathbf{A}_\gamma^n$ is analytic in γ:

$$\mathbf{A}_\gamma^n(\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(\mathbf{x}) + \gamma^2 \mathbf{C}^n(\mathbf{x}) + o(\gamma^2)\ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$.

Small-amplitude homogenisation

Consider

$$-\mathsf{div}\,(\mathbf{A}^n_\gamma \nabla u_n) = f\ ,$$

where $\mathbf{A}^n_\gamma$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(\Omega; \mathrm{M}_{d\times d})$, which is bounded from below; for small γ function $\mathbf{A}^n_\gamma$ is analytic in γ:

$$\mathbf{A}^n_\gamma(\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(\mathbf{x}) + \gamma^2 \mathbf{C}^n(\mathbf{x}) + o(\gamma^2)\ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).

Then (after passing to a subsequence, if needed)

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2)\ ;$$

the limit being measurable in $\mathbf{x}$, and analytic in γ.

$\mathbf{A}^\infty_\gamma$ is the effective conductivity.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) + \gamma^2 \mathbf{C}_0(\mathbf{x}) + o(\gamma^2) \ .$$

∎

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) + \gamma^2 \mathbf{C}_0(\mathbf{x}) + o(\gamma^2) \; .$$

∎

$\mathbf{C}_0$ depends only on a subsquence of $\mathbf{B}^n$ (and $\mathbf{A}_0$), and there is an explicit formula involving the H-measure of the above subsequence:

$$-\int \varphi \mathbf{C}_0 = \left\langle \boldsymbol{\mu}, \varphi \boxtimes \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{\mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \right\rangle \; .$$

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) + \gamma^2\mathbf{C}_0(\mathbf{x}) + o(\gamma^2)\ .$$

∎

$\mathbf{C}_0$ depends only on a subsquence of $\mathbf{B}^n$ (and $\mathbf{A}_0$), and there is an explicit formula involving the H-measure of the above subsequence:

$$-\int \varphi\mathbf{C}_0 = \left\langle \boldsymbol{\mu}, \varphi \boxtimes \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}} \right\rangle\ .$$

This might provide a precise sense for some formulas in the book by Landau & Lifschitz.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) + \gamma^2 \mathbf{C}_0(\mathbf{x}) + o(\gamma^2)\ .$$

∎

$\mathbf{C}_0$ depends only on a subsquence of $\mathbf{B}^n$ (and $\mathbf{A}_0$), and there is an explicit formula involving the H-measure of the above subsequence:

$$-\int \varphi \mathbf{C}_0 = \left\langle \boldsymbol{\mu}, \varphi \boxtimes \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{\mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \right\rangle\ .$$

This might provide a precise sense for some formulas in the book by Landau & Lifschitz.
The method also works on the system of linearised elasticity
(see Tartar's paper in the Proceedings of SIAM conference in Leesburgh, Dec 1988)

Our goal

What can be done for parabolic equations?

$$\begin{cases} \partial_t - \text{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(0,\cdot) = u_0 \ . \end{cases}$$

with some boundary conditions.

Our goal

What can be done for parabolic equations?

$$\begin{cases} \partial_t - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u(0,\cdot) = u_0 \ . \end{cases}$$

with some boundary conditions.

Things to check:

1. H-convergence and G-convergence (in particular, analytical dependence of the H-limit on a parameter)
2. Parabolic variant od H-measures
3. What result do we get for small-amplitude homogenisation in this case (possible applications)

Known results for elliptic equations

Homogenisation of parabolic equations

- H-convergence and G-convergence
- H-convergent sequence depending on a parameter

A parabolic variant of H-measures

- What are H-measures and variants ?
- A brief comparative description

Small-amplitude homogenisation

- Setting of the problem (parabolic case)
- Variant H-measures in small-amplitude homogenisation

Parabolic problems

If $\mathbf{A}$ does not depend on t, the problem reduces to the elliptic case.

Parabolic problems

If $\mathbf{A}$ does not depend on t, the problem reduces to the elliptic case.

For $\mathbf{A}$ depending on both t and $\mathbf{x}$, only a few papers (a few more than three, in fact):

1977 S. Spagnolo: Convergence of parabolic operators

1981 V.V. Žikov et al.: O G-shodimosti paraboličeskih operatorov

1997 A. Dall'Aglio, F. Murat: A corrector result . . .

Parabolic problems

If $\mathbf{A}$ does not depend on t, the problem reduces to the elliptic case.

For $\mathbf{A}$ depending on both t and $\mathbf{x}$, only a few papers (a few more than three, in fact):

1977 S. Spagnolo: Convergence of parabolic operators

1981 V.V. Žikov et al.: O G-shodimosti paraboličeskih operatorov

1997 A. Dall'Aglio, F. Murat: A corrector result ...

There are some interesting differences in comparison to the elliptic case.

Non-stationary diffusion

Consider a domain $Q = \langle 0, T\rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is open:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad\quad u(0,\cdot) = u_0 \; . \end{cases}$$

Non-stationary diffusion

Consider a domain $Q = \langle 0, T \rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is open:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad\quad u(0, \cdot) = u_0 \ . \end{cases}$$

More precisely: $V := \mathrm{H}^1_0(\Omega)$, $V' := \mathrm{H}^{-1}(\Omega)$ and $H := \mathrm{L}^2(\Omega)$, the Gel'fand triple: $V \hookrightarrow H \hookrightarrow V'$.

Non-stationary diffusion

Consider a domain $Q = \langle 0, T\rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is open:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(0,\cdot) = u_0 \ . \end{cases}$$

More precisely: $V := \mathrm{H}^1_0(\Omega)$, $V' := \mathrm{H}^{-1}(\Omega)$ and $H := \mathrm{L}^2(\Omega)$,
the Gel'fand triple: $V \hookrightarrow H \hookrightarrow V'$.
For time dependent functions: $\mathcal{V} := \mathrm{L}^2(0,T;V)$, $\mathcal{V}' := \mathrm{L}^2(0,T;V')$,
$\mathcal{W} = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}'\}$ and $\mathcal{H} := \mathrm{L}^2(0,T;H)$, again: $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$.

Non-stationary diffusion

Consider a domain $Q = \langle 0, T\rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is open:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(0,\cdot) = u_0 \ . \end{cases}$$

More precisely: $V := \mathrm{H}^1_0(\Omega)$, $V' := \mathrm{H}^{-1}(\Omega)$ and $H := \mathrm{L}^2(\Omega)$,
the Gel'fand triple: $V \hookrightarrow H \hookrightarrow V'$.
For time dependent functions: $\mathcal{V} := \mathrm{L}^2(0,T;V)$, $\mathcal{V}' := \mathrm{L}^2(0,T;V')$,
$\mathcal{W} = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}'\}$ and $\mathcal{H} := \mathrm{L}^2(0,T;H)$, again: $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$.
Additionally assume $\mathbf{A} \in \mathrm{L}^\infty(Q;\mathrm{M}_{d\times d})$ satisfies:

$$\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \alpha|\boldsymbol{\xi}|^2$$
$$\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \frac{1}{\beta}|\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}|^2 \ ,$$

i.e. it belongs to $\mathcal{M}(\alpha,\beta;Q)$.

Non-stationary diffusion

Consider a domain $Q = \langle 0, T\rangle \times \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is open:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(0,\cdot) = u_0 \; . \end{cases}$$

More precisely: $V := \mathrm{H}^1_0(\Omega)$, $V' := \mathrm{H}^{-1}(\Omega)$ and $H := \mathrm{L}^2(\Omega)$, the Gel'fand triple: $V \hookrightarrow H \hookrightarrow V'$.
For time dependent functions: $\mathcal{V} := \mathrm{L}^2(0,T;V)$, $\mathcal{V}' := \mathrm{L}^2(0,T;V')$, $\mathcal{W} = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}'\}$ and $\mathcal{H} := \mathrm{L}^2(0,T;H)$, again: $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$.
Additionally assume $\mathbf{A} \in \mathrm{L}^\infty(Q;\mathrm{M}_{d\times d})$ satisfies:

$$\begin{aligned} \mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} &\geqslant \alpha|\boldsymbol{\xi}|^2 \\ \mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} &\geqslant \frac{1}{\beta}|\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}|^2 \; , \end{aligned}$$

i.e. it belongs to $\mathcal{M}(\alpha,\beta;Q)$.
With such coefficients the problem is well posed:

$$\|u\|_{\mathcal{W}} \leqslant c_1\|u_0\|_H + c_2\|f\|_{\mathcal{V}'}.$$

Parabolic operators

Parabolic operator $\mathcal{P} \in \mathcal{L}(\mathcal{W}; \mathcal{V}')$

$$\mathcal{P}u := \partial_t u - \operatorname{div}(\mathbf{A}\nabla u)$$

is an isomorphisms of $\mathcal{W}_0 := \{u \in \mathcal{W} : u(0, \cdot) = 0\}$ onto $\mathcal{V}'$.

Parabolic operators

Parabolic operator $\mathcal{P} \in \mathcal{L}(\mathcal{W}; \mathcal{V}')$

$$\mathcal{P}u := \partial_t u - \mathrm{div}\,(\mathbf{A}\nabla u)$$

is an isomorphisms of $\mathcal{W}_0 := \{u \in \mathcal{W} : u(0,\cdot) = 0\}$ onto $\mathcal{V}'$.
Spagnolo introduced *G-convergence* for more general parabolic operators:

$$\mathcal{P}_{\mathcal{A}} := \partial_t + \mathcal{A} : \mathcal{W} \longrightarrow \mathcal{V}' \ ,$$

where $(\mathcal{A}u)(t) := A(t)u(t)$, with $A(t) \in \mathcal{L}(V; V')$ such that for $\varphi, \psi \in V$

$$\begin{gathered} t \mapsto \langle A(t)\varphi, \psi\rangle \quad \text{is measurable} \\ \lambda_0\|\varphi\|_V^2 \leqslant \langle A(t)\varphi, \varphi\rangle \leqslant \Lambda_0\|\varphi\|_V^2 \\ |\langle A(t)\varphi, \psi\rangle| \leqslant M\sqrt{\langle A(t)\varphi, \varphi\rangle}\sqrt{\langle A(t)\psi, \psi\rangle} \ , \end{gathered}$$

where λ_0, Λ_0 and M are some positive constants.

Parabolic operators

Parabolic operator $\mathcal{P} \in \mathcal{L}(\mathcal{W}; \mathcal{V}')$

$$\mathcal{P}u := \partial_t u - \mathrm{div}\,(\mathbf{A}\nabla u)$$

is an isomorphisms of $\mathcal{W}_0 := \{u \in \mathcal{W} : u(0,\cdot) = 0\}$ onto $\mathcal{V}'$.
Spagnolo introduced *G-convergence* for more general parabolic operators:

$$\mathcal{P}_{\mathcal{A}} := \partial_t + \mathcal{A} : \mathcal{W} \longrightarrow \mathcal{V}' \ ,$$

where $(\mathcal{A}u)(t) := A(t)u(t)$, with $A(t) \in \mathcal{L}(V; V')$ such that for $\varphi, \psi \in V$

$$\begin{gathered} t \mapsto \langle A(t)\varphi, \psi\rangle \quad \text{is measurable} \\ \lambda_0 \|\varphi\|_V^2 \leqslant \langle A(t)\varphi, \varphi\rangle \leqslant \Lambda_0 \|\varphi\|_V^2 \\ |\langle A(t)\varphi, \psi\rangle| \leqslant M\sqrt{\langle A(t)\varphi, \varphi\rangle}\sqrt{\langle A(t)\psi, \psi\rangle} \ , \end{gathered}$$

where λ_0, Λ_0 and M are some positive constants.
The set of all such operators $\mathcal{P}_{\mathcal{A}}$ we denote by $\mathcal{P}(\lambda_0, \Lambda_0, M)$.
For $A(t) = -\mathrm{div}\,(\mathbf{A}(t,\cdot),\cdot)$ we write $\mathcal{P}_{\mathbf{A}}$ instead of $\mathcal{P}_{\mathcal{A}}$.

G-convergence and compactness

A sequence $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ *G-converges* to $\mathcal{P}_{\mathcal{A}}$ (and we write $\mathcal{P}_{\mathcal{A}_n} \xrightharpoonup{G} \mathcal{P}_{\mathcal{A}}$) if for any $f \in \mathcal{V}'$

$$\mathcal{P}_{\mathcal{A}_n}^{-1} f \rightharpoonup \mathcal{P}_{\mathcal{A}}^{-1} f \quad \text{in } \mathcal{W}_0 \ .$$

G-convergence and compactness

A sequence $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ *G-converges* to $\mathcal{P}_{\mathcal{A}}$ (and we write $\mathcal{P}_{\mathcal{A}_n} \xrightarrow{G} \mathcal{P}_{\mathcal{A}}$) if for any $f \in \mathcal{V}'$

$$\mathcal{P}_{\mathcal{A}_n}^{-1} f \rightharpoonup \mathcal{P}_{\mathcal{A}}^{-1} f \quad \text{in } \mathcal{W}_0 \ .$$

If $V \hookrightarrow H \hookrightarrow V'$ (continuous inclusions), if they are also compact, Spagnolo proved the compactness of G-convergence:

For any $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ there is a subsequence $\mathcal{P}_{\mathcal{A}_{n'}}$ and a $\mathcal{P}_{\mathcal{A}} \in \mathcal{P}(\lambda_0, M^2\Lambda_0, \sqrt{\Lambda_0/\lambda_0}M)$, such that $\mathcal{P}_{\mathcal{A}_{n'}} \xrightarrow{G} \mathcal{P}_{\mathcal{A}}$.

G-convergence and compactness

A sequence $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ *G-converges* to $\mathcal{P}_{\mathcal{A}}$ (and we write $\mathcal{P}_{\mathcal{A}_n} \xrightarrow{G} \mathcal{P}_{\mathcal{A}}$) if for any $f \in \mathcal{V}'$

$$\mathcal{P}_{\mathcal{A}_n}^{-1} f \rightharpoonup \mathcal{P}_{\mathcal{A}}^{-1} f \quad \text{in } \mathcal{W}_0 \ .$$

If $V \hookrightarrow H \hookrightarrow V'$ (continuous inclusions), if they are also compact, Spagnolo proved the compactness of G-convergence:

For any $\mathcal{P}_{\mathcal{A}_n} \in \mathcal{P}(\lambda_0, \Lambda_0, M)$ there is a subsequence $\mathcal{P}_{\mathcal{A}_{n'}}$ and a $\mathcal{P}_{\mathcal{A}} \in \mathcal{P}(\lambda_0, M^2\Lambda_0, \sqrt{\Lambda_0/\lambda_0}M)$, such that $\mathcal{P}_{\mathcal{A}_{n'}} \xrightarrow{G} \mathcal{P}_{\mathcal{A}}$.

If each A_n is of the form: $A_n(t)u = -\text{div}\,(\mathbf{A}_n(t,\cdot)\nabla u) \quad , u \in V$, the limit is of the same form, where the matrix coefficients $\mathbf{A}$ satisfy the same type of bounds, but with different constants. Also, in such a case, on the subsequence we have the convergence

$$\mathbf{A}_{n'}\nabla u_{n'} \rightharpoonup \mathbf{A}\nabla u \quad \text{in } \mathrm{L}^2(Q; \mathbf{R}^d) \ .$$

H-convergence

The above motivates the following definition [DM, ŽKO]:

A sequence of matrix functions $\mathbf{A}_n \in \mathcal{M}(\alpha, \beta; Q)$ *H-converges to* $\mathbf{A} \in \mathcal{M}(\alpha', \beta'; Q)$ *if for any* $f \in \mathcal{V}'$ *and* $u_0 \in H$ *the solutions of parabolic problems*

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}_n \nabla u_n) = f \\ \qquad\qquad\qquad u_n(0, \cdot) = u_0 \ . \end{cases}$$

satisfy

$$\begin{aligned} u_n &\rightharpoonup u \quad \textit{in } \mathcal{V} \\ \mathbf{A}_n \nabla u_n &\rightharpoonup \mathbf{A} \nabla u \quad \textit{in } \mathrm{L}^2(Q; \mathbf{R}^d) \ . \end{aligned}$$

H-convergence

The above motivates the following definition [DM, ŽKO]:

A sequence of matrix functions $\mathbf{A}_n \in \mathcal{M}(\alpha, \beta; Q)$ *H-converges to* $\mathbf{A} \in \mathcal{M}(\alpha', \beta'; Q)$ *if for any* $f \in \mathcal{V}'$ *and* $u_0 \in H$ *the solutions of parabolic problems*

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}_n \nabla u_n) = f \\ \qquad\qquad\qquad u_n(0, \cdot) = u_0 \ . \end{cases}$$

satisfy

$$\begin{aligned} u_n &\longrightarrow u \quad \textit{in } \mathcal{V} \\ \mathbf{A}_n \nabla u_n &\longrightarrow \mathbf{A} \nabla u \quad \textit{in } \mathrm{L}^2(Q; \mathbf{R}^d) \ . \end{aligned}$$

Moreover, $\mathbf{A} \in \mathcal{M}(\alpha, \beta; Q)$.

H-convergence

The above motivates the following definition [DM, ŽKO]:

A sequence of matrix functions $\mathbf{A}_n \in \mathcal{M}(\alpha, \beta; Q)$ *H-converges to* $\mathbf{A} \in \mathcal{M}(\alpha', \beta'; Q)$ *if for any* $f \in \mathcal{V}'$ *and* $u_0 \in H$ *the solutions of parabolic problems*

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}_n \nabla u_n) = f \\ \qquad\qquad\qquad u_n(0, \cdot) = u_0 \ . \end{cases}$$

satisfy

$$u_n \longrightarrow u \quad \textit{in } \mathcal{V}$$

$$\mathbf{A}_n \nabla u_n \longrightarrow \mathbf{A} \nabla u \quad \textit{in } \mathrm{L}^2(Q; \mathbf{R}^d)\ .$$

Moreover, $\mathbf{A} \in \mathcal{M}(\alpha, \beta; Q)$.

H-convergence still has the advantage of the proper choice of bounds (the limit stays in the chosen set).

Two remarks

In the definition of H-convergence it is enough to consider $u_0 = 0$.

Two remarks

In the definition of H-convergence it is enough to consider $u_0 = 0$.

The parabolic H-convergence is generated by a topology.

Two remarks

In the definition of H-convergence it is enough to consider $u_0 = 0$.

The parabolic H-convergence is generated by a topology.

$$X := \bigcup_{n\in\mathbf{N}} \mathcal{M}(\frac{1}{n}, n; Q) \ ,$$

for $f \in \mathcal{V}'$, define $R_f : X \longrightarrow \mathcal{W}_0$ and $Q_f : X \longrightarrow \mathrm{L}^2(Q; \mathbf{R}^d)$:

$$R_f(\mathbf{A}) := u \ , \quad \text{where } u \text{ solves} \quad \begin{cases} u_t - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u(0,\cdot) = 0 \end{cases} ,$$

with the weak topology assumed on $\mathcal{W}_0$;
and $Q_f(\mathbf{A}) := \mathbf{A}\nabla u$, with the weak topology on $\mathrm{L}^2(Q; \mathbf{R}^d)$.

Two remarks

In the definition of H-convergence it is enough to consider $u_0 = 0$.

The parabolic H-convergence is generated by a topology.

$$X := \bigcup_{n\in\mathbf{N}} \mathcal{M}(\frac{1}{n}, n; Q) ,$$

for $f \in \mathcal{V}'$, define $R_f : X \longrightarrow \mathcal{W}_0$ and $Q_f : X \longrightarrow \mathrm{L}^2(Q; \mathbf{R}^d)$:

$$R_f(\mathbf{A}) := u , \quad \text{where } u \text{ solves} \quad \begin{cases} u_t - \operatorname{div}(\mathbf{A}\nabla u) = f \\ u(0,\cdot) = 0 \end{cases} ,$$

with the weak topology assumed on $\mathcal{W}_0$;
and $Q_f(\mathbf{A}) := \mathbf{A}\nabla u$, with the weak topology on $\mathrm{L}^2(Q; \mathbf{R}^d)$.
On X, define the weakest topology such that R_f and Q_f are continuous.
It is not metrisable.

However, the relative topology on $\mathcal{M}(\alpha, \beta; Q)$ is metrisable.

Analytical dependence

Theorem. *Let* $P \subseteq \mathbf{R}$ *be an open set and the sequence* $\mathbf{A}_n : Q \times P \to \mathrm{M}_{d\times d}(\mathbf{R})$ *such that* $\mathbf{A}_n(\cdot, p) \in \mathcal{M}(\alpha, \beta; Q)$ *for* $p \in P$. *Moreover, suppose that* $p \mapsto \mathbf{A}_n(\cdot, p)$ *is analytic mapping from* P *to* $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))$.
Then, there exists a subsequence $(\mathbf{A}_{n_k})$ *such that for every* $p \in P$

$$\mathbf{A}_{n_k}(\cdot, p) \xrightarrow{H} \mathbf{A}(\cdot, p) \ \text{ in } \mathcal{M}(\alpha, \beta; Q)\,,$$

and $p \mapsto \mathbf{A}(\cdot, p)$ *is analytic mapping from* P *to* $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d}(\mathbf{R}))$. ∎

What are H-measures ?

Objects introduced twenty years ago, by Luc Tartar and (independently) Patrick Gérard.

What are H-measures ?

Objects introduced twenty years ago, by Luc Tartar and (independently) Patrick Gérard.

To a L^2 weakly convergent sequence a measure defined on the product of physical space (variable $\mathbf{x}$) and the Fourier space (variable $\boldsymbol{\xi}$ — provides a direction) is associated.

H-measures generalise defect measures: they detect the difference between strong and weak convergence.

Why a parabolic variant?

Parabolic equations:

well studied, good theory

known explicit solutions

$1:2$ is a natural ratio to start with

Why a parabolic variant?

Parabolic equations:

- well studied, good theory
- known explicit solutions

$1:2$ is a natural ratio to start with

Possible applications to problems involving different scalings, and easy generalisations to other ratios.

Why a parabolic variant?

Parabolic equations:

well studied, good theory

known explicit solutions

$1:2$ is a natural ratio to start with

Possible applications to problems involving different scalings, and easy generalisations to other ratios.

Terminology: *classical* vs. *parabolic or variant* H-measures.

These variants were recently introduced by Martin Lazar and N.A.

Why a parabolic variant?

Parabolic equations:

well studied, good theory

known explicit solutions

$1:2$ is a natural ratio to start with

Possible applications to problems involving different scalings, and easy generalisations to other ratios.

Terminology: *classical* vs. *parabolic or variant* H-measures.

These variants were recently introduced by Martin Lazar and N.A.

Notation.

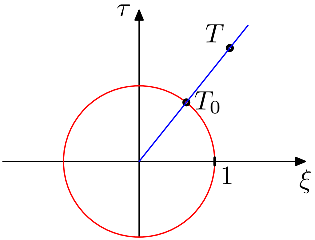
For simplicity (2D): $(t, x) = (x^0, x^1) = \mathbf{x}$ and $(\tau, \xi) = (\xi_0, \xi_1) = \boldsymbol{\xi}$

$\widehat{\ }$ or $\mathcal{F}$: the Fourier transform with $e^{-2\pi i(t\tau + x\xi)}$,

$\overline{\mathcal{F}}$: the inverse

Rough geometric idea

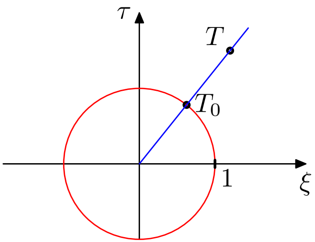
Take a sequence $\mathrm{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\hat{\mathrm{u}}_n|^2$ along rays and project to S^1



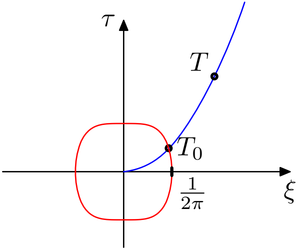
Rough geometric idea

Take a sequence $u_n \rightharpoonup 0$ in $L^2(\mathbf{R}^2)$, and integrate $|\hat{u}_n|^2$ along

rays and project to S^1



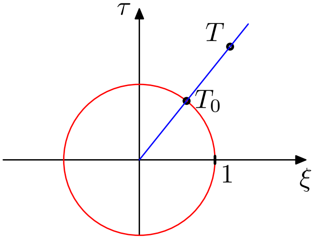
parabolas and project to P^1



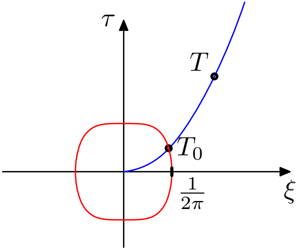
Rough geometric idea

Take a sequence $u_n \rightharpoonup 0$ in $L^2(\mathbf{R}^2)$, and integrate $|\hat{u}_n|^2$ along

rays and project to S^1



parabolas and project to P^1



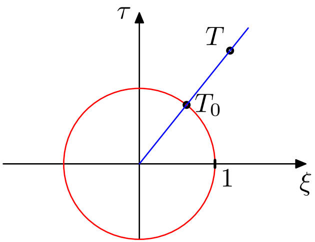
In $\mathbf{R}^2$ we have a compact surface:

$$S^1 \ldots \; r(\tau,\xi) \; := \sqrt{\tau^2+\xi^2} = 1$$

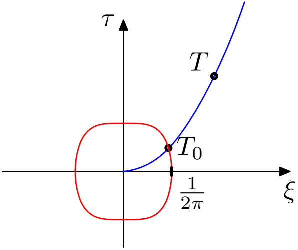
Rough geometric idea

Take a sequence $\mathrm{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\hat{\mathrm{u}}_n|^2$ along

rays and project to S^1



parabolas and project to P^1



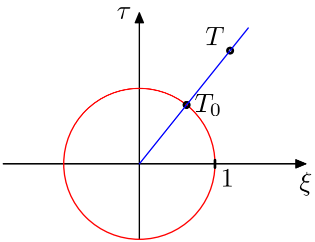
In $\mathbf{R}^2$ we have a compact surface:

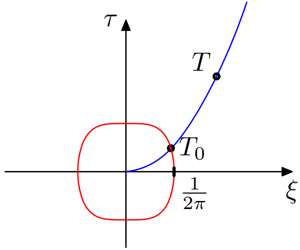
$$S^1 \ldots\ r(\tau,\xi) := \sqrt{\tau^2+\xi^2} = 1 \qquad P^1 \ldots\ \rho(\tau,\xi) := \sqrt[4]{(2\pi\tau)^2+(2\pi\xi)^4} = 1$$

Rough geometric idea

Take a sequence $u_n \rightharpoonup 0$ in $L^2(\mathbf{R}^2)$, and integrate $|\hat{u}_n|^2$ along

rays and project to S^1 parabolas and project to P^1





In $\mathbf{R}^2$ we have a compact surface:

$$S^1 \ldots \; r(\tau,\xi) \; := \sqrt{\tau^2+\xi^2} = 1 \qquad P^1 \ldots \; \rho(\tau,\xi) \; := \sqrt[4]{(2\pi\tau)^2+(2\pi\xi)^4} = 1$$

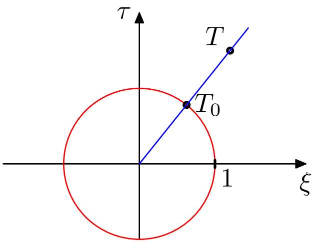
and a projection of $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ to the surface:

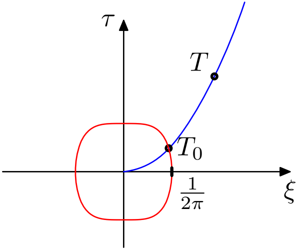
$$p(\tau,\xi) := \left(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\right)$$

Rough geometric idea

Take a sequence $\mathrm{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\hat{\mathrm{u}}_n|^2$ along

rays and project to S^1 | parabolas and project to P^1





In $\mathbf{R}^2$ we have a compact surface:

$$S^1 \ldots\ r(\tau,\xi) := \sqrt{\tau^2+\xi^2} = 1 \qquad P^1 \ldots\ \rho(\tau,\xi) := \sqrt[4]{(2\pi\tau)^2+(2\pi\xi)^4} = 1$$

and a projection of $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ to the surface:

$$p(\tau,\xi) := \left(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\right) \qquad \pi(\tau,\xi) := \left(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\right)$$

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x}) \quad ,$

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
Norm is again $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
Norm is again $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Tricky part: a is given only on S^1 or P^1.
We extend it by projections, p or π:

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x}) u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
Norm is again $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Tricky part: a is given only on S^1 or P^1.
We extend it by projections, p or π: if α is a function defined on the compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big)$$

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
Norm is again $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Tricky part: a is given only on S^1 or P^1.
We extend it by projections, p or π: if α is a function defined on the compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big) \qquad a(\tau, \xi) := \alpha\Big(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\Big)$$

Analytical view

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
Norm is again $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Tricky part: a is given only on S^1 or P^1.
We extend it by projections, p or π: if α is a function defined on the compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big) \qquad a(\tau, \xi) := \alpha\Big(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\Big)$$

Now we are ready to state the main theorem.

Existence of H-measures

Theorem. If $\mathsf{u}_n \longrightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there is a subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times S^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and any

$$\psi \in \mathrm{C}(S^{d-1})$$

we have

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p \ ) \, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi})$$

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there is a subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times P^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and any

$$\psi \in \mathrm{C}(P^{d-1})$$

we have

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ \pi)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times P^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \;.$$

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there is a subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times S^{d-1} \qquad \mathbf{R}^d \times P^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and any

$$\psi \in \mathrm{C}(S^{d-1}) \qquad \psi \in \mathrm{C}(P^{d-1})$$

we have

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ \pi)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \qquad = \int_{\mathbf{R}^d \times P^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \, .$$

Immediate properties

- $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ (hermitian)
- $\boldsymbol{\mu} \geqslant 0$ (positivity)
- $\mathsf{u}_n \otimes \mathsf{u}_n \longrightarrow \boldsymbol{\nu}$, then $\langle \boldsymbol{\nu}, \varphi \rangle = \langle \boldsymbol{\mu}, \varphi \boxtimes 1 \rangle$
- If $\mathsf{u}_{n'} \cdot \mathsf{e}_i$ have their supports in closed sets $K_i \subseteq \mathbf{R}^d$, then the support of $\boldsymbol{\mu}\mathsf{e}_i \cdot \mathsf{e}_j$ is contained in $(K_i \cap K_j) \times P^{d-1}$.

Immediate properties

- $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ (hermitian)
- $\boldsymbol{\mu} \geqslant 0$ (positivity)
- $\mathsf{u}_n \otimes \mathsf{u}_n \longrightarrow \boldsymbol{\nu}$, then $\langle \boldsymbol{\nu}, \varphi \rangle = \langle \boldsymbol{\mu}, \varphi \boxtimes 1 \rangle$
- If $\mathsf{u}_{n'} \cdot \mathsf{e}_i$ have their supports in closed sets $K_i \subseteq \mathbf{R}^d$, then the support of $\boldsymbol{\mu}\mathsf{e}_i \cdot \mathsf{e}_j$ is contained in $(K_i \cap K_j) \times P^{d-1}$.

Martin is going to say more about that tomorrow, and on the differences in the proofs for different variants.

Known results for elliptic equations

Homogenisation of parabolic equations

- H-convergence and G-convergence
- H-convergent sequence depending on a parameter

A parabolic variant of H-measures

- What are H-measures and variants ?
- A brief comparative description

Small-amplitude homogenisation

- Setting of the problem (parabolic case)
- Variant H-measures in small-amplitude homogenisation

Setting of the problem

A sequence of parabolic problems

$$(*) \qquad \begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}^n \nabla u_n) = f \\ u_n(0,\cdot) = u_0 \ . \end{cases}$$

where $\mathbf{A}^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(Q; \mathrm{M}_{d\times d})$, which is bounded from below;

Setting of the problem

A sequence of parabolic problems

$$(*) \qquad \begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}^n \nabla u_n) = f \\ \qquad\qquad u_n(0,\cdot) = u_0 \;. \end{cases}$$

where $\mathbf{A}^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(Q; \mathrm{M}_{d\times d})$, which is bounded from below; for small γ function $\mathbf{A}^n$ is analytic in γ:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma\mathbf{B}^n(t,\mathbf{x}) + \gamma^2\mathbf{C}^n(t,\mathbf{x}) + o(\gamma^2) \;,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).

Setting of the problem

A sequence of parabolic problems

$$(*) \qquad \begin{cases} \partial_t u_n - \mathrm{div}\,(\mathbf{A}^n \nabla u_n) = f \\ u_n(0,\cdot) = u_0 \ . \end{cases}$$

where $\mathbf{A}^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(Q; \mathrm{M}_{d\times d})$, which is bounded from below; for small γ function $\mathbf{A}^n$ is analytic in γ:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(t,\mathbf{x}) + \gamma^2 \mathbf{C}^n(t,\mathbf{x}) + o(\gamma^2) \ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).

Then (after passing to a subsequence if needed)

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2) \ ;$$

the limit being measurable in $t, \mathbf{x}$, and analytic in γ.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2 \mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2)\ .$$

∎

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2 \mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2)\ .$$

∎

Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \mathsf{div}\,(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2 \mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2)\ .$$

∎

Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}^1_0(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \mathsf{div}\,(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

Next, solve $(*)$ with $\mathbf{A}_\gamma^n$, f_γ and u_0, the solution u_γ^n.
Of course, f_γ and u_γ^n analytically depend on γ.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2\mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2)\ .$$

∎

Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \mathsf{div}\,(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

Next, solve $(*)$ with $\mathbf{A}_\gamma^n$, f_γ and u_0, the solution u_γ^n.
Of course, f_γ and u_γ^n analytically depend on γ.

Because of H-convergence, we have the weak convergences in $\mathrm{L}^2(Q)$:

$$(\dagger) \qquad \begin{aligned} \mathsf{E}_\gamma^n &:= \nabla u_\gamma^n \longrightarrow \nabla u \\ \mathsf{D}_\gamma^n &:= \mathbf{A}_\gamma^n \mathsf{E}_\gamma^n \longrightarrow \mathbf{A}_\gamma^\infty \nabla u\ . \end{aligned}$$

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2 \mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2) \ .$$

∎

Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \mathrm{div}\,(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

Next, solve $(*)$ with $\mathbf{A}_\gamma^n$, f_γ and u_0, the solution u_γ^n.
Of course, f_γ and u_γ^n analytically depend on γ.

Because of H-convergence, we have the weak convergences in $\mathrm{L}^2(Q)$:

$$(\dagger) \qquad \begin{aligned} \mathsf{E}_\gamma^n &:= \nabla u_\gamma^n \rightharpoonup \nabla u \\ \mathsf{D}_\gamma^n &:= \mathbf{A}_\gamma^n \mathsf{E}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u \ . \end{aligned}$$

Expansions in Taylor serieses (similarly for f_γ and u_γ^n):

$$\begin{aligned} \mathsf{E}_\gamma^n &= \mathsf{E}_0^n + \gamma \mathsf{E}_1^n + \gamma^2 \mathsf{E}_2^n + o(\gamma^2) \\ \mathsf{D}_\gamma^n &= \mathsf{D}_0^n + \gamma \mathsf{D}_1^n + \gamma^2 \mathsf{D}_2^n + o(\gamma^2) \ . \end{aligned}$$

No first-order term on the limit (cont.)

Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u \,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$
$$\mathsf{D}_1^n = \mathbf{A}_0 \mathsf{E}_1^n + \mathbf{B}^n \nabla u \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(Q) \,.$$

No first-order term on the limit (cont.)

Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u \,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$
$$\mathsf{D}_1^n = \mathbf{A}_0 \mathsf{E}_1^n + \mathbf{B}^n \nabla u \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(Q) \,.$$

Also, D_1^n converges to $\mathbf{B}_0 \nabla u$ (the term in expansion with γ^1)

$$\mathsf{D}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u = \mathbf{A}_0 \nabla u + \gamma \mathbf{B}_0 \nabla u + \gamma^2 \mathbf{C}_0 \nabla u + o(\gamma^2) \,.$$

No first-order term on the limit (cont.)

Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u \,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$
$$\mathsf{D}_1^n = \mathbf{A}_0 \mathsf{E}_1^n + \mathbf{B}^n \nabla u \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(Q) \,.$$

Also, D_1^n converges to $\mathbf{B}_0 \nabla u$ (the term in expansion with γ^1)

$$\mathsf{D}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u = \mathbf{A}_0 \nabla u + \gamma \mathbf{B}_0 \nabla u + \gamma^2 \mathbf{C}_0 \nabla u + o(\gamma^2) \,.$$

Thus $\mathbf{B}_0 \nabla u = 0$, and as $u \in \mathrm{L}^2([0,T]; \mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T \rangle; \mathrm{H}^{-1}(\Omega))$ was arbitrary, we conclude that $\mathbf{B}_0 = \mathbf{0}$.

No first-order term on the limit (cont.)

Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u\,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$
$$\mathsf{D}_1^n = \mathbf{A}_0 \mathsf{E}_1^n + \mathbf{B}^n \nabla u \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(Q)\,.$$

Also, D_1^n converges to $\mathbf{B}_0 \nabla u$ (the term in expansion with γ^1)

$$\mathsf{D}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u = \mathbf{A}_0 \nabla u + \gamma \mathbf{B}_0 \nabla u + \gamma^2 \mathbf{C}_0 \nabla u + o(\gamma^2)\,.$$

Thus $\mathbf{B}_0 \nabla u = 0$, and as $u \in \mathrm{L}^2([0,T]; \mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle; \mathrm{H}^{-1}(\Omega))$ was arbitrary, we conclude that $\mathbf{B}_0 = \mathbf{0}$.
For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \rightharpoonup \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u\,,$$

and this is the limit we still have to compute.

Expression for the quadratic correction

For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \rightharpoonup \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u \ ,$$

and this is the limit we shall express using only the parabolic variant H-measure $\boldsymbol{\mu}$.

Expression for the quadratic correction

For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \longrightarrow \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u \ ,$$

and this is the limit we shall express using only the parabolic variant H-measure $\boldsymbol{\mu}$.

u_1^n satisfies the equation $(*)$ with coefficients $\mathbf{A}_0$, div $(\mathbf{B}^n \nabla u)$ on the right hand side and the homogeneous innitial condition.

Expression for the quadratic correction

For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \longrightarrow \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u \ ,$$

and this is the limit we shall express using only the parabolic variant H-measure $\boldsymbol{\mu}$.

u_1^n satisfies the equation $(*)$ with coefficients $\mathbf{A}_0$, div $(\mathbf{B}^n \nabla u)$ on the right hand side and the homogeneous innitial condition.

By applying the Fourier transform (as if the equation were valid in the whole space), and multiplying by $2\pi i \boldsymbol{\xi}$, for $(\tau, \boldsymbol{\xi}) \neq (0, 0)$ we get

$$\widehat{\nabla u_1^n}(\tau, \boldsymbol{\xi}) = -\frac{(2\pi)^2 \, (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \, (\widehat{\mathbf{B}^n \nabla u})(\tau, \boldsymbol{\xi})}{2\pi i \tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} .$$

Expression for the quadratic correction (cont.)

As $(\boldsymbol{\xi} \otimes \boldsymbol{\xi})/(2\pi i\tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi})$ is constant along branches of paraboloids $\tau = c\boldsymbol{\xi}^2, c \in \overline{\mathbf{R}}$, we have $(\varphi \in \mathrm{C}_c^\infty(Q))$

$$\lim_n \Big\langle\, \varphi \mathbf{B}^n \mid \nabla u_1^n \,\Big\rangle = -\lim_n \Big\langle\, \widehat{\varphi \mathbf{B}^n} \mid \frac{(2\pi)^2\, (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\, (\widehat{\mathbf{B}^n \nabla u})}{2\pi i\tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \,\Big\rangle$$
$$= -\Big\langle \boldsymbol{\mu}, \varphi \frac{(2\pi)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \nabla u}{-2\pi i\tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

where $\boldsymbol{\mu}$ is the parabolic variant H-measure associated to $(\mathbf{B}^n)$, a measure with four indices (the first two of them not being contracted above).

Expression for the quadratic correction (cont.)

By varying function $u \in \mathrm{C}^1(Q)$ (e.g. choosing ∇u constant on $\langle 0, T\rangle \times \omega$, where $\omega \subseteq \Omega$) we get

$$\int_{\langle 0,T\rangle \times \omega} C_0^{ij}(t,\mathbf{x})\phi(t,\mathbf{x})dtd\mathbf{x} = -\Big\langle \boldsymbol{\mu}^{ij}, \phi \frac{(2\pi)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi}}{-2\pi i\tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

where $\boldsymbol{\mu}^{ij}$ denotes the matrix measure with components $\left(\boldsymbol{\mu}^{ij}\right)_{kl} = \mu_{iklj}$.

Expression for the quadratic correction (cont.)

By varying function $u \in \mathrm{C}^1(Q)$ (e.g. choosing ∇u constant on $\langle 0, T\rangle \times \omega$, where $\omega \subseteq \Omega$) we get

$$\int_{\langle 0,T\rangle\times\omega} C_0^{ij}(t,\mathbf{x})\phi(t,\mathbf{x})dtd\mathbf{x} = -\Big\langle \boldsymbol{\mu}^{ij}, \phi\frac{(2\pi)^2\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{-2\pi i\tau + (2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}\Big\rangle,$$

where $\boldsymbol{\mu}^{ij}$ denotes the matrix measure with components $\left(\boldsymbol{\mu}^{ij}\right)_{kl} = \mu_{iklj}$.

Remark. For the periodic example of small-amplitude homogenisation, we have got the same results by applying the variant H-measures, as with direct calculations.