THE CLOSURE OF TWO-SIDED MULTIPLICATIONS ON C*-ALGEBRAS AND PHANTOM LINE BUNDLES

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ABSTRACT. For a C^*-algebra A we consider the problem of when the set $\mathrm{TM}_0(A)$ of all two-sided multiplications $x \mapsto axb$ $(a, b \in A)$ on A is norm closed, as a subset of $\mathcal{B}(A)$. We first show that $\mathrm{TM}_0(A)$ is norm closed for all prime C^*-algebras A. On the other hand, if $A \cong \Gamma_0(\mathcal{E})$ is an n-homogeneous C^*-algebra, where $\mathcal{E}$ is the canonical $\mathbb{M}_n$-bundle over the primitive spectrum X of A, we show that $\mathrm{TM}_0(A)$ fails to be norm closed if and only if there exists a σ-compact open subset U of X and a phantom complex line subbundle $\mathcal{L}$ of $\mathcal{E}$ over U (i.e. $\mathcal{L}$ is not globally trivial, but is trivial on all compact subsets of U). This phenomenon occurs whenever $n \geq 2$ and X is a CW-complex (or a topological manifold) of dimension $3 \leq d < \infty$.

1. Introduction

Let A be a C^*-algebra and let $\mathrm{IB}(A)$ (resp. $\mathrm{ICB}(A)$) denote the set of all bounded (resp. completely bounded) maps $\phi : A \to A$ that preserve (closed two-sided) ideals of A (i.e. $\phi(I) \subseteq I$ for all ideals I of A). The most prominent class of maps $\phi \in \mathrm{ICB}(A) \subset \mathrm{IB}(A)$ are *elementary operators*, i.e. those that can be expressed as finite sums of *two-sided multiplications* $M_{a,b} : x \mapsto axb$, where a and b are elements of the multiplier algebra $M(A)$.

Elementary operators play an important role in modern quantum information and quantum computation theory. In particular, maps $\phi : \mathbb{M}_n \to \mathbb{M}_n$ ($\mathbb{M}_n$ are $n \times n$ matrices over $\mathbb{C}$) of the form $\phi = \sum_{i=1}^{\ell} M_{a_i^*, a_i}$ ($a_i \in \mathbb{M}_n$ such that $\sum_{i=1}^{\ell} a_i^* a_i = 1$) represent the (trace-duals of) quantum channels, which are mathematical models of the evolution of an 'open' quantum system (see e.g. [21]). Elementary operators also provide ways to study the structure of C^*-algebras (see [2]).

Let $\mathcal{E}\ell(A)$, $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$ denote, respectively, the sets of all elementary operators on A, two-sided multiplications on A and two-sided multiplications on A with coefficients in A (i.e. $\mathrm{TM}_0(A) = \{M_{a,b} : a, b \in A\}$).

The elementary operators are always dense in $\mathrm{IB}(A)$ in the topology of pointwise convergence (by [23, Corollary 2.3]). However, more subtle considerations enter in when one asks if $\phi \in \mathrm{ICB}(A)$ can be approximated pointwise by elementary operators of cb-norm at most $\|\phi\|_{cb}$ ([24] shows that nuclearity of A suffices; see also [26]).

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It is an interesting problem to describe those operators $\phi \in \mathrm{IB}(A)$ (resp. $\phi \in \mathrm{ICB}(A)$) that can be approximated in operator norm (resp. cb-norm) by elementary operators. Earlier works, which we cite below, revealed that this is an intricate question in general, and can involve many and varied properties of A and ϕ. In this paper, we show that the apparently much simpler problems of describing the norm closures of $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$ can have complicated answers even for rather well-behaved C^*-algebras.

In some cases $\mathcal{E}\ell(A) = \mathrm{IB}(A)$ (which implies $\mathcal{E}\ell(A) = \mathrm{ICB}(A)$); or $\mathcal{E}\ell(A)$ is norm dense in $\mathrm{IB}(A)$; or $\mathcal{E}\ell(A) \subset \mathrm{ICB}(A)$ is dense in cb-norm. The conditions just mentioned are in fact all equivalent for separable C^*-algebras A. More precisely, Magajna [25] shows that for separable C^*-algebras A, the property that $\mathcal{E}\ell(A)$ is norm (resp. cb-norm) dense in $\mathrm{IB}(A)$ (resp. $\mathrm{ICB}(A)$) characterizes finite direct sums of homogeneous C^*-algebras with the finite type property. Moreover, in this situation we already have the equality $\mathrm{ICB}(A) = \mathrm{IB}(A) = \mathcal{E}\ell(A)$.

It can happen that $\mathcal{E}\ell(A)$ is already norm closed (or cb-norm closed). In [13, 14], the first author showed that for a unital separable C^*-algebra A, if $\mathcal{E}\ell(A)$ is norm (or cb-norm) closed then A is necessarily subhomogeneous, the homogeneous subquotients of A must have the finite type property and established further necessary conditions on A. In [14, 15] he gave some partial converse results.

There is a considerable literature on derivations and inner derivations of C^*-algebras. Inner derivations d_a on a C^*-algebra A, (i.e. those of the form $d_a(x) = ax - xa$ with $a \in M(A)$) are important examples of elementary operators. In [35, Corollary 4.6] Somerset shows that if A is unital, $\{d_a : a \in A\}$ is norm closed if and only if $\mathrm{Orc}(A) < \infty$, where $\mathrm{Orc}(A)$ is a constant defined in terms of a certain graph structure on Prim(A) (the primitive spectrum of A). If $\mathrm{Orc}(A) = \infty$, the structure of outer derivations that are norm limits of inner derivations remains undescribed. In addition, if A is unital and separable, then by [19, Theorem 5.3] and [35, Corollary 4.6] $\mathrm{Orc}(A) < \infty$ if and only if the set $\{M_{u,u^*} : u \in A, u \text{ unitary}\}$ of inner automorphisms is norm closed.

In [12, 15] the first author considered the problem of which derivations on unital C^*-algebras A can be cb-norm approximated by elementary operators. By [15, Theorem 1.5] every such a derivation is necessarily inner in a case when every Glimm ideal of A is prime. When this fails, it is possible to produce examples which have outer derivations that are simultaneously elementary operators ([12, Example 6.1]).

While considering derivations d that are elementary operators and/or norm limits of inner derivations, we realized that they are sometimes expressible in the form $d = M_{a,b} - M_{b,a}$ even though they are not inner. We have not been able to decide when all such d are of this form, but this led us to the seemingly simpler question of considering the closures of $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$. In this paper, we see that nontrivial considerations enter into these questions about two-sided multiplications. Of course the left multiplications $\{M_{a,1} : a \in M(A)\}$ are already norm closed, as are the right multiplications. So $\mathrm{TM}(A)$ is a small subclass of $\mathcal{E}\ell(A)$, and seems to be the basic case to study.

This paper is organized as follows. We begin in §2 with some generalities and an explanation that the set of elementary operators of length at most ℓ has the same completion in the operator and cb-norms (for each $\ell \geq 1$).

In §3, we show that for a prime C^*-algebra A, we always have $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$ both norm closed.

In §4, we recall the description of (n-)homogeneous C^*-algebras A as sections $\Gamma_0(\mathcal{E})$ of $\mathbb{M}_n$-bundles $\mathcal{E}$ over $X = \mathrm{Prim}(A)$ and some general results about $\mathrm{IB}(A)$, $\mathrm{ICB}(A)$ and $\mathcal{E}\ell(A)$ for such A.

In §5, for homogeneous C^*-algebras $A = \Gamma_0(\mathcal{E})$, we consider subclasses $\mathrm{IB}_1(A)$ and $\mathrm{IB}_{0,1}(A)$ of $\mathrm{IB}(A)$ that seem (respectively) to be the most obvious choices for the norm closure of $\mathrm{TM}(A)$ and of $\mathrm{TM}_0(A)$, extrapolating from fibrewise restrictions on $\phi \in \mathrm{TM}(A)$. For each $\phi \in \mathrm{IB}_1(A)$ we associate a complex line subbundle $\mathcal{L}_\phi$ of the restriction $\mathcal{E}|_U$ to an open subset $U \subseteq X = \mathrm{Prim}(A)$, where U is determined by ϕ as the cozero set of ϕ (U identifies the fibres of $\mathcal{E}$ on which ϕ acts by a nonzero operator). For separable A, the main result of this section is Theorem 5.15, where we characterize the condition $\phi \in \mathrm{TM}(A)$ in terms of triviality of the bundle $\mathcal{L}_\phi$. We close §5 with Remark 5.21 comparing our bundle considerations to slightly similar results in the literature for innerness of $C(X)$-linear automorphisms when X is compact, or for some more general unital A.

Our final §6 is the main section of this paper. For homogeneous C^*-algebras $A = \Gamma_0(\mathcal{E})$, we characterize operators ϕ in the norm closure of $\mathrm{TM}_0(A)$ as those operators in $\mathrm{IB}_{0,1}(A)$ for which the associated complex line bundle $\mathcal{L}_\phi$ is trivial on each compact subset of U, where U is as above (Theorem 6.9). As a consequence, we obtain that $\mathrm{TM}_0(A)$ fails to be norm closed if and only if there exists a σ-compact open subset U of X and a phantom complex line subbundle $\mathcal{L}$ of $\mathcal{E}|_U$ (i.e. $\mathcal{L}$ is not globally trivial, but is trivial on each compact subset of U). Using this and some algebraic topological ideas, we show that $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$ both fail to be norm closed whenever A is n-homogeneous with $n \geq 2$ and X contains an open subset homeomorphic to $\mathbb{R}^d$ for some $d \geq 3$ (Theorem 6.18).

2. Preliminaries

Throughout this paper A will denote a C^*-algebra. By an *ideal* of A we always mean a closed two-sided ideal. As usual, by $Z(A)$ we denote the centre of A, by $M(A)$ the *multiplier algebra* of A, and by $\mathrm{Prim}(A)$ the *primitive spectrum* of A (i.e. the set of kernels of all irreducible representations of A equipped with the Jacobson topology).

Every $\phi \in \mathrm{IB}(A)$ is linear over $Z(M(A))$ and, for any ideal I of A, ϕ induces a map

$$\phi_I : A/I \to A/I, \quad \text{which sends} \quad a + I \text{ to } \phi(a) + I. \tag{2.1}$$

It is easy to see that the norm (resp. cb-norm) of an operator $\phi \in \mathrm{IB}(A)$ (resp. $\phi \in \mathrm{ICB}(A)$) can be computed via the formulae

$$\begin{aligned} \|\phi\| &= \sup\{\|\phi_P\| \ : \ P \in \mathrm{Prim}(A)\} \quad \text{resp.} \\ \|\phi\|_{cb} &= \sup\{\|\phi_P\|_{cb} \ : \ P \in \mathrm{Prim}(A)\}. \end{aligned} \tag{2.2}$$

The *length* of a non-zero elementary operator $\phi \in \mathcal{E}\ell(A)$ is the smallest positive integer $\ell = \ell(\phi)$ such that $\phi = \sum_{i=1}^{\ell} M_{a_i,b_i}$ for some $a_i, b_i \in M(A)$. We also define $\ell(0) = 0$. We write $\mathcal{E}\ell_\ell(A)$ for the elementary operators of length at most ℓ. Thus $\mathcal{E}\ell_1(A) = \mathrm{TM}(A)$.

We will also consider the following subsets of $\mathrm{TM}(A)$:

$$\begin{aligned} \mathrm{TM}_{cp}(A) &= \{M_{a,a^*} : a \in M(A)\}, \\ \mathrm{InnAut}_{\mathrm{alg}}(A) &= \{M_{a,a^{-1}} : a \in M(A),\ a \text{ invertible}\}, \text{ and} \\ \mathrm{InnAut}(A) &= \{M_{u,u^*} : u \in M(A),\ u \text{ unitary}\} \end{aligned} \tag{2.3}$$

(where cp and alg signify, respectively, completely positive and algebraic). Note that $\mathrm{InnAut}(A) = \mathrm{TM}_{cp}(A) \cap \mathrm{InnAut}_{\mathrm{alg}}(A)$.

It is well known that elementary operators are completely bounded with the following estimate for their cb-norm:

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h , \tag{2.4}$$

where $\|\cdot\|_h$ is the Haagerup tensor norm on the algebraic tensor product $M(A) \otimes M(A)$, i.e.

$$\|u\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} \ : \ u = \sum_i a_i \otimes b_i \right\}.$$

By inequality (2.4) the mapping

$$(M(A) \otimes M(A), \|\cdot\|_h) \to (\mathcal{E}\ell(A), \|\cdot\|_{cb}) \quad \text{given by} \quad \sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}.$$

defines a well-defined contraction. Its continuous extension to the completed Haagerup tensor product $M(A) \otimes_h M(A)$ is known as a *canonical contraction* from $M(A) \otimes_h M(A)$ to $\mathrm{ICB}(A)$ and is denoted by Θ_A.

We have the following result (see [2, Proposition 5.4.11]):

Theorem 2.1 (Mathieu)**.** *Θ_A is isometric if and only if A is a prime C^*-algebra.*

The next result is a combination of [36, Corollary 3.8], (2.2), [37, Corollary 2.4], and the facts that for $\phi = \sum_{i=1}^{\ell} M_{a_i,b_i}$, we have $\|\phi_\pi\| = \|\phi_{\ker \pi}\|$ and $\|\phi_\pi\|_{cb} = \|\phi_{\ker \pi}\|_{cb}$ where for irreducible representation $\pi\colon A \to \mathcal{B}(H_\pi)$, $\phi_\pi = \sum_{i=1}^{\ell} M_{\pi(a_i),\pi(b_i)} \in \mathcal{E}\ell(\mathcal{B}(H_\pi))$ (as in [36, §4] or [37, §2]).

Theorem 2.2 (Timoney)**.** *For A a C^*-algebra and arbitrary $\phi \in \mathcal{E}\ell(A)$ of length ℓ we have*

$$\|\phi\|_{cb} = \|\phi^{(\ell)}\| \leq \sqrt{\ell}\|\phi\|,$$

where $\phi^{(\ell)}$ denotes the ℓ-th amplification of ϕ on $M_\ell(A)$, $\phi^{(\ell)} : [x_{i,j}] \mapsto [\phi(x_{i,j})]$.

In particular, on each $\mathcal{E}\ell_\ell(A)$ the metric induced by the cb-norm is equivalent to the metric induced by the operator norm.

3. Two-sided multiplications on prime C^*-algebras

If A is a prime C^*-algebra, we prove here (Theorem 3.4) that $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$ must be closed in $\mathcal{B}(A)$.

The crucial step is the following lemma.

Lemma 3.1. *Let a, b, c and d be norm-one elements of an operator space V. If*

$$\|a \otimes b - c \otimes d\|_h < \varepsilon \leq 1/3,$$

then there exists a complex number μ of modulus one such that

$$\max\{\|a-\mu c\|, \|b-\overline{\mu}d\|\} < 4\varepsilon.$$

Proof. First we dispose of the simpler cases where a and c are linearly dependent or where b and d are linearly dependent. If a and c are dependent then $a = \mu c$ with $|\mu| = 1$. So $a \otimes b - c \otimes d = c \otimes (\mu b - d)$ and $\|a \otimes b - c \otimes d\|_h = \|\mu b - d\| < \varepsilon < 4\varepsilon$. Similarly if b and d are dependent, $b = \bar{\mu} d$ with $|\mu| = 1$ and $\|\bar{\mu} a - c\| < \varepsilon$.

Leaving aside these cases, $a \otimes b - c \otimes d$ is a tensor of rank 2. By [5, Lemma 2.3] there is an invertible matrix $S \in \mathbb{M}_2$ such that

$$\| \begin{bmatrix} a & -c \end{bmatrix} S \| < \varepsilon \quad \text{and} \quad \left\| S^{-1} \begin{bmatrix} b \\ d \end{bmatrix} \right\| < \varepsilon.$$

Write $\alpha_{i,j}$ for the i,j entry of S and $\beta_{i,j}$ for the i,j entry of S^{-1}.

Since $\alpha_{1,1}\beta_{1,1} + \alpha_{1,2}\beta_{2,1} = 1$, at least one of the absolute values $|\alpha_{1,1}|$, $|\beta_{1,1}|$, $|\alpha_{1,2}|$ or $|\beta_{2,1}|$ must be at least $1/\sqrt{2}$. We treat the four cases separately, by very similar arguments.

The case $|\alpha_{1,1}| \geq 1/\sqrt{2}$.: From

$$\begin{bmatrix} a & -c \end{bmatrix} S = \begin{bmatrix} \alpha_{1,1}a - \alpha_{2,1}c & \alpha_{1,2}a - \alpha_{2,2}c \end{bmatrix}$$

we have $\|\alpha_{1,1}a - \alpha_{2,1}c\| < \varepsilon$, so

$$\left\| a - \frac{\alpha_{2,1}}{\alpha_{1,1}} c \right\| < \frac{\varepsilon}{|\alpha_{1,1}|} \leq \sqrt{2}\varepsilon.$$

(Hence $\alpha_{2,1} \neq 0$ as $\varepsilon \leq 1/3$ and $\|a\| = 1$.) Let $\lambda = \alpha_{2,1}/\alpha_{1,1}$. Then

$$|1 - |\lambda|| = |\|a\| - |\lambda|\|c\|| \leq \|a - \lambda c\| \leq \sqrt{2}\varepsilon,$$

and so $|\lambda| \in [1 - \sqrt{2}\varepsilon, 1 + \sqrt{2}\varepsilon]$. Also

$$\left| \frac{\lambda}{|\lambda|} - \lambda \right| = |1 - |\lambda|| \leq \sqrt{2}\varepsilon.$$

So for $\mu = \frac{\lambda}{|\lambda|}$ we have $|\mu| = 1$ and

$$\|a - \mu c\| \leq \|a - \lambda c\| + |\lambda - \mu|\|c\| < \sqrt{2}\varepsilon + \sqrt{2}\varepsilon < 3\varepsilon.$$

Then

$$\begin{aligned} a \otimes b - c \otimes d &= a \otimes b - (\mu c \otimes \bar{\mu} d) \\ &= (a \otimes b) - (a \otimes \bar{\mu} d) + (a \otimes \bar{\mu} d) - (\mu c \otimes \bar{\mu} d) \\ &= (a \otimes (b - \bar{\mu} d)) + ((a - \mu c) \otimes \bar{\mu} d) \end{aligned}$$

and thus

$$\begin{aligned} \|b - \bar{\mu} d\| &= \|a \otimes (b - \bar{\mu} d)\|_h \\ &\leq \|a \otimes b - c \otimes d\|_h + \|(a - \mu c) \otimes \bar{\mu} d\|_h \\ &< \varepsilon + 3\varepsilon = 4\varepsilon. \end{aligned}$$

Case $|\alpha_{1,2}| \geq 1/\sqrt{2}$.: We start now with $\|\alpha_{1,2}a - \alpha_{2,2}c\| < \varepsilon$ and proceed in the same way (with $\lambda = \alpha_{2,2}/\alpha_{1,2}$).

Case $|\beta_{1,1}| \geq 1/\sqrt{2}$.: We use

$$S^{-1}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \beta_{1,1}b + \beta_{1,2}d \\ \beta_{2,1}b + \beta_{2,2}d \end{bmatrix}$$

and $\|\beta_{1,1}b+\beta_{1,2}d\| < \varepsilon$, leading to a similar argument (with $\lambda = -\beta_{1,2}/\beta_{1,1}$ and b taking the role of a).

Case $|\beta_{2,1}| \geq 1/\sqrt{2}$.: Use $\|\beta_{2,1}b + \beta_{2,2}d\| < \varepsilon$. □

Corollary 3.2. *If V is an operator space, the set $S_1 = \{a \otimes b : a, b \in V\}$ of all elementary tensors forms a closed subset of $V \otimes_h V$.*

Proof. Suppose $a_n \otimes b_n \to u \in V \otimes_h V$ (for $a_n, b_n \in V$). If $u = 0$, certainly $u \in S_1$ and otherwise we may assume $\|u\|_h = 1$ and also that

$$\|a_n \otimes b_n\|_h = 1 = \|a_n\| = \|b_n\| \quad (n \geq 1).$$

Passing to a subsequence, we may suppose

$$\|a_n \otimes b_n - a_{n+1} \otimes b_{n+1}\|_h \leq \frac{1}{4 \cdot 2^n} \quad (n \geq 1).$$

By Lemma 3.1, we may multiply a_n and b_n by complex conjugate modulus one scalars chosen inductively to get a'_n and b'_n such that

$$a_n \otimes b_n = a'_n \otimes b'_n, \quad \|a'_n - a'_{n+1}\| \leq 1/2^n \quad \text{and} \quad \|b'_n - b'_{n+1}\| \leq 1/2^n \quad (n \geq 1).$$

In this way we find $a = \lim_{n\to\infty} a'_n$ and $b = \lim_{n\to\infty} b'_n$ in V with $u = a \otimes b \in S_1$. □

Question 3.3. If V is an operator space and $\ell > 1$, is the set

$$S_\ell = \left\{ \sum_{i=1}^{\ell} a_i \otimes b_i : a_i, b_i \in V \right\}$$

of all tensors of rank at most ℓ closed in $V \otimes_h V$? In particular, can we extend Lemma 3.1 as follows.

Let V be an operator space and let $\mathbf{a} \odot \mathbf{b}$ and $\mathbf{c} \odot \mathbf{d}$ be two norm-one tensors of the same (finite) rank ℓ in $V \otimes_h V$, where $\mathbf{a}, \mathbf{c}$ and $\mathbf{b}, \mathbf{d}$ are, respectively, $1 \times \ell$ and $\ell \times 1$ matrices with entries in V. Suppose that $\|\mathbf{a} \odot \mathbf{b} - \mathbf{c} \odot \mathbf{d}\|_h < \varepsilon$ for some $\varepsilon > 0$. Can we find absolute constants C and δ (which depend only on ℓ and ε) so that $\delta \to 0$ as $\varepsilon \to 0$ with the following property:

There exists an invertible matrix $S \in \mathbb{M}_\ell$ such that

$$\|S\|, \|S^{-1}\| \leq C, \quad \|\mathbf{a}S^{-1} - \mathbf{c}\| < \delta \quad \text{and} \quad \|S\mathbf{b} - \mathbf{d}\| < \delta?$$

Theorem 3.4. *If A is a prime C^*-algebra, then both $\mathrm{TM}(A)$ and $\mathrm{TM}_0(A)$ are norm closed.*

Proof. By Theorem 2.2 we may work with the cb-norm instead of the (operator) norm. Since A is prime, by Mathieu's theorem (Theorem 2.1) the canonical map $\Theta : M(A) \otimes_h M(A) \to \mathrm{ICB}(A)$, $\Theta : a \otimes b \mapsto M_{a,b}$, is isometric. By Corollary 3.2, the set S_1 of all elementary tensors in $M(A) \otimes_h M(A)$ is closed in the Haagerup norm. Therefore, $\mathrm{TM}(A) = \Theta(S)$ is closed in the cb-norm.

For the case of $\mathrm{TM}_0(A)$, we use the same argument but work with the restriction of Θ to $A \otimes_h A$. □

Corollary 3.5. *If A is a prime C^*-algebra, then the sets $\mathrm{TM}_{cp}(A)$, $\mathrm{InnAut}_{\mathrm{alg}}(A)$, and $\mathrm{InnAut}(A)$ (see (2.3)) are all norm closed.*

Proof. Suppose that an operator ϕ in the norm closure of any of these sets. Then, by Theorem 3.4 there are $b, c \in M(A)$ such that $\phi = M_{b,c}$. Let $\varepsilon > 0$.

Suppose that ϕ is in the closure of $\mathrm{TM}_{cp}(A)$. By Theorem 2.2 we may work with the cb-norm instead of the (operator) norm. We may also assume that $\|\phi\|_{cb} = 1 = \|b\| = \|c\|$. Then there is $a \in M(A)$ such that

$$\|M_{b,c} - M_{a,a^*}\|_{cb} = \|b \otimes c - a \otimes a^*\|_h < \varepsilon$$

(Theorem 2.1). If $\varepsilon \leq 1/3$, by Lemma 3.1 we can find a complex number μ of modulus one such that $\|b - \mu a\| < 4\varepsilon$ and $\|c - \overline{\mu} a^*\| < 4\varepsilon$. Then $\|b - c^*\| \leq 8\varepsilon$. Hence $c = b^*$, so $\phi = M_{b,c} \in \mathrm{TM}_{cp}(A)$.

Suppose that ϕ is in the closure of $\mathrm{InnAut}_{\mathrm{alg}}(A)$. Then there is an invertible element $a \in M(A)$ such that $\|M_{b,c} - M_{a,a^{-1}}\| < \varepsilon$. Since A is an essential ideal in $M(A)$, this implies $\|bxc - axa^{-1}\| < \varepsilon$ for all $x \in M(A)$, $\|x\| \leq 1$. Letting $x = 1$ we obtain $\|bc - 1\| < \varepsilon$. Hence $c = b^{-1}$, so $\phi = M_{b,c} \in \mathrm{InnAut}_{\mathrm{alg}}(A)$.

$\mathrm{InnAut}(A)$ is norm closed as an intersection $\mathrm{TM}_{cp}(A) \cap \mathrm{InnAut}_{\mathrm{alg}}(A)$ of two closed sets. □

4. On homogeneous C^*-algebras

We recall that a C^*-algebra A is called *n-homogeneous* (where n is finite) if every irreducible representation of A acts on an n-dimensional Hilbert space. We say that A is *homogeneous* if it is n-homogeneous for some n. We will use the following definitions and facts about homogeneous C^*-algebras:

Remark 4.1. Let A be an n-homogeneous C^*-algebra. By [20, Theorem 4.2] $\mathrm{Prim}(A)$ is a (locally compact) Hausdorff space. If there is no danger of confusion, we simply write X for $\mathrm{Prim}(A)$.

(a) A well-known theorem of Fell [10, Theorem 3.2], and Tomiyama-Takesaki [38, Theorem 5] asserts that for any n-homogeneous C^*-algebra, A, there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $PU(n) = \mathrm{Aut}(\mathbb{M}_n)$ such that A is isomorphic to the C^*-algebra $\Gamma_0(\mathcal{E})$ of continuous sections of $\mathcal{E}$ which vanish at infinity.

Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with primitive spectra X_i $(i = 1, 2)$ are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ (the pullback bundle) as bundles over X_1 (see [38, Theorem 6]). Thus, we may identify A with $\Gamma_0(\mathcal{E})$.

(b) For $a \in A$ and $t \in X$ we define $\pi_t(a) = a(t)$. Then, after identifying the fibre $\mathcal{E}_t$ with $\mathbb{M}_n$, $\pi_t : a \mapsto \pi_t(a)$ (for $t \in X$) gives all irreducible representations of A (up to the equivalence).

For a closed subset $S \subseteq X$ we define

$$I_S = \bigcap_{t \in S} \ker \pi_t = \{a \in A \;:\; a(t) = 0 \text{ for all } t \in S\}.$$

By [11, VII 8.7.] any closed two-sided ideal of A is of the form I_S for some closed subset $S \subset X$. Further, by the the *generalized Tietze Extension Theorem* we may identify $A_S = A/I_S$ with $\Gamma_0(\mathcal{E}|_S)$ (see [11, II. 14.8. and VII 8.6.]). If $S = \{t\}$ we just write A_t.

(c) If $\phi \in \mathrm{IB}(A)$ and $S \subset X$ closed, we write ϕ_S for the operator ϕ_{I_S} on A_S (see (2.1)). If $S = \{t\}$ we just write ϕ_t. If A is trivial (i.e. $A = C_0(X, \mathbb{M}_n)$), we will consider ϕ_t as an operator : $\mathbb{M}_n \to \mathbb{M}_n$ (after identifying A_t with $\mathbb{M}_n$ in the obvious way).

If $U \subset X$ is open, we can regard $B = \Gamma_0(\mathcal{E}|_U)$ as the ideal $I_{X\setminus U}$ of A (by extending sections to be zero outside U) and for $\phi \in \mathrm{IB}(A)$, we then have a restriction $\phi|_U \in \mathrm{IB}(B)$ of ϕ to this ideal (with $(\phi|_U)_t = \phi_t$ for $t \in U$).

(d) $\mathrm{IB}(A) = \mathrm{ICB}(A)$. Indeed, for $\phi \in \mathrm{IB}(A)$ and $t \in X$ we have $\|\phi_t\|_{cb} \le n\|\phi_t\|$ ([29, p. 114]), so by (2.2) we have $\|\phi\|_{cb} \le n\|\phi\|$. Hence $\phi \in \mathrm{ICB}(A)$.

(e) Since each $a \in Z(A)$ has $a(t)$ a multiple of the identity in the fibre $\mathcal{E}_t$ for each $t \in X$, we can identify $Z(A)$ with $C_0(X)$. Observe that A is quasi-central (i.e. no primitive ideal of A contains $Z(A)$).

(f) By [25, Lemma 3.2] we can identify $M(A)$ with $\Gamma_b(\mathcal{E})$ (the C^*-algebra of bounded continuous sections of $\mathcal{E}$). As usual, we will identify $Z(M(A))$ with $C_b(X)$ (using the Dauns-Hofmann theorem [33, Theorem A.34]).

If $A = C_0(X, \mathbb{M}_n)$, it is well known that $M(A) = C_b(X, \mathbb{M}_n) = C(\beta X, \mathbb{M}_n)$ [1, Corollary 3.4], where βX denotes the Stone-Čech compactification.

(g) On each fibre $\mathcal{E}_t$ we can introduce an inner product $\langle \cdot, \cdot \rangle_2$ as follows.

Choose an open covering $\{U_\alpha\}$ of X such that each $\mathcal{E}|_{U_\alpha}$ is isomorphic to $U_\alpha \times \mathbb{M}_n$ (as an $\mathbb{M}_n$-bundle), say via isomorphism Φ_α. Let

$$\langle \xi, \eta \rangle_2 = \mathrm{tr}(\Phi_\alpha(\xi)\Phi_\alpha(\eta)^*) \qquad (\xi, \eta \in \mathcal{E}_t), \tag{4.1}$$

where α is chosen so that $t \in U_\alpha$ and $\mathrm{tr}(\cdot)$ is the standard trace on $\mathbb{M}_n$. This is independent of the choice of α since all automorphisms of $\mathbb{M}_n$ are inner and $\mathrm{tr}(\cdot)$ is invariant under conjugation by unitaries. If $a, b \in M(A) = \Gamma_b(\mathcal{E})$ then $t \mapsto \langle a(t), b(t) \rangle_2$ is in $C_b(X)$.

The norm $\|\cdot\|_2$ on $\mathcal{E}_t$ associated with $\langle \cdot, \cdot \rangle_2$ satisfies

$$\|\xi\| \le \|\xi\|_2 \le \sqrt{n}\|\xi\| \qquad (\xi \in \mathcal{E}_t). \tag{4.2}$$

In the terminology of [8], $(\mathcal{E}, \langle \cdot, \cdot \rangle_2)$ is a (complex continuous) Hilbert bundle of rank n^2 with fibre norms equivalent to the original C^*-norms (by (4.2)).

(h) A is said to have the *finite type property* if $\mathcal{E}$ can be trivialized over some finite open cover of X. By [25, Remark 3.3] $M(A)$ is homogeneous if and only if A has the finite type property. When this fails, it is possible to have $\mathrm{Prim}(M(A))$ non-Hausdorff [4, Theorem 2.1]. On the other hand, $M(A)$ is always quasi-standard (see [3, Corollary 4.10]).

For completeness we include a proof of the following.

Proposition 4.2. *Let X be a locally compact Hausdorff space and $A = C_0(X, \mathbb{M}_n)$.*

(a) $\mathrm{IB}(A)$ *can be identified with $C_b(X, \mathcal{B}(\mathbb{M}_n))$ by a mapping which sends an operator $\phi \in \mathrm{IB}(A)$ to the function $(t \mapsto \phi_t)$.*

(b) *Any $\phi \in \mathrm{IB}(A)$ can be written in the form*

$$\phi = \sum_{i,j=1}^{n} M_{e_{i,j}, a_{i,j}}, \tag{4.3}$$

where $(e_{i,j})_{i,j=1}^n$ are standard matrix units of $\mathbb{M}_n$ (considered as constant functions in $C_b(X, \mathbb{M}_n) = M(A)$) and $a_{i,j} \in M(A)$ depend on ϕ.

Thus, we have

$$\mathrm{IB}(A) = \mathrm{ICB}(A) = C_b(X, \mathcal{B}(\mathbb{M}_n)) = \mathcal{E}\ell(A) = \mathcal{E}\ell_{n^2}(A).$$

Proof. Let $\phi \in \mathrm{IB}(A)$.

(a) Suppose that the function $t \mapsto \phi_t \colon X \to \mathcal{B}(\mathbb{M}_n)$ is discontinuous at some point $t_0 \in X$. Then there is a net (t_α) in X converging to t_0 such that $\|\phi_{t_\alpha} - \phi_{t_0}\| \geq \delta > 0$ for all α. So there is $u_\alpha \in \mathbb{M}_n$ of norm at most 1 with $\|\phi_{t_\alpha}(u_\alpha) - \phi_{t_0}(u_\alpha)\| \geq \delta$. Passing to a subnet we may suppose $u_\alpha \to u$ and then (since $\|\phi_{t_\alpha}\| \leq \|\phi\|$ and $\|\phi_{t_0}\| \leq \|\phi\|$) we must have

$$\|\phi_{t_\alpha}(u) - \phi_{t_0}(u)\| > \delta/2$$

for α large enough.

Now choose $f \in C_0(X)$ equal to 1 on a neighbourhood of t_0 and put $a(t) = f(t)u$. We then have $a \in A$ and

$$\pi_{t_\alpha}(\phi(a)) = f(t_\alpha)\phi_{t_\alpha}(u) = \phi_{t_\alpha}(u)$$

for large α and this contradicts continuity of $\phi(a)$ at t_0.

So $t \mapsto \phi_t$ must be continuous (and also bounded by $\|\phi\|$).

Conversely, assume that the function $t \mapsto \phi_t$ is continuous and uniformly bounded by some $M > 0$. Then for $a \in A$, $t \mapsto \phi_t(\pi_t(a))$ is continuous, bounded and vanishes at infinity, hence in A. So there is an associated mapping $\phi \colon A \to A$ which is easily seen to be bounded and linear. Moreover $\phi \in \mathrm{IB}(A)$ since all ideals of A are of the form I_S for some closed $S \subset X$.

(b) First assume that A is unital, so that X is compact. Then each $x \in A$ is a linear combination over $C(X) = Z(A)$ of the $e_{i,j}$ and since ϕ is $C(X)$-linear, we have

$$x = \sum_{i,j=1}^{n} x_{i,j} e_{i,j} \quad \Rightarrow \quad \phi(x) = \sum_{i,j=1}^{n} x_{i,j} \phi(e_{i,j}).$$

We may write

$$\phi(e_{i,j}) = \sum_{k,\ell=1}^{n} \phi_{i,j,k,\ell} e_{k,\ell} = \sum_{k,r=1}^{n} e_{k,r} e_{i,j} \left(\sum_{s,\ell=1}^{n} \phi_{r,s,k,\ell} e_{s,\ell} \right)$$

where $\phi_{i,j,k,\ell} \in C(X)$. It follows that

$$\begin{aligned}
\phi(x) &= \sum_{i,j=1}^{n} x_{i,j} \left(\sum_{k,r=1}^{n} e_{k,r} e_{i,j} \left(\sum_{s,\ell=1}^{n} \phi_{r,s,k,\ell} e_{s,\ell} \right) \right) \\
&= \sum_{k,r=1}^{n} e_{k,r} \left(\left(\sum_{i,j=1}^{n} x_{i,j} e_{i,j} \right) \sum_{s,\ell=1}^{n} \phi_{r,s,k,\ell} e_{s,\ell} \right) \\
&= \sum_{k,r=1}^{n} e_{k,r} x \left(\sum_{s,\ell=1}^{n} \phi_{r,s,k,\ell} e_{s,\ell} \right).
\end{aligned}$$

Hence, ϕ is of the form (4.3), where $a_{i,j} = \sum_{s,\ell=1}^{n} \phi_{j,s,i,\ell} e_{s,\ell} \in M(A)$.

Now suppose that A is non-unital (so that X is non-compact). By (a) we can identify ϕ with the function $t \mapsto \phi_t \colon X \to \mathcal{B}(\mathbb{M}_n)$, which can be then uniquely extended to a continuous function $\beta X \to \mathcal{B}(\mathbb{M}_n)$. This extension defines an operator in $\mathrm{IB}(C(\beta X, \mathbb{M}_n)) = \mathrm{IB}(M(A))$, which we also denote by ϕ. By the first part of the proof, ϕ can be represented as (4.3). $\square$

Remark 4.3. In fact, in the case of general separable C^*-algebras A, Magajna [25] establishes the equivalence of the following properties:

(a) $\mathrm{IB}(A) = \mathcal{E}\ell(A)$.
(b) $\mathcal{E}\ell(A)$ is norm dense in $\mathrm{IB}(A)$.
(c) A is a finite direct sum of homogeneous C^*-algebras with the finite type property.

Analyzing the arguments in [25], for the implication (c) $\Rightarrow$ (a) it is sufficient to assume that X is paracompact.

Since any n-homogeneous C^*-algebra is locally of the form $C(K, \mathbb{M}_n)$ for some compact subset K of X with $K^\circ \neq \emptyset$, we have the following consequence of Proposition 4.2:

Corollary 4.4. *If A is a homogeneous C^*-algebra, then for any $\phi \in \mathrm{IB}(A)$ the function $t \mapsto \|\phi_t\|$ is continuous on X. Hence the* cozero set $\mathrm{coz}(\phi) = \{t \in X : \phi_t \neq 0\}$ *is open in X.*

5. Fibrewise length restrictions

Here we consider a homogeneous C^*-algebra $A = \Gamma_0(\mathcal{E})$ and operators $\phi \in \mathrm{IB}(A)$ such that ϕ_t is a two-sided multiplication on each fibre A_t (with $t \in X$, and $X = \mathrm{Prim}(A)$ as usual). We will write $\phi \in \mathrm{IB}_1(A)$ for this hypothesis. For separable A, the main result in this section (Theorem 5.15) characterizes when all such operators ϕ are two-sided multiplications, in terms of triviality of complex line subbundles of $\mathcal{E}|_U$ for $U \subset X$ open.

In addition to $\mathrm{IB}_1(A)$, we introduce various subsets $\mathrm{IB}_1^{\mathrm{nv}}(A)$ and $\mathrm{IB}_{0,1}(A)$ (Notation 5.5) which are designed to facilitate the description of $\mathrm{TM}(A)$, $\mathrm{TM}_0(A)$ and both of their norm closures in terms of complex line bundles. The sufficient condition that ensures $\mathrm{IB}_1^{\mathrm{nv}}(A) \subset \mathrm{TM}(A)$ is that X is paracompact with vanishing second integral Čech cohomology group $\check{H}^2(X;\mathbb{Z})$ (Corollary 5.11). For X compact of finite covering dimension d and $A = C(X, \mathbb{M}_n)$ we show that $\mathrm{TM}(A) \subsetneq \mathrm{IB}_1(A)$ provided $\check{H}^2(X;\mathbb{Z}) \neq 0$ and $n^2 \geq (d+1)/2$ (Proposition 5.12). We get the same conclusion $\mathrm{TM}(A) \subsetneq \mathrm{IB}_1(A)$ for σ-unital n-homogeneous C^*-algebras $A = \Gamma_0(\mathcal{E})$ with $n \geq 2$ provided X has a nonempty open subset homeomorphic to (an open set in) $\mathbb{R}^d$ with $d \geq 3$ (Corollary 5.16).

Notation 5.1. Let A be an n-homogeneous C^*-algebra. For $\ell \geq 1$ we write

$$\mathrm{IB}_\ell(A) = \{\phi \in \mathrm{IB}(A) : \phi_t \in \mathcal{E}\ell_\ell(A_t) \text{ for all } t \in X\}.$$

Lemma 5.2. *Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra and $\phi \in \mathrm{IB}_\ell(A)$.*

If $t_0 \in X$ is such that $\phi_{t_0} \in \mathcal{E}\ell_\ell(A_{t_0}) \setminus \mathcal{E}\ell_{\ell-1}(A_{t_0})$ (that is, such that ϕ_{t_0} has length exactly the maximal ℓ), then there are $a_1, \dots, a_\ell, b_1, \dots, b_\ell \in A$ and a compact neighbourhood N of t_0 such that ϕ agrees with the elementary operator $\sum_{i=1}^\ell M_{a_i,b_i}$ modulo the ideal I_N, that is

$$\phi(x) - \sum_{i=1}^{\ell} a_i x b_i \in I_N \qquad \text{for all } x \in A.$$

Moreover, we can choose N so that $\phi_t \in \mathcal{E}\ell_\ell(A_t) \setminus \mathcal{E}\ell_{\ell-1}(A_t)$ for all $t \in N$ (that is, ϕ_t is of the maximal length ℓ for t in a neighbourhood of t_0).

Proof. Choose a compact neighbourhood K of t_0 such that $A_K \cong C(K, \mathbb{M}_n)$ and let ϕ_K be the induced operator (Remark 4.1 (b), (c)).

Then, for $x \in A_K$ we have $\phi_K(x) = \sum_{i=1}^{n^2} c_i x d_i$ for some $c_i, d_i \in A_K$ (by Proposition 4.2 (b)). Moreover we can assume that $\{c_1(t), \ldots, c_{n^2}(t)\}$ are linearly independent for each $t \in K$, and even independent of t. Since $(\phi_K)_{t_0} = \phi_{t_0}$ has length ℓ, we must be able to write (in $\mathbb{M}_n \otimes \mathbb{M}_n$)

$$\sum_{i=1}^{n^2} c_i(t_0) \otimes d_i(t_0) = \sum_{j=1}^{\ell} c'_j \otimes d'_j.$$

We can choose $d'_1, \ldots, d'_\ell$ to be a maximal linearly independent subsequence of $d_1(t_0), \ldots, d_{n^2}(t_0)$. Then, via elementary linear algebra, there is a matrix α of size $n^2 \times \ell$ and another matrix β of size $\ell \times n^2$ so that

$$\begin{bmatrix} d_1(t_0) \\ \vdots \\ d_{n^2}(t_0) \end{bmatrix} = \alpha \begin{bmatrix} d'_1 \\ \vdots \\ d'_\ell \end{bmatrix}, \qquad \begin{bmatrix} d'_1 \\ \vdots \\ d'_\ell \end{bmatrix} = \beta \begin{bmatrix} d_1(t_0) \\ \vdots \\ d_{n^2}(t_0) \end{bmatrix}$$

and $\beta\alpha$ the identity. We have

$$\begin{bmatrix} c'_1 & \cdots & c'_\ell \end{bmatrix} = \begin{bmatrix} c_1(t_0) & \cdots & c_{n^2}(t_0) \end{bmatrix} \alpha.$$

If we define

$$\begin{bmatrix} d'_1(t) \\ \vdots \\ d'_\ell(t) \end{bmatrix} = \beta \begin{bmatrix} d_1(t) \\ \vdots \\ d_{n^2}(t) \end{bmatrix}$$

then $d'_1(t), \ldots, d'_\ell(t)$ must be linearly independent for all t in some compact neighbourhood N of t_0. Thus for $t \in N$ we have (in $\mathbb{M}_n \otimes \mathbb{M}_n$)

$$\sum_{i=1}^{n^2} c_i(t) \otimes d_i(t) = \sum_{i=1}^{n^2} c_i(t_0) \otimes d_i(t) = \sum_{j=1}^{\ell} c'_j \otimes d'_j(t).$$

By Remark 4.1 (b) we can find elements $a_j, b_j \in A$ ($1 \le j \le \ell$) such that $a_j(t) = c'_j$ and $b_j(t) = d'_j(t)$ for all $t \in N$.

Since for each $t \in N$ both of the sets $\{a_1(t), \ldots, a_\ell(t)\}$ and $\{b_1(t), \ldots, b_\ell(t)\}$ are linearly independent, we get that $\phi_t = \sum_{j=1}^{\ell} M_{a_j(t), b_j(t)}$ has length exactly ℓ for all $t \in N$ as required. □

Corollary 5.3. *Let A be a homogeneous C^*-algebra and $\phi \in \mathrm{IB}_1(A)$. If $t_0 \in X$ is such that $\phi_{t_0} \neq 0$ then there is a compact neighbourhood N of t_0 and $a, b \in A$ such that $a(t) \neq 0$ and $b(t) \neq 0$ for all $t \in N$ and ϕ agrees with $M_{a,b}$ modulo the ideal I_N.*

Remark 5.4. Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra, $a, b \in M(A) = \Gamma_b(\mathcal{E})$ and $\phi = M_{a,b}$.

We may replace a and b by

$$t \mapsto \sqrt{\frac{\|b(t)\|}{\|a(t)\|}}\, a(t) \quad \text{and} \quad t \mapsto \sqrt{\frac{\|a(t)\|}{\|b(t)\|}}\, b(t)$$

without changing ϕ so as to ensure that $\|a(t)\| = \|b(t)\|$ for each $t \in X$ and that $\|\phi_t\| = \|a(t)\|^2 = \|b(t)\|^2$ for $t \in X$.

Notation 5.5. Let A be a homogeneous C^*-algebra. We write

$$\mathrm{IB}_1^{\mathrm{nv}}(A) = \{\phi \in \mathrm{IB}(A) \ : \ 0 \neq \phi_t \in \mathrm{TM}(A_t) \text{ for all } t \in X\}$$

(where nv signifies nowhere-vanishing).

We also use

$$\begin{array}{rcl} \mathrm{IB}_0(A) & = & \{\phi \in \mathrm{IB}(A) \ : (t \mapsto \|\phi_t\|) \in C_0(X)\}, \\ \mathrm{IB}_{0,1}(A) & = & \mathrm{IB}_0(A) \cap \mathrm{IB}_1(A), \\ \mathrm{IB}_{0,1}^{\mathrm{nv}}(A) & = & \mathrm{IB}_1^{\mathrm{nv}}(A) \cap \mathrm{IB}_0(A), \text{ and} \\ \mathrm{TM}^{\mathrm{nv}}(A) & = & \mathrm{TM}(A) \cap \mathrm{IB}_1^{\mathrm{nv}}(A). \end{array}$$

By Remark 5.4, $\mathrm{TM}_0(A) = \mathrm{TM}(A) \cap \mathrm{IB}_0(A)$.

Proposition 5.6. *Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra and suppose $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A)$. Then there is a canonically associated complex line subbundle $\mathcal{L}_\phi$ of $\mathcal{E}$ with the property that*

$$\phi \in \mathrm{TM}(A) \quad \Longleftrightarrow \quad \mathcal{L}_\phi \text{ is a trivial bundle.}$$

Proof. By Corollary 5.3, locally ϕ is a two-sided multiplication. That is, given $t_0 \in X$ there is a compact neighbourhood N of t_0 and $a, b \in A$ such that $\phi_t = M_{a(t),b(t)}$ for all $t \in N$.

We define

$$\mathcal{L}_\phi \cap (\mathcal{E}|_N) \quad \text{to be} \quad \{(t, \lambda a(t)) : t \in N, \lambda \in \mathbb{C}\}.$$

Then $\mathcal{L}_\phi$ is well-defined since if N' is another neighbourhood of a possibly different $t_0' \in X$ and $a', b' \in A$ have $\phi_t = M_{a'(t),b'(t)}$ for all $t \in N'$, then there is $\mu(t) \in \mathbb{C} \setminus \{0\}$ such that $a'(t) = \mu(t)a(t)$ for $t \in N \cap N'$.

The definition we gave of $\mathcal{L}_\phi \cap (\mathcal{E}|_N)$ shows that $\mathcal{L}_\phi$ is a locally trivial complex line subbundle of $\mathcal{E}$. The map

$$: N \times \mathbb{C} \to \mathcal{L}_\phi \cap (\mathcal{E}|_N) \quad \text{given by} \quad (t, \lambda) \mapsto (t, \lambda a(t)).$$

provides a local trivialization.

If $\phi \in \mathrm{TM}(A)$, then clearly $\mathcal{L}_\phi$ is a trivial bundle. Conversely, If $\mathcal{L}_\phi$ is a trivial bundle, choose a continuous nowhere vanishing section $s : X \to \mathcal{L}_\phi$. Then for any neighbourhood N as above there is a continuous map $\zeta \colon N \to \mathbb{C} \setminus \{0\}$ such that $a(t) = \zeta(t)s(t)$. If we define $s' \colon X \to \mathcal{E}$ by $s'(t) = (1/\zeta(t))b(t)$ for $t \in N$, then we have $s, s' \in \Gamma(\mathcal{E})$ well-defined and $\phi_t(x(t)) = s(t)x(t)s'(t)$ for all $x \in A$. Normalizing s and s' as in Remark 5.4, we get $c, d \in \Gamma_b(\mathcal{E}) = M(A)$ (Remark 4.1 (f)) with $\phi = M_{c,d}$. □

Notation 5.7. If $A = \Gamma_0(\mathcal{E})$ is homogeneous and $\phi \in \mathrm{IB}_1(A)$, we consider the cozero set $U = \mathrm{coz}(\phi)$ (open by Corollary 4.4) and then, for $B = \Gamma_0(\mathcal{E}|_U)$, $\phi|_U \in \mathrm{IB}_1^{\mathrm{nv}}(B)$ (see Remark 4.1 (c)). We occasionally use $\mathcal{L}_\phi$ for the subbundle $\mathcal{L}_{\phi|_U}$ of $\mathcal{E}|_U$.

Proposition 5.8. *Let $A = \Gamma_0(\mathcal{E})$ be an n-homogeneous C^*-algebra such that X is σ-compact. If $\mathcal{L}$ is a complex line subbundle of $\mathcal{E}$, then there is $\phi \in \mathrm{IB}_{0,1}^{\mathrm{nv}}(A)$ with $\mathcal{L}_\phi = \mathcal{L}$.*

Proof. Let $\langle \cdot, \cdot \rangle_2$ be as in Remark 4.1 (g). With respect to this inner product we have a complementary subbundle $\mathcal{L}^\perp$ of $\mathcal{E}$ such that $\mathcal{L} \oplus \mathcal{L}^\perp = \mathcal{E}$.

Note that, by local compactness, X has a base consisting of σ-compact open sets. (If $t_0 \in U \subset X$ with U open, choose a compact neighborhood N of t_0 contained in U and a function $f \in C_0(X)$ supported in N with $f(t_0) = 1$. Take $V = \{t \in X : |f(t)| > 0\}$.)

Since X is σ-compact (and since every σ-compact space is Lindelöf), we can find a countable open cover $\{U_i\}_{i=1}^\infty$ of X such that each restriction $\mathcal{E}|_{U_i}$ is trivial and each U_i is σ-compact. Then we can find n^2 norm-one sections $(e_j^i)_{j=1}^{n^2}$ of $\Gamma_0(\mathcal{E}|_{U_i}) \cong C_0(U_i, \mathbb{M}_n)$ such that

$$\operatorname{span}\{e_1^i(t), \cdots, e_{n^2}^i(t)\} = \mathcal{E}_t \cong \mathbb{M}_n \quad \text{for all } t \in U_i.$$

By extending outside U_i with 0 we may assume that e_j^i are globally defined, so that $e_j^i \in A$. Define $f_j^i(t)$ as the orthogonal projection of e_j^i into the fibre $\mathcal{L}_t$, so that $f_j^i \in A$. We define

$$\phi : A \to A \quad \text{by} \quad \phi = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\sum_{j=1}^{n^2} M_{f_j^i, (f_j^i)^*} \right).$$

Note that $\phi \in \mathrm{IB}_0(A)$ as a sum of an absolutely convergent series of operators in $\mathrm{IB}_0(A)$ (and $\mathrm{IB}_0(A)$ is norm closed). We claim that $\phi \in \mathrm{IB}_{0,1}^{\mathrm{nv}}(A)$ and $\mathcal{L}_\phi = \mathcal{L}$. Indeed, for an arbitrary point $t \in X$ choose a norm-one (in C^*-norm) vector $s \in \mathcal{L}_t$. Then there are scalars λ_j^i with $f_j^i(t) = \lambda_j^i \cdot s$ and $|\lambda_j^i| = \|f_j^i(t)\| \le \sqrt{n}\|e_j^i(t)\| = \sqrt{n}$ (by (4.2)). Then

$$\phi_t = \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \left(\sum_{j=1}^{n^2} |\lambda_j^i|^2 \right) \right) \cdot M_{s,s^*}.$$

This shows that $\phi \in \mathrm{IB}_{0,1}^{\mathrm{nv}}(A)$ and that for all $t \in X$ we have $\phi_t = M_{a(t),a^*(t)}$ for some $a(t) \in \mathcal{L}_t$. By the proof of Proposition 5.6 we conclude $\mathcal{L}_\phi = \mathcal{L}$. $\square$

In the sequel, by $\lceil \cdot \rceil$ we denote the 'ceiling function' (i.e. if $x \in \mathbb{R}$ then $\lceil x \rceil$ is the smallest integer greater or equal to x).

Remark 5.9. Let $\mathcal{L}$ be a complex line bundle over a locally compact Hausdorff space X.

(a) $\mathcal{L}$ is isomorphic to a subbundle of some $\mathbb{M}_2$-bundle $\mathcal{E}$. Indeed, let $\mathcal{F} = \mathcal{L} \oplus (X \times \mathbb{C})$. Then $\mathcal{E} = \mathrm{Hom}(\mathcal{F}, \mathcal{F}) = \mathcal{F} \otimes \mathcal{F}^*$ is an $\mathbb{M}_2$-bundle with the desired property (see [32, Example 3.5]).

Further, if X is σ-compact, then $A = \Gamma_0(\mathcal{E})$ (with $\mathcal{E}$ as above) is an example of a 2-homogeneous C^*-algebra with $\mathrm{Prim}(A) = X$ that allows an operator $\phi \in \mathrm{IB}_{0,1}^{\mathrm{nv}}(A)$ such that $\mathcal{L}_\phi \cong \mathcal{L}$ (by Proposition 5.8).

(b) Suppose that $\mathcal{L}$ is a subbundle of a trivial bundle $X \times \mathbb{C}^m$. If $p = \lceil \sqrt{m} \rceil$, then for each $n \ge p$ we can regard $\mathcal{L}$ as a subbundle of a trivial matrix bundle $X \times \mathbb{M}_n$, using some linear embedding $\mathbb{C}^m \hookrightarrow \mathbb{M}_n$.

Remark 5.10. If the space X is paracompact, it is well-known that locally trivial complex line bundles over X are classified by the homotopy classes of maps from X to $\mathbb{C}P^\infty$ and/or by the elements of the second integral Čech cohomology $\check{H}^2(X; \mathbb{Z})$ (see e.g. [18, Corollary 3.5.6 and Theorem 3.4.7] and [33, Proposition 4.53 and Theorem 4.42].) By [18], we know that complex line bundles over X are pullbacks of the canonical bundle over $\mathbb{C}P^\infty$ (via a map from X to $\mathbb{C}P^\infty$).

In light of Proposition 5.6 and Remark 5.10, for a given a homogeneous C^*-algebra $A = \Gamma_0(\mathcal{E})$ we define a map

$$\theta : \mathrm{IB}_1^{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z}) \tag{5.1}$$

which sends an operator $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A)$ to the corresponding class of $\mathcal{L}_\phi$ in $\check{H}^2(X;\mathbb{Z})$. By Proposition 5.6 we have $\theta^{-1}(0) = \mathrm{TM}^{\mathrm{nv}}(A)$. As a direct consequence of this observation we have:

Corollary 5.11. *Let A be a homogeneous C^*-algebra such that X is paracompact. If $\check{H}^2(X;\mathbb{Z}) = 0$ then $\mathrm{IB}_1^{\mathrm{nv}}(A) = \mathrm{TM}^{\mathrm{nv}}(A)$.*

We will now give some sufficient conditions on a trivial homogeneous C^*-algebra A that will ensure the surjectivity of the map θ. To do this, first recall that a topological space X is said to have the *Lebesgue covering dimension* $d < \infty$ if d is the smallest non-negative integer with the property that each finite open cover of X has a refinement in which no point of X is included in more than $d+1$ elements (see e.g. [9]). In this case we write $d = \dim X$.

Proposition 5.12. *Let X be a compact Hausdorff space with $\dim X \leq d < \infty$. For $n \geq 1$ let $A_n = C(X, \mathbb{M}_n)$. If $p = \left\lceil \sqrt{(d+1)/2} \right\rceil$ then for any $n \geq p$ the mapping θ from (5.1) is surjective. In particular, if $\check{H}^2(X;\mathbb{Z}) \neq 0$, then $\mathrm{TM}^{\mathrm{nv}}(A_n) \subsetneq \mathrm{IB}_1^{\mathrm{nv}}(A_n)$ for all $n \geq p$.*

To prove this will use the following fact (which may be known):

Lemma 5.13. *Let X be a CW-complex with $\dim X = d$. Then each complex line bundle $\mathcal{L}$ over X is isomorphic to a line subbundle of $X \times \mathbb{C}^m$ with $m = \lceil (d+1)/2 \rceil$.*

Proof. We consider $\mathbb{C}P^\infty$ as a CW-complex in the usual way (see [16, Example 0.6]). Let $\Psi : X \to \mathbb{C}P^\infty$ be the classifying map of the bundle $\mathcal{L}$ (Remark 5.10). Using the cellular approximation theorem [16, Theorem 4.8] and Remark 5.10 we may assume that the map Ψ is cellular, so that Ψ takes the k-skeleton of X to the k-skeleton of $\mathbb{C}P^\infty$ for all k. Since $\mathbb{C}P^\infty$ has one cell in each even dimension, $\Psi(X)$ is contained in the d-skeleton of $\mathbb{C}P^\infty$, which is the $(d-1)$-skeleton if d is odd, and is $\mathbb{C}P^{m-1}$.

Hence $\mathcal{L}$ is isomorphic to the pullback $\Psi^*(\gamma)$ of the canonical line bundle γ on $\mathbb{C}P^{m-1}$ (Remark 5.10), a subbundle of the trivial bundle $\mathbb{C}P^{m-1} \times \mathbb{C}^m$. □

Proof of Proposition 5.12. Let $\mathcal{L}$ be any complex line bundle over X. By the proof of [32, Lemma 2.3] there exists a finite complex Y with $\dim Y \leq d$, a continuous function $f : X \to Y$, and a line bundle $\mathcal{L}'$ over Y such that $\mathcal{L} \cong f^*(\mathcal{L}')$. By Lemma 5.13 we conclude that $\mathcal{L}'$ is isomorphic to a line subbundle of $Y \times \mathbb{C}^m$, with $m = \lceil (d+1)/2 \rceil$. Hence, $\mathcal{L}$ is isomorphic to a line subbundle of $X \times \mathbb{C}^m$. By Remark 5.9 (b) if $n \geq p = \lceil \sqrt{m} \rceil = \left\lceil \sqrt{(d+1)/2} \right\rceil$ ($\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$ for all $x \geq 0$), we can assume that $\mathcal{L}$ is already a subbundle of $X \times \mathbb{M}_n$. By the proof of Proposition 5.8 we can find an operator $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A_n)$ such that $\mathcal{L}_\phi = \mathcal{L}$. By Remark 5.10 we conclude that the map θ is surjective. That $\mathrm{TM}^{\mathrm{nv}}(A) \subsetneq \mathrm{IB}_1^{\mathrm{nv}}(A_n)$ ($n \geq p$) when $H^2(X;\mathbb{Z}) \neq 0$ follows directly from previous observations and Proposition 5.6. □

Example 5.14. Note that if X is either the 2-sphere, the 2-torus or the Klein bottle, then it is well-known that $\check{H}^2(X;\mathbb{Z}) \neq 0$. In particular, if $A = C(X, \mathbb{M}_n)$ ($n \geq 2$) then Proposition 5.12 shows that $\mathrm{TM}^{\mathrm{nv}}(A) \subsetneq \mathrm{IB}_1^{\mathrm{nv}}(A)$.

Theorem 5.15. *Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. Consider the following conditions:*

(a) *For every open subset $U \subset X$, each complex line subbundle of $\mathcal{E}|_U$ is trivial.*
(b) $\mathrm{IB}_1(A) = \mathrm{TM}(A)$.
(c) $\mathrm{IB}_{0,1}(A) = \mathrm{TM}_0(A)$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). If A is separable, conditions (a), (b) and (c) are equivalent.

Proof. (a) $\Rightarrow$ (b): Assume (a) holds and $\phi \in \mathrm{IB}_1(A)$. Let $U = \mathrm{coz}(\phi)$ (open by Corollary 4.4). By Proposition 5.6 we may assume that $U \neq X$. Let $B = \Gamma_0(\mathcal{E}|_U)$ and let $\phi|_U$ be the restriction of ϕ to B. Then $\phi|_U \in \mathrm{IB}_1^{\mathrm{nv}}(B)$. By (a), $\mathcal{L}_\phi$ is trivial (on U) and by Proposition 5.6 we have $\phi|_U \in \mathrm{TM}(B)$, that is $\phi|_U = M_{c,d}$ for some $c, d \in M(B) = \Gamma_b(\mathcal{E}|_U)$. By Remark 5.4, we can suppose that $\|c(t)\|^2 = \|d(t)\|^2 = \|\phi_t\|$ for $t \in U$, so that $c, d \in B$. We can then define $a, b \in A$ by $a(t) = b(t) = 0$ for $t \in X \setminus U$ and, for $t \in U$, $a(t) = c(t)$, $b(t) = d(t)$. Then we have $\phi = M_{a,b} \in \mathrm{TM}_0(A) \subseteq \mathrm{TM}(A)$.

(b) $\Rightarrow$ (c): Take intersections with $\mathrm{IB}_0(A)$.

Now assume that A is separable, so that X is second-countable.

(c) $\Rightarrow$ (b): If $\phi \in \mathrm{IB}_1(A)$, take a strictly positive function $f \in C_0(X)$ and define $\psi \in \mathrm{IB}_{0,1}(A)$ by $\psi_t = f(t)^2\phi_t$. By (c) and Remark 5.4 we have $\psi = M_{c,d}$ for $c, d \in A$ with $\|c(t)\|^2 = \|d(t)\|^2 = \|\psi_t\|$. We can define $a, b \in M(A) = \Gamma_b(\mathcal{E})$ by $a(t) = c(t)/f(t)$ and $b(t) = d(t)/f(t)$ to get $\phi = M_{a,b} \in \mathrm{TM}(A)$.

(b) $\Rightarrow$ (a): Assume (b) holds. Let U be an open subset of X and $\mathcal{L}$ a complex line subbundle of $\mathcal{E}|_U$. By Proposition 5.8 applied to $B = \Gamma_0(\mathcal{E}|_U)$ (U is σ-compact since X is second-countable), there is $\psi \in \mathrm{IB}_0(B)$ with $\mathcal{L}_\psi = \mathcal{L}$. Since $(t \mapsto \|\psi_t\|) \in C_0(U)$, we can define $\phi \in \mathrm{IB}_0(A)$ by $\phi_t = \psi_t$ for $t \in U$ and $\phi_t = 0$ for $t \in X \setminus U$. By (b), $\phi = M_{a,b}$ for $a, b \in M(A) = \Gamma_b(\mathcal{E})$ and then $a|_U$ defines a nowhere vanishing section of $\mathcal{L}$. $\square$

Corollary 5.16. *Let $A = \Gamma_0(\mathcal{E})$ be an n-homogeneous C^*-algebra with $n \geq 2$.*

(a) *If X is second-countable with $\dim X < 2$, or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then*

$$\mathrm{IB}_{0,1}(A) = \mathrm{TM}_0(A) \quad \text{and} \quad \mathrm{IB}_1(A) = \mathrm{TM}(A).$$

(b) *If X is σ-compact and contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then*

$$\mathrm{IB}_{0,1}(A) \setminus \mathrm{TM}_0(A) \neq \emptyset \quad \text{and} \quad \mathrm{IB}_1(A) \setminus \mathrm{TM}(A) \neq \emptyset.$$

Remark 5.17. By a d-manifold we always mean a second-countable topological manifold of dimension d.

To prove this we will use the following facts (which are well-known to topologists).

Remark 5.18. If X is a metrizable space with $\dim X = d < \infty$, then any locally trivial fibre bundle over X can be trivialized over some open cover of X consisting of at most $d + 1$ elements. This follows from Dowker's and Ostrand's theorems [9, Theorems 3.2.1 and 3.2.4].

Lemma 5.19. *Let Y be a metrizable space with $\dim Y = d < \infty$ and let X be a closed subset of Y. Then any map $f : X \to \mathbb{C}P^\infty$ can be, up to homotopy, continuously extended to some open neighbourhood of X in Y.*

Proof. Let $\mathcal{L}$ be a complex line bundle over X defined by f (Remark 5.10). By [9, Theorem 3.1.4], we have $\dim X \le \dim Y = d$. By Remark 5.18 $\mathcal{L}$ can be trivialized over some open cover of X consisting of (at most) $d+1$ elements. In particular, $\mathcal{L}$ is determined by some map $g : X \to \mathbb{C}P^d$ (see e.g. [18, § 3.5]) and by Remark 5.10 g is homotopic to f. By [17, Theorem V.7.1], finite dimensional manifolds (in particular $\mathbb{C}P^d$) are ANR spaces and so by [17, Theorem III.3.2], g extends (continuously) to some open neighbourhood of X in Y. □

Proposition 5.20. *Suppose that X is a locally compact subset of a non-compact connected 2-manifold M. Then $\check{H}^2(X;\mathbb{Z}) = 0$.*

Proof. First assume that $X = M$. Then by [28, Theorem 2.2], since every 2-manifold admits a smooth structure (a classical result for which we have failed to find a complete modern reference), X is homotopy equivalent to a CW-complex of dimension $d < 2$. Using Lemma 5.13 (and Remark 5.10) we conclude that $\check{H}^2(X;\mathbb{Z}) = 0$.

Now let X be an open subset of M. Since the previous argument applies to each connected component of X, we again have $\check{H}^2(X;\mathbb{Z}) = 0$.

If X is a locally compact subset of M, then X is open in its closure $\overline{X}$. Let $Y = \overline{X} \setminus X$. Then $N = M \setminus Y$ is open in M and X is closed in N. Suppose that $\check{H}^2(X;\mathbb{Z}) \ne 0$ and let $f : X \to \mathbb{C}P^\infty$ be any non-null-homotopic map (Remark 5.10). By Lemma 5.19 f extends, up to homotopy, to a map defined on some open neighbourhood U of X in M. In particular, $\check{H}^2(U;\mathbb{Z}) \ne 0$ which contradicts the second part of the proof. □

Proof of Corollary 5.16. For (a) it suffices to show that $\check{H}^2(U;\mathbb{Z}) = 0$ for all open subsets U of X (by Theorem 5.15 and Remark 5.10).

By Proposition 5.20 this is true if X is a subset of a non-compact connected 2-manifold. Suppose that X is second-countable with $\dim X < 2$. Then for each open subset U of X we have $\dim U \le \dim X$ (by the 'subset theorem' [9, Theorem 3.1.19]), so $\check{H}^2(U;\mathbb{Z}) = 0$ (see e.g. [9, p. 94–95]).

For (b) we first choose an open subset $U \subset X$ for which $\mathcal{E}|_U \cong U \times \mathbb{M}_n$ and such that U can be considered as an open set in $\mathbb{R}^d$ ($d \ge 3$). We use the simple fact that U contains an open subset that has the homotopy type of the 2-sphere $\mathbb{S}^2$. So, replacing U by such a subset, we can find a non-trivial line subbundle $\mathcal{L}$ of $U \times \mathbb{C}^2$. By Remark 5.9 (b) we may assume that $\mathcal{L}$ is a subbundle of $U \times \mathbb{M}_n \cong \mathcal{E}|_U$. The assertion now follows from the proof of Theorem 5.15. □

Remark 5.21. In the literature there are somewhat similar phenomena that arise for unital C^*-algebras A of sections of a C^*-bundle over a (second-countable) compact Hausdorff space X. The question was to describe when the set $\mathrm{Aut}_{C(X)}(A)$ of all $C(X)$-linear automorphisms of such A coincides with the inner automorphisms of A (see e.g. [22, 34, 30, 31]). For example, if A is any separable unital continuous trace C^*-algebra with (primitive) spectrum X, there always exists an exact sequence

$$0 \longrightarrow \mathrm{InnAut}(A) \longrightarrow \mathrm{Aut}_{C(X)}(A) \xrightarrow{\eta} \check{H}^2(X;\mathbb{Z})$$

of abelian groups. In general, η does not need to be surjective unless A is stable [30, Theorem 2.1]. If A is n-homogeneous then the image of η is contained in the torsion subgroup of $\check{H}^2(X;\mathbb{Z})$ [30, 2.19]. In particular, $A = C(\mathbb{S}^2, \mathbb{M}_2)$ shows that it can happen that $\mathrm{TM}^{\mathrm{nv}}(A) \subsetneq \mathrm{IB}_1^{\mathrm{nv}}(A)$ even though $\mathrm{Aut}_{C(\mathbb{S}^2)}(A) = \mathrm{InnAut}(A)$

(since $\check{H}^2(\mathbb{S}^2;\mathbb{Z}) \cong \mathbb{Z}$ is torsion free). Our Proposition 5.12 shows that the map θ from (5.1) is surjective in this case. In contrast to η, there is no obvious group structure on the domain $\mathrm{IB}_1^{\mathrm{nv}}(A)$ of θ.

6. Closure of $\mathrm{TM}(A)$ on homogeneous C^*-algebras

Here we continue to work with n-homogeneous algebras $A = \Gamma_0(\mathcal{E})$. The class $\mathrm{IB}_1(A)$ considered in §5 is rather obviously designed to capture a restriction on the closure of $\mathrm{TM}(A)$ (and similarly $\mathrm{IB}_{0,1}(A)$ should relate to the closure of $\mathrm{TM}_0(A)$). We verify right away (Proposition 6.1) that $\mathrm{IB}_1(A)$ and $\mathrm{IB}_{0,1}(A)$ are indeed closed. However, further restrictions on the operators ϕ in the closure of $\mathrm{TM}(A)$ arise because triviality of the line bundles $\mathcal{L}_\psi$ associated with $\psi \in \mathrm{TM}(A)$ is still present for the line bundle $\mathcal{L}_\phi$ provided $U = \mathrm{coz}(\phi)$ is compact (see Corollary 6.5). If U is not compact, this triviality is evident on compact subsets of U (see Theorem 6.9, where we characterize the closure of $\mathrm{TM}_0(A)$). However $\mathcal{L}_\phi$ need not be trivial globally on U (so that $\phi \notin \mathrm{TM}(A)$ is possible) and this led us to define the concept of a phantom bundle (Definition 6.11). The terminology is by analogy with the well known concept of a phantom map (see [27]). Thus, in Corollary 6.12, we see that finding ϕ in the norm closure of $\mathrm{TM}_0(A)$ with $\phi \notin \mathrm{TM}_0(A)$ is directly related to finding suitable phantom complex line bundles.

For these to exist, we need U to have a rather complicated algebraic topological structure, and we find examples with $\pi_1(U) \cong \mathbb{Q}$ (Proposition 6.17). In fact, we can also find such examples when X contains (a copy of) an open subset of $\mathbb{R}^d$ with $d \geq 3$ and $n \geq 2$ (Theorem 6.18).

Proposition 6.1. *Let A be a homogeneous C^*-algebra. Then $\mathrm{IB}_1(A)$ and $\mathrm{IB}_{0,1}(A)$ are norm closed subsets of $\mathcal{B}(A)$.*

Proof. If $(\phi_k)_{k=1}^\infty$ is a sequence in $\mathrm{IB}_1(A)$ that converges in operator norm to $\phi \in \mathcal{B}(A)$, then it is clear that $\phi(I) \subset I$ for each ideal I of A. Thus $\phi \in \mathrm{IB}(A)$.

By (2.2) we have $\|\phi - \phi_k\| = \sup_{t\in X} \|\phi_t - (\phi_k)_t\|$ and so $\lim_{k\to\infty}(\phi_k)_t = \phi_t \in \mathcal{B}(A_t)$ (for $t \in X$). Since $A_t \cong \mathbb{M}_n$, invoking Theorem 3.4, we have $\phi_t \in \mathrm{TM}(A_t)$ (for $t \in X$) and hence $\phi \in \mathrm{IB}_1(A)$.

If $\phi_k \in \mathrm{IB}_{0,1}(A)$ for each k, then $\|(\phi_k)_t\| \to \|\phi_t\|$ uniformly for $t \in X$. As $(t \mapsto \|(\phi_k)_t\|) \in C_0(X)$, it follows that $(t \mapsto \|\phi_t\|) \in C_0(X)$ and so $\phi \in \mathrm{IB}_0(A)$. □

Lemma 6.2. *Let A be a homogeneous C^*-algebra, and let $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A)$. Then there is $\psi \in \mathrm{IB}_1^{\mathrm{nv}}(A)$ with $\psi_t = \phi_t/\|\phi_t\|$ for each $t \in X$.*

Moreover $\phi \in \mathrm{TM}(A) \iff \psi \in \mathrm{TM}(A)$.

Proof. Since $t \mapsto \|\phi_t\|$ is continuous by Corollary 4.4, we can define $\psi_t = \phi_t/\|\phi_t\|$ and get $\psi \in \mathrm{IB}(A)$ via local applications of Proposition 4.2. Clearly $\psi \in \mathrm{IB}_1^{\mathrm{nv}}(A)$.

If $\phi = M_{a,b} \in \mathrm{TM}(A)$ for $a, b \in M(A) = \Gamma_b(\mathcal{E})$, then we can normalize a and b as in Remark 5.4 and then take $c, d \in A$ with $c(t) = a(t)/\sqrt{\|\phi_t\|}$, $d(t) = b(t)/\sqrt{\|\phi_t\|}$ to get $\psi = M_{c,d}$. So $\psi \in \mathrm{TM}(A)$. We can reverse this argument. □

Remark 6.3. Let $\overline{\overline{\mathrm{TM}(A)}}$ denote the operator norm closure of $\mathrm{TM}(A)$, and similarly for $\overline{\overline{\mathrm{TM}_0(A)}}$. If A is homogeneous, then Proposition 6.1 gives $\overline{\overline{\mathrm{TM}(A)}} \subset \mathrm{IB}_1(A)$ and $\overline{\overline{\mathrm{TM}_0(A)}} \subset \mathrm{IB}_{0,1}(A)$.

Proposition 6.4. *Let $A = \Gamma_0(\mathcal{E})$ be an n-homogeneous C^*-algebra. Suppose that $\phi \in \overline{\overline{\mathrm{TM}(A)}}$ such that $\inf_{t\in X} \|\phi_t\| = \delta > 0$. Then $\phi \in \mathrm{TM}(A)$.*

Proof. Let $(\phi_k)_{k=1}^\infty$ be a sequence in $\mathrm{TM}(A)$ with $\lim_{k\to\infty}\phi_k = \phi \in \mathcal{B}(A)$. For k large enough that $\|\phi_k - \phi\| < \delta/2$ we must have $\|(\phi_k)_t\| > \delta/2$ for each $t \in X$ (and hence $\phi_k \in \mathrm{IB}_1^{\mathrm{nv}}(A)$). With no loss of generality we may assume that this holds for all $k \geq 1$.

Since

$$\sup_{t\in X} |\|(\phi_k)_t\| - \|\phi_t\|| \leq \sup_{t\in X} \|(\phi_k)_t - \phi_t\| = \|\phi_k - \phi\|,$$

we may use Lemma 6.2 to normalise each ϕ_k and ϕ and assume that

$$1 = \|\phi\| = \|\phi_t\| = \|(\phi_k)_t\| = \|\phi_k\|$$

holds for all $k \geq 1$ and $t \in X$ (and still $\lim_{k\to\infty}\phi_k = \phi$).

We now write $\phi_k = M_{a_k,b_k}$ for $a_k, b_k \in M(A) = \Gamma_b(\mathcal{E})$ such that $\|a_k(t)\| = \|b_k(t)\| = 1$ (for all $t \in X$ and all k). We consider the line bundle $\mathcal{L}_\phi$ associated with ϕ according to Proposition 5.6 which is locally expressible as $\{(t, \lambda a(t))\}$, where $\phi_t = M_{a(t),b(t)}$ locally. We assume, as we can, that $\|a(t)\| = \|b(t)\| = 1$ (locally).

Let $0 < \varepsilon < (18n)^{-1/2}$.

By Remark 4.1 (d), for k suitably large (but fixed) and $t \in X$ arbitrary, we have $\|(\phi_k)_t - \phi_t\|_{cb} < \varepsilon$. Since, by Mathieu's theorem (Theorem 2.1), we locally have

$$\|(\phi_k)_t - \phi_t\|_{cb} = \|M_{a_k(t),b_k(t)} - M_{a(t),b(t)}\|_{cb} = \|a_k(t) \otimes b_k(t) - a(t) \otimes b(t)\|_h,$$

by Lemma 3.1, we can locally find a scalar $\mu_k(t)$ of modulus 1 such that

$$\|a_k(t) - \mu_k(t)a(t)\| < 4\varepsilon$$

(note that $(18n)^{-1/2} < 1/3$ for all $n \geq 1$).

Consider the inner product $\langle\cdot,\cdot\rangle_2$ defined in Remark 4.1 (g). We claim that locally $\langle a_k(t), a(t)\rangle_2 \neq 0$. Indeed, first note that (locally)

$$|\langle a_k(t), a(t)\rangle_2| = |\langle a_k(t), \mu_k(t)a(t)\rangle_2|$$

and by (4.2)

$$\|a_k(t)\|_2 \geq 1, \quad \|\mu_k(t)a(t)\|_2 \geq 1, \quad \|a_k(t) - \mu_k(t)a(t)\|_2 < 6\sqrt{n}\varepsilon.$$

Since any two vectors v and w of norm at least 1 in a Hilbert space satisfy

$$\|v - w\|_2^2 \geq \|v\|_2^2 + \|w\|_2^2 - 2|\langle v, w\rangle_2| \geq 2(1 - |\langle v, w\rangle_2|),$$

letting $v = a_k(t)$ and $w = \mu_k(t)a(t)$, we have (locally)

$$\begin{aligned} |\langle a_k(t), \mu_k(t)a(t)\rangle_2| &\geq 1 - \frac{1}{2}\|a_k(t) - \mu_k(t)a(t)\|_2^2 > 1 - 18n\varepsilon^2 \\ &> 0. \end{aligned}$$

We can therefore define $a'(t)$ locally as the normalised (in operator norm) orthogonal projection

$$a'(t) = \frac{\langle a_k(t), a(t)\rangle_2}{|\langle a_k(t), a(t)\rangle_2|} \cdot a(t).$$

Then $t \mapsto a'(t)$ is locally well-defined and continuous (by Remark 4.1 (g)).

As $a'(t)$ is independent of multiplying $a(t)$ by unit scalars, it defines a nowhere vanishing global section of $\mathcal{L}_\phi$. By Proposition 5.6, we must have $\phi \in \mathrm{TM}(A)$, as required. □

Corollary 6.5. *Let A be a unital homogeneous C^*-algebra. Then*

$$\overline{\overline{\mathrm{TM}(A)}} \cap \mathrm{IB}_1^{\mathrm{nv}}(A) \subset \mathrm{TM}(A).$$

Proof. Let $\phi \in \overline{\overline{\mathrm{TM}(A)}} \cap \mathrm{IB}_1^{\mathrm{nv}}(A)$. Since $t \mapsto \|\phi_t\|$ is continuous (Corollary 4.4) and never vanishing on X (which is compact, as A is unital), it has a minimum value $\delta > 0$. By Proposition 6.4, $\phi \in \mathrm{TM}(A)$. □

Example 6.6. Let $A = C(X, \mathbb{M}_n)$ $(n \geq 2)$, where X is any compact Hausdorff space with $\dim X \leq 7$ and $\check{H}^2(X;\mathbb{Z}) \neq 0$. Then $\overline{\overline{\mathrm{TM}(A)}} \subsetneq \mathrm{IB}_1(A)$. Indeed, by Proposition 5.12 there exists $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A) \setminus \mathrm{TM}(A)$. By Corollary 6.5, $\phi \notin \overline{\overline{\mathrm{TM}(A)}}$. (Since A is unital, $\mathrm{TM}_0(A) = \mathrm{TM}(A)$ and $\mathrm{IB}_{0,1}(A) = \mathrm{IB}_1(A)$.)

Corollary 6.7. *If $A = \Gamma_0(\mathcal{E})$ is a homogeneous C^*-algebra, then both* $\mathrm{InnAut}_{\mathrm{alg}}(A)$ *and* $\mathrm{InnAut}(A)$ *(see (2.3)) are norm closed.*

Proof. If $M_{a,a^{-1}} \in \mathrm{InnAut}_{\mathrm{alg}}(A)$, then for all $t \in X$ we have $\|(M_{a,a^{-1}})_t\| = \|a(t)\|\|a(t)^{-1}\| \geq 1$. Hence if ϕ is in the norm closure of $\mathrm{InnAut}_{\mathrm{alg}}(A)$, we have $\|\phi_t\| \geq 1$ for each $t \in X$. By Proposition 6.4, $\phi = M_{b,c}$ for some $b, c \in M(A)$. Since $\phi_t(1) = 1$, $c(t) = b(t)^{-1}$ for each t and so $c = b^{-1} \in M(A) = \Gamma_b(\mathcal{E})$.

The proof for the $\mathrm{InnAut}(A)$ is similar. □

Remark 6.8. The results that $\mathrm{InnAut}(A)$ is norm closed if the C^*-algebra A is prime or homogeneous (in Corollaries 3.5 and 6.7) can also be deduced from [35, 19, 3]. To explain the deductions, we first identify $\mathrm{InnAut}(A)$ with $\mathrm{InnAut}(M(A))$.

If A is prime, then $M(A)$ is also prime (by [2, Lemma 1.1.7]). In particular, $\mathrm{Orc}(M(A)) = 1$ (in the sense of [35, §2]), so by [35, Corollary 4.6] inner derivations of $M(A)$ are norm closed. Then [19, Theorem 5.3] implies that $\mathrm{InnAut}(M(A))$ is also norm closed.

If A is homogeneous (or more generally quasi-central and quasi-standard in the sense of [3]), then $M(A)$ is quasi-standard [3, Corollary 4.10]. Thus we have $\mathrm{Orc}(M(A)) = 1$, and we may conclude as in the prime case.

Theorem 6.9. *Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. For an operator $\phi \in \mathcal{B}(A)$, the following two conditions are equivalent:*

(a) $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$.

(b) $\phi \in \mathrm{IB}_{0,1}(A)$ *and for* $U = \mathrm{coz}(\phi)$ *(open by Corollary 4.4)* $\mathcal{L}_\phi$ *is trivial on each compact subset of* U*.*

Proof. (a) ⇒ (b): Let $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$, so that $\phi \in \mathrm{IB}_{0,1}(A)$ (Remark 6.3). For each compact subset $K \subset U$, we have $\phi_K \in \overline{\overline{\mathrm{TM}(A_K)}}$ (recall that $A_K = \Gamma(\mathcal{E}|_K)$ by Remark 4.1 (b)). By Corollary 6.5 we have $\phi_K \in \mathrm{TM}(A_K)$, so that $\mathcal{L}_\phi$ must be trivial on K (by Proposition 5.6).

(b) ⇒ (a): Let $\phi \in \mathrm{IB}_{0,1}(A)$, so that $t \mapsto \|\phi_t\|$ is in $C_0(X)$.

For any sequence $\delta_n > 0$ decreasing strictly to 0 (for instance $\delta_n = 1/n$) let

$$K_n = \{t \in X \ : \ \|\phi_t\| \geq \delta_n\}.$$

Then each K_n is compact, $K_n \subset K_{n+1}^\circ$ and $\bigcup_{n=1}^\infty K_n = U$. By Proposition 5.6, $\psi_{K_n} \in \mathrm{TM}(A_{K_n}) = \mathrm{TM}(\Gamma(\mathcal{E}|_{K_n}))$ and so there are $a_n, b_n \in A_{K_n}$ with $\psi_{K_n} = M_{a_n,b_n}$. Using Remark 5.4, we may assume $\|a_n(t)\| = \|b_n(t)\| = \sqrt{\|\phi_t\|}$ for $t \in K_n$. By Remark 4.1 (b) we may extend a_n to $c_n \in A$ with $c_n(t) = 0$ for $t \in X \setminus K_{n+1}^\circ$ and $\|c_n(t)\|^2 \leq \delta_n$ for all $t \in X \setminus K_n$. Similarly we extend b_n to $d_n \in A$ supported in K_{n+1}° with $\|d_n(t)\|^2 \leq \delta_n$ for $t \in X \setminus K_n$. Then $(M_{c_n,d_n} - \phi)_t$ has norm at most $2\delta_n$ for all $t \in X$ and hence $\lim_{n\to\infty} M_{c_n,d_n} = \phi$. Thus $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$. □

Corollary 6.10. *For a homogeneous C^*-algebra $A = \Gamma_0(\mathcal{E})$ the following conditions are equivalent:*

(a) $\overline{\overline{\mathrm{TM}_0(A)}} = \mathrm{IB}_{0,1}(A)$.
(b) *For each σ-compact open subset U of X, every complex line subbundle of $\mathcal{E}|_U$ is trivial on all compact subsets of U.*

Proof. (a) $\Rightarrow$ (b): Let U be a σ-compact open subset of X, $B = \Gamma_0(\mathcal{E}|_U)$ and $\mathcal{L}$ a complex line subbundle of $\mathcal{E}|_U$. By Proposition 5.8 we can find an operator $\phi \in \mathrm{IB}^{\mathrm{nv}}_{0,1}(B)$ such that $\mathcal{L}_\phi = \mathcal{L}$. By extending ϕ to be zero outside U, we may assume that $\phi \in \mathrm{IB}_{0,1}(A)$, so that $U = \mathrm{coz}(\phi)$. By assumption, $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$, so by Theorem 6.9 $\mathcal{L}$ is trivial on all compact subsets of U.

(b) $\Rightarrow$ (a): If $\phi \in \mathrm{IB}_{0,1}(A)$ then $U = \mathrm{coz}(\phi)$ is an open, necessarily σ-compact subset of X (since $t \mapsto \|\phi_t\|$ is in $C_0(X)$). By assumption, $\mathcal{L}_\phi$ is trivial on every compact subset of U. Hence, $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$ by Theorem 6.9. □

Definition 6.11. A locally trivial fibre bundle $\mathcal{F}$ over a locally compact Hausdorff space X is said to be a *phantom bundle* if $\mathcal{F}$ is not globally trivial, but is trivial on each compact subset of X.

Corollary 6.12. *Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. Then* $\mathrm{TM}_0(A)$ *fails to be norm closed in $\mathcal{B}(A)$ if and only if there exists a σ-compact open subset U of X and a phantom complex line subbundle of $\mathcal{E}|_U$.*

If these equivalent conditions hold, then $\mathrm{TM}(A)$ *fails to be norm closed.*

Proof. If $\mathrm{TM}_0(A)$ fails to be norm closed, there is $\phi \in \overline{\overline{\mathrm{TM}_0(A)}} \setminus \mathrm{TM}_0(A)$. Note that $\phi \in \mathrm{IB}_{0,1}(A)$ by Proposition 6.1. By Theorem 6.9, for $U = \mathrm{coz}(\phi)$ (open and σ-compact), $\mathcal{L}_\phi$ is trivial on each compact subset of U. Moreover $\phi|_U \in \mathrm{IB}^{\mathrm{nv}}_{0,1}(B)$ for $B = \Gamma_0(\mathcal{E}|_U)$. By Proposition 5.6, if $\mathcal{L}_\phi$ is globally trivial, then $\phi|_U \in \mathrm{TM}(B) \cap \mathrm{IB}_0(B) = \mathrm{TM}_0(B)$. So $\phi|_U = M_{a,b}$ for $a, b \in B$. Since B can be considered as an ideal of A (Remark 4.1 (c)), we treat $a, b \in A$. Hence $\phi = M_{a,b} \in \mathrm{TM}_0(A)$, a contradiction. Thus $\mathcal{L}_\phi$ is a phantom bundle.

Conversely, suppose that $U \subset X$ is open and σ-compact and that $\mathcal{L}$ is a phantom complex line subbundle of $\mathcal{E}|_U$. Then, taking $B = \Gamma_0(\mathcal{E}|_U)$, Proposition 5.8 provides $\psi \in \mathrm{IB}^{\mathrm{nv}}_{0,1}(B)$ with $\mathcal{L}_\psi = \mathcal{L}$. As $\mathcal{L}$ is a phantom bundle, by Proposition 5.6, $\psi \notin \mathrm{TM}(B)$. We may define $\phi \in \mathrm{IB}_{0,1}(A)$ by $\phi_t = \psi_t$ for $t \in U$ and $\phi_t = 0$ for $t \in X \setminus U$. From $\psi = \phi|_U \notin \mathrm{TM}(B)$, we have $\phi \notin \mathrm{TM}(A)$ but $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$ by Theorem 6.9. □

We now describe below a class of homogeneous C^*-algebras A for which $\mathrm{TM}_0(A)$ and $\mathrm{TM}(A)$ both fail to be norm closed. We first explain some preliminaries.

Remark 6.13. Let G be a group and n a positive integer. Recall that a space X is called an *Eilenberg-MacLane* space of type $K(G, n)$, if it's n-th homotopy group $\pi_n(X)$ is isomorphic to G and all other homotopy groups trivial. If $n > 1$ then G must be abelian (since for all $n > 1$, the homotopy groups $\pi_n(X)$ are abelian). We state some basic facts and examples about Eilenberg-MacLane spaces:

(a) There exists a CW-complex $K(G, n)$ for any group G at $n = 1$, and abelian group G at $n > 1$. Moreover such a CW-complex is unique up to homotopy type. Hence, by abuse of notation, it is common to denote any such space by $K(G, n)$ [16, p. 365-366].

(b) Given a CW-complex X, there is a bijection between its cohomology group $H^n(X;G)$ and the homotopy classes $[X, K(G,n)]$ of maps from X to $K(G,n)$ [16, Theorem 4.57].

(c) $K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$ [16, Example 4.50]. In particular, by (b) and Remark 5.10, for each CW-complex X there is a bijection between $[X, K(\mathbb{Z},2)]$ and isomorphism classes of complex line bundles over X.

Proposition 6.14. *If X is a locally compact CW-complex of type $K(\mathbb{Q},1)$, then every non-trivial complex line bundle over X is a phantom bundle. Moreover there are uncountably many non-isomorphic such bundles.*

Proof. The standard model of $K(\mathbb{Q},1)$ is the mapping telescope Δ of the sequence

$$\mathbb{S}^1 \xrightarrow{f_1} \mathbb{S}^1 \xrightarrow{f_2} \mathbb{S}^1 \xrightarrow{f_3} \cdots, \tag{6.1}$$

where $f_n : \mathbb{S}^1 \to \mathbb{S}^1$ is given by $z \mapsto z^{n+1}$ (see e.g. [7, Example 1.9] and [16, Section 3.F]).

We first consider the case when $X = \Delta$. Applying $H_1(-;\mathbb{Z})$ to the levels of the mapping telescope (6.1) gives the system

$$\mathbb{Z} \xrightarrow{(f_1)_*} \mathbb{Z} \xrightarrow{(f_2)_*} \mathbb{Z} \xrightarrow{(f_3)_*} \cdots,$$

where $(f_n)_* : \mathbb{Z} \to \mathbb{Z}$ is given by $k \mapsto (n+1)k$ (see [16, Section 3.F]). The colimit of this system is (by [16, Proposition 3.33]) $H_1(\Delta;\mathbb{Z}) = \mathbb{Q}$ and all other integral homology groups are trivial. By the universal coefficient theorem for cohomology [16, Theorem 3.2] (see also [16, §3.F]) each integral cohomology group of Δ is trivial, except for $\check{H}^2(\Delta;\mathbb{Z})$ which is isomorphic to $\mathrm{Ext}(\mathbb{Q};\mathbb{Z})$. By [39] $\mathrm{Ext}(\mathbb{Q};\mathbb{Z})$ is isomorphic to the additive group of real numbers. Hence, by Remark 5.10, there exists uncountably many non-isomorphic complex line bundles over Δ. We claim that each non-trivial such bundle $\mathcal{L}$ is a phantom bundle. Indeed, for $n \geq 1$ let Δ_n denote the n-the level of the mapping telescope (6.1). If K be an arbitrary compact subset of Δ then K is contained in some Δ_n. Since all Δ_n's are homotopy equivalent to $\mathbb{S}^1$, and since $\check{H}^2(\mathbb{S}^1;\mathbb{Z}) = 0$, we conclude that $\mathcal{L}|_{\Delta_n}$ is trivial. Then $\mathcal{L}|_K$ is also trivial, since $K \subset \Delta_n$.

If X is another locally compact CW-complex of type $K(\mathbb{Q},1)$, then by Remark 6.13 (a), there are maps $f\colon \Delta \to X$ and $g\colon X \to \Delta$ such that $g \circ f$ and $f \circ g$ are homotopic (respectively) to the identity maps (on Δ and X, respectively). If $\mathcal{L}$ is a non-trivial complex line bundle over Δ, then $g^*(\mathcal{L})$ is non-trivial over X (Remark 5.10). Moreover $g^*(\mathcal{L})$ is a phantom bundle because $K \subset X$ compact implies $g(K) \subset \Delta$ compact and $g^*(\mathcal{L})|_K$ is a restriction of $g^*(\mathcal{L})|_{g^{-1}(g(K))} = g^*(\mathcal{L}|_{g(K)})$, which is a trivial bundle. Since g is a homotopy equivalence, every non-trivial complex line bundle over X must be isomorphic to $g^*(\mathcal{L})$ for some $\mathcal{L}$. □

Remark 6.15. With the same notation as in the proof of Proposition 6.14, one can show that for each compact subset K of Δ we have $\check{H}^2(K;\mathbb{Z}) = 0$. To sketch the proof, choose an arbitrary complex line bundle $\mathcal{L}$ over K. Then using Lemma 5.19 (and Remark 5.10) $\mathcal{L}$ can be extended to an open neighbourhood U of K. The assertion can now be established via an argument with triangulations of Δ. There is a triangulation of Δ where Δ_1 has 3 triangles and each Δ_{n+1} has $n+3$ more triangles than Δ_n. We may subdivide the triangles that touch K to get finitely many that cover K and are all contained in U. Now consider the union T of the triangles that touch K. It is enough to show $\mathcal{L}|_T$ is trivial. We can deformation retract T to a union of 1-simplices. To do so, work on one triangle (2-cell) at a time, starting with any 2-cell in Δ_1 with a 'free' edge not in the boundary of Δ_1 relative to Δ_2 (where 'free' means the edge does not bound a second 2-cell). After each step, consider the remaining 2-cells, edges and vertices. Move on to Δ_2 once all 2-cells in Δ_1 are exhausted, etc, so as to arrive at a 1-simplex after finitely many steps. As all complex line bundles over 1-simplices are trivial, we have that $\mathcal{L}|_T$ is trivial.

In private correspondence, Mladen Bestvina informed us that we can find phantom bundles even over some open subset of $\mathbb{R}^3$, and referred us to [6]. We outline the construction of such a subset.

Proposition 6.16. *There exists an open subset Ω of $\mathbb{R}^3$ of type $K(\mathbb{Q}, 1)$.*

Proof. In [6], a construction is given of dense open sets U in the 3-sphere $\mathbb{S}^3$ with fundamental groups $\pi_1(U)$ that are large subgroups of $\mathbb{Q}$. Given a sequence n_i of natural numbers $n_i > 1$, $\pi_1(U)$ can be $\{p/q \in \mathbb{Q} : p \in \mathbb{Z}, q = \prod_{i=1}^k n_i \text{ for some } k\}$. In particular we will take $n_i = i + 1$ and then $\pi_1(U) = \mathbb{Q}$.

The construction defines U as a union of closed solid tori $U = \bigcup_{i=1}^\infty S_i$. For each i, both S_i and the complement of its interior $T_i = \mathbb{S}^3 \setminus S_i^\circ$ are solid tori with intersection $S_i \cap T_i$ a (2-dimensional) torus. At each step, T_{i+1} is constructed inside T_i as an unknotted solid torus of smaller cross-sectional area that winds n_i times around the meridian circle of T_i. Since T_{i+1} can be unfolded to a standard embedding of a torus via an ambient isotopy of $\mathbb{S}^3$, S_{i+1} must be a solid torus.

Let $f : \mathbb{S}^n \to U$ be an arbitrary map. Then f maps $\mathbb{S}^n$ into one of the solid tori S_i and these are homotopic to their meridian circle. In particular $\pi_n(U) = 0$ for all $n > 1$. By Remark 6.13 U has the type $K(\mathbb{Q}, 1)$.

Choose any point $t \in \mathbb{S}^3 \setminus U$. Since $\mathbb{S}^3 \setminus \{t\}$ is homeomorphic to $\mathbb{R}^3$, say via the homeomorphism F, then $\Omega = F(U)$ is an open subset of $\mathbb{R}^3$ of the type $K(\mathbb{Q}, 1)$. $\square$

Proposition 6.17. *Let X be any locally compact σ-compact CW-complex of type $K(\mathbb{Q}, 1)$ (e.g. $X = \Delta$). Then the C^*-algebra $A = C_0(X, \mathbb{M}_n)$ ($n \geq 2$) has the following property:*

> *There exists an operator $\phi \in \mathrm{IB}_{0,1}^{\mathrm{nv}}(A) \setminus \mathrm{TM}(A)$ such that ϕ is in the norm closure of $\mathrm{TM}_{cp}(A) \cap \mathrm{TM}_0(A) = \{M_{a,a^*} : a \in A\}$.*

In particular, $\mathrm{TM}_0(A)$, $\mathrm{TM}(A)$ and $\mathrm{TM}_{cp}(A)$ all fail to be norm closed.

Further, if $X = \Delta$, we have $\overline{\overline{\mathrm{TM}_0(A)}} = \mathrm{IB}_{0,1}(A)$.

Proof. Choose any phantom complex line bundle $\mathcal{L}$ over X (Proposition 6.14). Since, by Remark 6.13 (a), X has the same homotopy type as the space Δ of Proposition 6.14 (which is a 2-dimensional complex), using Remark 5.10 and Lemma 5.13 we may assume that $\mathcal{L}$ is a subbundle of the trivial bundle $X \times \mathbb{C}^2$. We also realise $\mathbb{C}^2$ as a subset of $\mathbb{M}_n$ as $\{z_1 e_{1,1} + z_2 e_{1,2} : z_1, z_2 \in \mathbb{C}\}$, and in this way consider $\mathcal{L}$ a subbundle of $X \times \mathbb{M}_n$. By the proof of Proposition 5.8 we can find two sections a, b of $\mathcal{L}$ vanishing at infinity (so that $a, b \in A$) such that $\mathrm{span}\{a(t), b(t)\} = \mathcal{L}_t$ for each $t \in X$.

We define a map

$$\phi : A \to A \quad \text{by} \quad \phi = M_{a,a^*} + M_{b,b^*}.$$

Then ϕ defines a completely positive elementary operator on A of length at most 2. Clearly, $\phi_t \neq 0$ for all $t \in X$, so $\phi \in \mathrm{IB}_{0,1}^{\mathrm{nv}}(A)$. Also, $\mathcal{L}_\phi = \mathcal{L}$. Since the bundle $\mathcal{L}$ is non-trivial, by Proposition 5.6 we have $\phi \notin \mathrm{TM}(A)$. On the other hand, since $\mathcal{L}$ is a phantom bundle, Theorem 6.9 implies $\phi \in \overline{\overline{\mathrm{TM}_0(A)}}$. Thus $\phi \in \overline{\overline{\mathrm{TM}_0(A)}} \setminus \mathrm{TM}(A)$ and consequently ϕ has length 2.

We have ϕ_K completely positive on $A_K = \Gamma(\mathcal{E}|_K)$ for each compact $K \subset X$. Since $\mathcal{L}|_K$ is a trivial bundle, $\phi_K = M_{a,b}$ for some $a, b \in A_K$ and we may suppose $\|a(t)\| = \|b(t)\|$ holds for all $t \in K$. It follows from positivity of ϕ_t that $b(t) = a(t)^*$ (for $t \in K$). By the proof of Theorem 6.9, ϕ is in the norm closure of $\mathrm{TM}_{cp}(A) \cap \mathrm{TM}_0(A)$.

Now suppose that $X = \Delta$. Then by Remark 6.15 $\check{H}^2(K;\mathbb{Z}) = 0$ for all compact subsets K of Δ. By Corollary 6.10 (and Remark 5.10) we conclude that $\overline{\overline{\mathrm{TM}_0(A)}} = \mathrm{IB}_{0,1}(A)$. □

Recalling that Corollary 5.16 (a) and Proposition 6.1 deal with the cases where X is second-countable with $\dim X < 2$ or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, we now add the opposite conclusion for higher dimensions.

Theorem 6.18. *Let $A = \Gamma_0(\mathcal{E})$ be an n-homogeneous C^*-algebra with $n \geq 2$.*

If there is a nonempty open subset of X homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{TM}_0(A)$ and $\mathrm{TM}(A)$ both fail to be norm closed.

Proof. We first choose an open subset $U \subset X$ for which $\mathcal{E}|_U$ is trivial and such that U can be considered as an open set in $\mathbb{R}^d$ ($d \geq 3$). Choose any open subset V of U that has the homotopy type of the set Ω of Proposition 6.16. In particular, V is of type $K(\mathbb{Q}, 1)$, so it allows a phantom complex line bundle (Proposition 6.14). Now apply Corollary 6.12. □

Remark 6.19. Suppose that $A = \Gamma_0(\mathcal{E})$ is a separable n-homogeneous C^*-algebra with $n \geq 2$ such that $\dim X = d < \infty$. By Remark 5.18 (applied to an $\mathbb{M}_n$-bundle $\mathcal{E}$) A has the finite type property. Hence, by [25, Theorem 1.1], we have $\mathrm{IB}(A) = \mathcal{E}\ell(A)$. If X is either a CW-complex or a subset of a d-manifold, the following relations between $\mathrm{TM}_0(A)$, $\overline{\overline{\mathrm{TM}_0(A)}}$ and $\mathrm{IB}_{0,1}(A)$ occur:

(a) If $d < 2$ we always have $\mathrm{TM}_0(A) = \overline{\overline{\mathrm{TM}_0(A)}} = \mathrm{IB}_{0,1}(A)$ (Corollary 5.16 (a)).

(b) If $d = 2$ we have four possibilities:

  (i) $\mathrm{TM}_0(A) = \overline{\overline{\mathrm{TM}_0(A)}} = \mathrm{IB}_{0,1}(A)$. This happens e.g. whenever X is a subset of a non-compact connected 2-manifold (Corollary 5.16 (a)).

  (ii) $\mathrm{TM}_0(A) = \overline{\overline{\mathrm{TM}_0(A)}} \subsetneq \mathrm{IB}_{0,1}(A)$. This happens e.g. for $A = C(X, \mathbb{M}_n)$, where $X = \mathbb{S}^2$ (by Example 5.14 and since any proper open subset U of $\mathbb{S}^2$ is homeomorphic to an open subset of $\mathbb{R}^2$, so $\check{H}^2(U;\mathbb{Z}) = 0$ by Proposition 5.20).

  (iii) $\mathrm{TM}_0(A) \subsetneq \overline{\overline{\mathrm{TM}_0(A)}} = \mathrm{IB}_{0,1}(A)$. This happens e.g. for $A = C_0(X, \mathbb{M}_n)$, where $X = \Delta$ is the standard model of $K(\mathbb{Q}, 1)$ (Proposition 6.17).

  (iv) $\mathrm{TM}_0(A) \subsetneq \overline{\overline{\mathrm{TM}_0(A)}} \subsetneq \mathrm{IB}_{0,1}(A)$. This happens e.g. for $A = C_0(X, \mathbb{M}_n)$, where X is the topological disjoint union $\mathbb{S}^2 \sqcup \Delta$ (by Proposition 6.17, Corollary 6.10 and Example 5.14).

(c) If $d > 2$ we always have $\mathrm{TM}_0(A) \subsetneq \overline{\overline{\mathrm{TM}_0(A)}} \subsetneq \mathrm{IB}_{0,1}(A)$ (by Theorem 6.18 and the fact that X must contain an open subset homeomorphic to $\mathbb{R}^d$ —

if X is a subset of a d-manifold, this follows from [9, Theorems 1.7.7, 1.8.9 and 4.1.9]).

Similar relations occur between $\mathrm{TM}(A)$, $\overline{\overline{\mathrm{TM}(A)}}$ and $\mathrm{IB}_1(A)$ in parts (a) and (c) of the above cases.

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