ELEMENTARY OPERATORS ON HILBERT MODULES OVER PRIME C^*-ALGEBRAS

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Abstract. Let X be a right Hilbert module over a C^*-algebra A equipped with the canonical operator space structure. We define an elementary operator on X as a map $\phi : X \to X$ for which there exists a finite number of elements u_i in the C^*-algebra $\mathbb{B}(X)$ of adjointable operators on X and v_i in the multiplier algebra $M(A)$ of A such that $\phi(x) = \sum_i u_i x v_i$ for $x \in X$. If $X = A$ this notion agrees with the standard notion of an elementary operator on A. In this paper we extend Mathieu's theorem for elementary operators on prime C^*-algebras by showing that the completely bounded norm of each elementary operator on a non-zero Hilbert A-module X agrees with the Haagerup norm of its corresponding tensor in $\mathbb{B}(X) \otimes M(A)$ if and only if A is a prime C^*-algebra.

1. Introduction

An operator on a C^*-algebra A is called an elementary operator if it can be expressed as a finite sum of two-sided multiplications $M_{a,b} : x \mapsto axb$, where a and b are elements of the multiplier algebra $M(A)$. In other words, an elementary operator on A is a map $\phi : A \to A$ of the form $\phi : x \mapsto \sum_i a_i x b_i$ for some finite collections of $a_i, b_i \in M(A)$. Obviously, such a representation of an elementary operator is not unique.

It is well-known that elementary operators on C^*-algebras are completely bounded mappings with the following estimate for their cb-norm:

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h ,$$

where $\| \cdot \|_h$ is the Haagerup tensor norm on the algebraic tensor product $M(A) \otimes M(A)$, i.e.

$$\|t\|_h = \inf \left\{ \left\| \sum_i u_i u_i^* \right\|^{\frac{1}{2}} \left\| \sum_i v_i^* v_i \right\|^{\frac{1}{2}} \ : \ t = \sum_i u_i \otimes v_i \right\}.$$

Hence, if $\mathrm{CB}(A)$ denotes the set of all completely bounded maps on A, the above inequality ensures that the mapping

$$(M(A) \otimes M(A), \| \cdot \|_h) \to (\mathrm{CB}(A), \| \cdot \|_{cb}) \quad \text{given by} \quad \sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}$$

Date: January 12, 2020.

2010 *Mathematics Subject Classification.* Primary 46L08, 46L07, Secondary 47L25.

Key words and phrases. C^*-algebra, prime, Hilbert C^*-module, elementary operator, completely bounded map.

This work has been fully supported by the Croatian Science Foundation under the project IP-2016-06-1046.

is a well-defined contraction. Its continuous extension to the Haagerup tensor product $M(A) \otimes_h M(A)$ (which is the completion of $M(A) \otimes M(A)$ in $\|\cdot\|_h$) is known as the canonical contraction from $M(A) \otimes_h M(A)$ to $\mathrm{CB}(A)$ and is denoted by Θ_A.

An interesting and a non-trivial question is to characterize the case when Θ_A is isometric or injective. The obvious necessary condition for the injectivity of Θ_A is that A is a prime C^*-algebra. It turns out that the primeness of A is also a sufficient condition for Θ_A to be isometric. First, Haagerup showed in [11] that Θ_A is isometric if A is the C^*-algebra of all bounded linear operators on a Hilbert space. Then Chatterjee and Sinclair showed in [7] that Θ_A is isometric if A is a separably-acting von Neumann factor. Finally, Mathieu completed the answer to this problem [1, Proposition 5.4.11]:

Theorem 1.1 (Mathieu)**.** *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) *Θ_A is isometric.*
(ii) *Θ_A is injective.*
(iii) *A is a prime C^*-algebra.*

If a C^*-algebra A is unital, but not necessarily prime, one can construct a central Haagerup tensor product $A \otimes_{Z,h} A$ and consider the induced contraction $\Theta_A^Z : A \otimes_{Z,h} A \to \mathrm{CB}(A)$. The analogous questions about Θ_A^Z were treated in [17, 2, 3].

It is also an interesting problem to consider which classes of maps (like derivations or automorphisms) on C^*-algebras can be approximated by two-sided multiplications or elementary operators in the operator or completely bounded norm. For results on this subject we refer to [9, 10] and the references within.

The purpose of this paper is to extend Theorem 1.1 to the class of operators on Hilbert C^*-modules which generalize elementary operators on C^*-algebras.

2. Preliminaries

Throughout the paper A will be a C^*-algebra. By an ideal of A we always mean a closed two-sided ideal. An ideal I of A is said to be *essential* if for any $a \in A$, $aI = \{0\}$ (or $Ia = \{0\}$) implies $a = 0$.

A C^*-algebra A is said to be *prime* if the product of any two non-zero ideals of A is non-zero. Equivalently, A is prime if for $a, b \in A$ such that $aAb = \{0\}$ it follows that $a = 0$ or $b = 0$ (see e.g. [5, Lemma 2.17]).

A *Hilbert C^*-module over A* (or a *Hilbert A-module*) is a right A-module X equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : X \times X \to A$ such that X is a Banach space with respect to the norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Recall that the inner product on X has the properties

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,
(2) $\langle x, ya\rangle = \langle x, y\rangle a$,
(3) $\langle x, y\rangle = \langle y, x\rangle^*$,
(4) $\langle x, x\rangle \geq 0$; $\langle x, x\rangle = 0 \Leftrightarrow x = 0$,

that are satisfied for all $x, y, z \in X$, $a \in A$ and $\alpha, \beta \in \mathbb{C}$. In a similar way a left Hilbert A-module is defined; the only differences are that we have a left module action and an inner product is linear and A-linear in the first variable instead of in the second variable.

For a Hilbert A-module X we denote by $\langle X, X\rangle$ the closed linear span of the set $\{\langle x, y\rangle : x, y \in X\}$. Clearly, $\langle X, X\rangle$ is an ideal of A. If $\langle X, X\rangle = A$, X is said to be *full*. We will say that X is *essentially full* if $\langle X, X\rangle$ is an essential ideal of A.

Every C^*-algebra can be regarded as a Hilbert C^*-module over itself with respect to the inner product $\langle a, b\rangle = a^*b$. Also, if I is an ideal in a C^*-algebra A then I can be regarded as a Hilbert A-module with the same inner product. Further, if $X_1, \ldots, X_n$ are Hilbert A-modules, then $X_1 \oplus \ldots \oplus X_n$ is a Hilbert A-module with respect to the module action given as

$$(x_1 \oplus \ldots \oplus x_n)a = x_1 a \oplus \ldots \oplus x_n a$$

and the inner product

$$\langle x_1 \oplus \ldots \oplus x_n, y_1 \oplus \ldots \oplus y_n\rangle = \sum_{i=1}^{n} \langle x_i, y_i\rangle.$$

If X and Y are Hilbert A-modules we denote by $\mathbb{B}(X, Y)$ the Banach space of all *adjointable operators* from X to Y, that is, those $u : X \to Y$ for which there is $u^* : Y \to X$ with the property

$$\langle ux, y\rangle = \langle x, u^*y\rangle \qquad \forall x \in X,\, y \in Y.$$

It is well-known that all adjointable operators are bounded and A-linear (i.e. $u(xa) = (ux)a$ for all $x \in X$ and $a \in A$). By $\mathbb{K}(X, Y)$ we denote the Banach subspace of $\mathbb{B}(X, Y)$ generated by the maps

$$\theta_{y,x} : X \to Y, \qquad z \mapsto y\langle x, z\rangle,$$

where $x \in X$ and $y \in Y$ are arbitrary. If $X = Y$ we write $\mathbb{B}(X)$ and $\mathbb{K}(X)$ (or $\mathbb{B}_A(X)$ and $\mathbb{K}_A(X)$ when we want to emphasize the underlying C^*-algebra A), and these are C^*-algebras. Moreover, $\mathbb{B}(X)$ is the multiplier C^*-algebra of $\mathbb{K}(X)$ ([15, Corollary 2.54]). If we regard a C^*-algebra A as a Hilbert module over itself, then $\mathbb{B}(A)$ is actually the multiplier C^*-algebra $M(A)$ of A.

If X is a Hilbert A-module then, regarding A as a Hilbert A-module, $A \oplus X$ becomes a Hilbert A-module in above-mentioned way, so the C^*-algebras $\mathbb{K}(A \oplus X)$ and $\mathbb{B}(A \oplus X)$ are well defined. The first of them, i.e. $\mathbb{K}(A \oplus X)$, is known as the *linking algebra* of X; we denote it by $\mathcal{L}(X)$ ([6, p. 350]). Then we can write

$$\mathcal{L}(X) = \begin{bmatrix} \mathbb{K}(A) & \mathbb{K}(X, A) \\ \mathbb{K}(A, X) & \mathbb{K}(X) \end{bmatrix} = \left\{ \begin{bmatrix} T_a & l_y \\ r_x & u \end{bmatrix} : a \in A,\, x, y \in X,\, u \in \mathbb{K}(X) \right\},$$

where $T_a(b) = ab$ and $r_x(b) = xb$ for all $b \in A$, while $l_y(z) = \langle y, z\rangle$ for all $z \in X$. Thereby, $a \mapsto T_a$ is an isomorphism of C^*-algebras A and $\mathbb{K}(A)$, $y \mapsto l_y$ is an isometric conjugate linear isomorphism between Banach spaces X and $\mathbb{K}(X, A)$, and $x \mapsto r_x$ is an isometric linear isomorphism between Banach spaces X and $\mathbb{K}(A, X)$.

For more details about Hilbert C^*-modules we refer the reader to [13, 14, 15, 18].

If X is an operator space we write $\mathrm{CB}(X)$ for the Banach algebra of all completely bounded maps on X. For details about operator spaces, their tensor products and completely bounded maps we refer to [4, 8, 16].

3. Results

Let X be a Hilbert A-module. Besides the linking algebra $\mathcal{L}(X)$, we need another subalgebra of $\mathbb{B}(A \oplus X)$, larger than $\mathcal{L}(X)$.

We define an *extended linking algebra* of X as

$$\mathcal{L}_{\text{ext}}(X) = \begin{bmatrix} \mathbb{B}(A) & \mathbb{K}(X,A) \\ \mathbb{K}(A,X) & \mathbb{B}(X) \end{bmatrix} = \left\{ \begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix} : v \in M(A),\, x, y \in X,\, u \in \mathbb{B}(X) \right\},$$

where, similarly as before, for $v \in M(A)$, $T_v : A \to A$ is defined by $T_v(a) = va$.

Let us first show that $\mathcal{L}_{\text{ext}}(X)$ is a C^*-algebra. For that we shall need the following remark.

Remark 3.1. Let X be a Hilbert A-module. If B is any C^*-algebra that contains A as an ideal, then X can be also regarded as a Hilbert B-module with respect to the same inner product (which takes values in $A \subseteq B$), while the right action of B on X is defined as follows. For $x \in X$, $a \in A$ and $b \in B$, set

$$(xa)b := x(ab)$$

(see e.g. [4, 8.1.4 (4)]). Obviously, $\mathbb{B}_B(X) = \mathbb{B}_A(X)$ and $\mathbb{K}_A(X) = \mathbb{K}_B(X)$, so all $u \in \mathbb{B}_A(X)$ are also B-linear. In particular, by taking $B = M(A)$, any Hilbert A-module X can be regarded as a Hilbert $M(A)$-module.

Lemma 3.2. *Let X be a Hilbert A-module. $\mathcal{L}_{\text{ext}}(X)$ is a C^*-subalgebra of $\mathbb{B}(A \oplus X)$ which contains $\mathcal{L}(X)$ as an essential ideal.*

Proof. Clearly $\mathcal{L}_{\text{ext}}(X)$ is a linear subspace of $\mathbb{B}(A \oplus X)$. If

$$S = \begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix} \in \mathcal{L}_{\text{ext}}(X),$$

one can easily verify that the adjoint of S in $\mathbb{B}(A \oplus X)$ is given by

$$\begin{bmatrix} T_{v^*} & l_x \\ r_y & u^* \end{bmatrix},$$

so $S^* \in \mathcal{L}_{\text{ext}}(X)$. Further, for all $v_1, v_2 \in M(A)$, $x_1, x_2, y_1, y_2 \in X$ and $u_1, u_2 \in \mathbb{B}(X)$ we have

$$\begin{bmatrix} T_{v_1} & l_{y_1} \\ r_{x_1} & u_1 \end{bmatrix} \begin{bmatrix} T_{v_2} & l_{y_2} \\ r_{x_2} & u_2 \end{bmatrix} = \begin{bmatrix} T_{v_1 v_2 + \langle y_1, x_2 \rangle} & l_{y_2 v_1^* + u_2^* y_1} \\ r_{x_1 v_2 + u_1 x_2} & \theta_{x_1, y_2} + u_1 u_2 \end{bmatrix} \in \mathcal{L}_{\text{ext}}(X),$$

since X can be regarded as a Hilbert $M(A)$-module (Remark 3.1) and hence $y_2 v_1^*, x_1 v_2 \in X$. This shows that $\mathcal{L}_{\text{ext}}(X)$ is a self-adjoint subalgebra of $\mathbb{B}(A \oplus X)$. Using the similar arguments as in the proof of [15, Lemma 3.20] we also conclude that $\mathcal{L}_{\text{ext}}(X)$ is norm closed and hence a C^*-subalgebra of $\mathbb{B}(A \oplus X)$.

Finally, using the fact that $\mathcal{L}(X)$ is an essential ideal of (its multiplier C^*-algebra) $\mathbb{B}(A \oplus X)$, we conclude that $\mathcal{L}(X)$ is an essential ideal of $\mathcal{L}_{\text{ext}}(X)$. □

In the introduction we gave the notion of essentially full Hilbert modules: a Hilbert A-module X is essentially full if $\langle X, X \rangle$ is an essential ideal of A. As we show in the next lemma, essential fullness guarantees some kind of nondegeneracy of X regarded as a Hilbert C^*-module over any C^*-algebra which contains $\langle X, X \rangle$ as an essential ideal.

Lemma 3.3. *For a non-zero Hilbert A-module X the following conditions are equivalent:*

(i) *X is essentially full.*
(ii) *For each non-zero element $a \in A$ there exists $x \in X$ such that $xa \neq 0$.*

Proof. (i) $\implies$ (ii). Assume X is essentially full and let $a \in A$ be such that $xa = 0$ for all $x \in X$. Then

$$\langle y, x\rangle a = \langle y, xa\rangle = 0 \qquad \forall x, y \in X,$$

which implies $\langle X, X\rangle a = \{0\}$. Since X is essentially full, we conclude that $a = 0$.

(ii) $\implies$ (i). Let $a \in A$, $a \neq 0$. By assumption, there exists $x \in X$ such that $xa \neq 0$. Then

$$\langle xa, x\rangle a = \langle xa, xa\rangle \neq 0,$$

so $\langle X, X\rangle a \neq \{0\}$. Therefore, X is essentially full. $\square$

In the following proposition we give several equivalent descriptions of Hilbert C^*-modules over prime C^*-algebras.

Proposition 3.4. *Let X be a non-zero Hilbert A-module. The following conditions are equivalent:*

(i) *A is prime.*
(ii) *X is essentially full and $\mathbb{K}(X)$ is prime.*
(iii) *The linking algebra $\mathcal{L}(X)$ is prime.*
(iv) *The extended linking algebra $\mathcal{L}_{\mathrm{ext}}(X)$ is prime.*
(v) *If $a \in A$ and $u \in \mathbb{K}(X)$ are such that $uxa = 0$ for all $x \in X$, then $a = 0$ or $u = 0$.*
(vi) *X is essentially full and if $x_1, x_2 \in X$ are such that $x_1\langle x, x_2\rangle = 0$ for all $x \in X$, then $x_1 = 0$ or $x_2 = 0$.*

Proof. (i) $\implies$ (ii), (iii). Assume that A is prime. Then any non-zero (two-sided) ideal of A is essential (see e.g. [1, Lemma 1.1.2]), so in particular X is essentially full. Observe that, in order to get that $\mathcal{L}(X)$ is prime, it is enough to show that if A is prime then $\mathbb{K}(X)$ is also prime. Namely, the linking algebra $\mathcal{L}(X)$ is defined as $\mathbb{K}(A \oplus X)$. Since $A \oplus X$ is a Hilbert C^*-module over the same C^*-algebra A, it will then follow that $\mathcal{L}(X)$ is prime whenever A is prime.

Assume there exist non-zero $u_1, u_2 \in \mathbb{K}(X)$ such that $u_1\mathbb{K}(X)u_2 = \{0\}$. Then there are $x_1, x_2 \in X$ such that $u_1x_1 \neq 0$ and $u_2x_2 \neq 0$. By assumption,

$$u_1\, \theta_{x_1a, u_2x_2}\, u_2 = 0 \qquad \forall a \in A.$$

Then

$$\begin{aligned}
\langle u_1x_1, u_1x_1\rangle a\langle u_2x_2, u_2x_2\rangle &= \langle u_1x_1, u_1(x_1a\langle u_2x_2, u_2x_2\rangle)\rangle \\
&= \langle u_1x_1, (u_1\, \theta_{x_1a, u_2x_2}\, u_2)(x_2)\rangle \\
&= 0
\end{aligned}$$

for all $a \in A$, which is a contradiction with the assumption that A is prime, since both $\langle u_1x_1, u_1x_1\rangle$ and $\langle u_2x_2, u_2x_2\rangle$ are non-zero. Therefore, $u_1\mathbb{K}(X)u_2 = \{0\}$ can happen only when $u_1 = 0$ or $u_2 = 0$, which shows that $\mathbb{K}(X)$ is prime.

(ii) $\implies$ (i). Assume that X is essentially full and that A is not prime. Then there exist non-zero elements $a_1, a_2 \in A$ such that $a_1Aa_2 = \{0\}$. Then by Lemma 3.3 there are $x_1, x_2 \in X$ such that $x_1a_1 \neq 0$ and $x_2a_2 \neq 0$. By assumption,

$$a_1\langle x_1a_1, ux_2\rangle a_2 = 0 \qquad \forall u \in \mathbb{K}(X).$$

Then for all $x \in X$ and $u \in \mathbb{K}(X)$ we have

$$\begin{aligned}(\theta_{x_1a_1,x_1a_1}\, u\, \theta_{x_2a_2,x_2a_2})(x) &= x_1a_1\langle x_1a_1, u\,\theta_{x_2a_2,x_2a_2}(x)\rangle \\ &= x_1a_1\langle x_1a_1, u(x_2a_2\langle x_2a_2, x\rangle)\rangle \\ &= x_1a_1\langle x_1a_1, ux_2\rangle a_2\langle x_2a_2, x\rangle \\ &= x_1(a_1\langle x_1a_1, ux_2\rangle a_2)\langle a_2x_2, x\rangle \\ &= 0.\end{aligned}$$

Thus,

$$\theta_{x_1a_1,x_1a_1}\, \mathbb{K}(X)\, \theta_{x_2a_2,x_2a_2} = \{0\}.$$

Since both $\theta_{x_1a_1,x_1a_1}$ and $\theta_{x_2a_2,x_2a_2}$ are non-zero, we conclude that $\mathbb{K}(X)$ is not prime.

(iii) $\Longrightarrow$ (iv). This follows directly from Lemma 3.2 and the fact that any C^*-algebra that contains a prime essential ideal must be prime itself.

(iv) $\Longrightarrow$ (v). Assume that $\mathcal{L}_{\text{ext}}(X)$ is prime. Then for non-zero elements $a_0 \in A$ and $u_0 \in \mathbb{K}(X)$ there are elements $v \in M(A)$, $x, y \in X$ and $u \in \mathbb{B}(X)$ such that

$$0 \neq \begin{bmatrix} 0 & 0 \\ 0 & u_0 \end{bmatrix} \begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix} \begin{bmatrix} T_{a_0} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r_{u_0xa_0} & 0 \end{bmatrix}.$$

Thus, $u_0xa_0 \neq 0$ for some $x \in X$.

(v) $\Longrightarrow$ (vi). Suppose first that there exists $a \in A$, $a \neq 0$, such that $xa = 0$ for all $x \in X$. Then $uxa = 0$ for all $x \in X$ and $u \in \mathbb{K}(X)$. By assumption, it follows that $u = 0$ for all $u \in \mathbb{K}(X)$, which is not since $X \neq \{0\}$. Therefore, X is essentially full.

Let $x_1, x_2 \in X$ be such that $x_1\langle x, x_2\rangle = 0$ for all $x \in X$. Then

$$\theta_{x_1,x_1}(x)\langle x_2, x_2\rangle = x_1\langle x_1, x\rangle\langle x_2, x_2\rangle = x_1\langle x_2\langle x, x_1\rangle, x_2\rangle = 0 \qquad \forall x \in X.$$

Hence, by assumption, $\theta_{x_1,x_1} = 0$ or $\langle x_2, x_2\rangle = 0$, that is, $x_1 = 0$ or $x_2 = 0$.

(vi) $\Longrightarrow$ (i). Assume (vi) holds but A is not prime. Then there are non-zero elements $a_1, a_2 \in A$ such that $a_1Aa_2 = \{0\}$. By assumption X is essentially full, so by Lemma 3.3 there exist $x_1, x_2 \in X$ such that $x_1a_1 \neq 0$ and $x_2a_2 \neq 0$. But then

$$x_1a_1\langle x, x_2a_2\rangle = x_1a_1\langle x, x_2\rangle a_2 = 0 \qquad \forall x \in X,$$

which contradicts our assumption. □

Remark 3.5. In particular, Proposition 3.4 shows (probably the well-known fact) that the primeness is an invariant property under Morita equivalence (see e.g. [15, Chapter 3]). Indeed, if X is an $A - B$ imprimitivity bimodule, then by definition X is full both as a left Hilbert A-module and as a right Hilbert B-module. Then $A \cong \mathbb{K}(X)$ by [15, Proposition 3.8], so the equivalence of (i) and (ii) in Proposition 3.4 says that A is prime if and only if B is prime. For the other interesting properties that are invariant under Morita equivalence we refer to [12].

The next simple example demonstrates the necessity of the assumption that X if essentially full in both conditions (ii) and (vi) of Proposition 3.4.

Example 3.6. Let A be any non-prime C^*-algebra that contains a prime non-zero ideal I (e.g. $A = \mathbb{C} \oplus \mathbb{C}$ and $I = \mathbb{C} \oplus \{0\}$). Consider $X = I$ as a Hilbert A-module in the usual way. Then $\mathbb{K}(X) = I$ is a prime C^*-algebra, while A is not.

Further, if $x_1, x_2 \in X$ satisfy $0 = x_1\langle x, x_2\rangle = x_1 x^* x_2$ for all $x \in X$, the primeness of I implies $x_1 = 0$ or $x_2 = 0$. Therefore, the second condition in (vi) is satisfied, but (i) does not hold.

If X is a Hilbert A-module, we can introduce the operator space structure on X via the operator space structure of its linking algebra $\mathcal{L}(X)$ (or extended linking algebra $\mathcal{L}_{\text{ext}}(X)$), after identifying X as the $2-1$ corner in $\mathcal{L}(X)$ (or $\mathcal{L}_{\text{ext}}(X)$), via the isometric isomorphism $X \cong \mathbb{K}(A, X)$, $x \mapsto r_x$. That is, for all $n \in \mathbb{N}$ and $\left[x_{ij}\right] \in \mathrm{M}_n(X)$ we define

$$\left\| \left[x_{ij}\right] \right\|_{\mathrm{M}_n(X)} := \left\| \left[\begin{bmatrix} 0 & 0 \\ r_{x_{ij}} & 0 \end{bmatrix} \right] \right\|_{\mathrm{M}_n(\mathcal{L}(X))} = \left\| \left[\begin{bmatrix} 0 & 0 \\ r_{x_{ij}} & 0 \end{bmatrix} \right] \right\|_{\mathrm{M}_n(\mathcal{L}_{\text{ext}}(X))},$$

so that the canonical embedding

$$\iota_X : X \hookrightarrow \mathcal{L}_{\text{ext}}(X), \qquad \iota_X : x \mapsto \begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}$$

becomes a complete isometry. This structure is called the *canonical operator space structure* on X (for details we refer to [4, Section 8.2]). Further, since the canonical embeddings

$$\iota_{M(A)} : M(A) \hookrightarrow \mathcal{L}_{\text{ext}}(X), \qquad \iota_{M(A)} : v \mapsto \begin{bmatrix} T_v & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\iota_{\mathbb{B}(X)} : \mathbb{B}(X) \hookrightarrow \mathcal{L}_{\text{ext}}(X), \qquad \iota_{\mathbb{B}(X)} : u \mapsto \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix}$$

are injective $*$-homomorphisms between C^*-algebras, they are also completely isometric.

We record the next simple fact:

Lemma 3.7. *Let X be a Hilbert A-module. For each $\phi \in \mathrm{CB}(X)$ we define a map*

$$\widetilde{\phi} : \mathcal{L}_{\text{ext}}(X) \to \mathcal{L}_{\text{ext}}(X) \qquad \textit{by} \qquad \widetilde{\phi}\left(\begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix} \right) := \begin{bmatrix} 0 & 0 \\ r_{\phi(x)} & 0 \end{bmatrix}.$$

Then $\widetilde{\phi} \in \mathrm{CB}(\mathcal{L}_{\text{ext}}(X))$ and $\|\widetilde{\phi}\|_{cb} = \|\phi\|_{cb}$.

Proof. For all $n \in \mathbb{N}$, $\left[v_{ij}\right] \in \mathrm{M}_n(M(A))$, $\left[x_{ij}\right], \left[y_{ij}\right] \in \mathrm{M}_n(X)$ and $\left[u_{ij}\right] \in \mathrm{M}_n(\mathbb{B}(X))$ we have

$$\begin{aligned} \left\| \widetilde{\phi}_n \left(\left[\begin{bmatrix} T_{v_{ij}} & l_{y_{ij}} \\ r_{x_{ij}} & u_{ij} \end{bmatrix} \right] \right) \right\|_{\mathrm{M}_n(\mathcal{L}_{\text{ext}}(X))} &= \left\| (\iota_X)_n \left(\left[\phi(x_{ij})\right] \right) \right\|_{\mathrm{M}_n(\mathcal{L}_{\text{ext}}(X))} \\ &= \left\| \left[\phi(x_{ij})\right] \right\|_{\mathrm{M}_n(X)} \\ &= \left\| \phi_n \left(\left[x_{ij}\right] \right) \right\|_{\mathrm{M}_n(X)}. \end{aligned}$$

□

Remark 3.8. By Remark 3.1 any Hilbert A-module X can be considered as a Hilbert $M(A)$-module and every $u \in \mathbb{B}(X)$ is $M(A)$-linear. Now for all $u \in \mathbb{B}(X)$, $x \in X$ and $v \in M(A)$ we have $u(xv) = (ux)v$, so in this way X becomes a Banach $\mathbb{B}(X) - M(A)$-bimodule (in particular, the product uxv is unambiguously defined).

Moreover, it is straightforward to check that each matrix space $\mathrm{M}_n(X)$ $(n \in \mathbb{N})$ is a Banach $\mathrm{M}_n(\mathbb{B}(X)) - \mathrm{M}_n(M(A))$-bimodule in the canonical way. That is,

$$\left\| \left[u_{ij}\right] \left[x_{ij}\right] \right\|_{\mathrm{M}_n(X)} \leq \left\| \left[u_{ij}\right] \right\|_{\mathrm{M}_n(\mathbb{B}(X))} \left\| \left[x_{ij}\right] \right\|_{\mathrm{M}_n(X)}$$

and

$$\left\| \left[x_{ij}\right] \left[v_{ij}\right] \right\|_{\mathrm{M}_n(X)} \leq \left\| \left[x_{ij}\right] \right\|_{\mathrm{M}_n(X)} \left\| \left[v_{ij}\right] \right\|_{\mathrm{M}_n(M(A))}$$

for all $n \in \mathbb{N}$, $\left[u_{ij}\right] \in \mathrm{M}_n(\mathbb{B}(X))$, $\left[v_{ij}\right] \in \mathrm{M}_n(M(A))$ and $\left[x_{ij}\right] \in \mathrm{M}_n(X)$.

Let us now introduce the class of elementary operators on Hilbert C^*-modules.

If X is a Hilbert A-module, then first, following the C^*-algebraic case, for each $u \in \mathbb{B}(X)$ and $v \in M(A)$ we define a map

$$M_{u,v} : X \to X \qquad \text{by} \qquad M_{u,v} : x \mapsto uxv.$$

Definition 3.9. By an *elementary operator* on a Hilbert A-module X we mean a map $\phi : X \to X$ for which there exists a finite number of elements $u_1, \ldots, u_k \in \mathbb{B}(X)$ and $v_1, \ldots, v_k \in M(A)$ such that

$$\phi = \sum_{i=1}^{k} M_{u_i,v_i}. \tag{3.1}$$

Example 3.10. If a C^*-algebra A is considered as a Hilbert A-module in the standard way, then $\mathbb{B}(A)$ and $M(A)$ coincide, so elementary operators on A, as a Hilbert A-module, agree with the usual notion of elementary operators on A, as a C^*-algebra.

Similarly as in the C^*-algebraic case, if X is a Hilbert A-module, then using the operator space axioms, Remark 3.8 and the C^*-identity, it is easy to verify that elementary operators on X are completely bounded and that their cb-norm is dominated by the Haagerup norm of their corresponding tensor in $\mathbb{B}(X) \otimes M(A)$. That is, if an elementary operator $\phi : X \to X$ is represented as in (3.1) then

$$\|\phi\|_{cb} \leq \left\| \sum_{i=1}^{k} u_i \otimes v_i \right\|_h$$

(see [1, p. 207]). Therefore, the mapping

$$(\mathbb{B}(X) \otimes M(A), \|\cdot\|_h) \to (\mathrm{CB}(X), \|\cdot\|_{cb}) \quad \text{given by} \quad \sum_{i=1}^{k} u_i \otimes v_i \mapsto \sum_{i=1}^{k} M_{u_i,v_i},$$

is a well-defined contraction, so we can continuously extend it to the map

$$\Theta_X : (\mathbb{B}(X) \otimes_h M(A), \|\cdot\|_h) \to (\mathrm{CB}(X), \|\cdot\|_{cb}),$$

where $\mathbb{B}(X) \otimes_h M(A)$ is the completion of $\mathbb{B}(X) \otimes M(A)$ with respect to $\|\cdot\|_h$.

Lemma 3.11. *Using the same notation as in Lemma 3.7, for each $t \in \mathbb{B}(X) \otimes_h M(A)$ we have*

$$\widetilde{\Theta_X(t)} = \Theta_{\mathcal{L}_{\mathrm{ext}}(X)}((\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)).$$

Proof. By [4, Proposition 1.5.6] there exist sequences (u_k) in $\mathbb{B}(X)$ and (v_k) in $M(A)$ such that the series $\sum_{k=1}^{\infty} u_k u_k^*$ and $\sum_{k=1}^{\infty} v_k^* v_k$ are norm convergent and

$t = \sum_{k=1}^{\infty} u_k \otimes v_k$. Then the series $\sum_{k=1}^{\infty} u_k x v_k$ is norm convergent for every $x \in X$ and for all $v \in M(A)$, $x, y \in X$ and $u \in \mathbb{B}(X)$ we have

$$\begin{aligned} \widetilde{\Theta_X(t)}\left(\begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix}\right) &= \begin{bmatrix} 0 & 0 \\ \sum_{k=1}^{\infty} r_{u_k x v_k} & 0 \end{bmatrix} = \sum_{k=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & u_k \end{bmatrix} \begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix} \begin{bmatrix} T_{v_k} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \Theta_{\mathcal{L}_{\mathrm{ext}}(X)}((\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)) \left(\begin{bmatrix} T_v & l_y \\ r_x & u \end{bmatrix}\right). \end{aligned}$$

□

We are now ready to prove the main result of this paper, the generalization of Theorem 1.1 in the context of Hilbert C^*-modules.

Theorem 3.12. *Let X be a non-zero Hilbert A-module. The following conditions are equivalent:*

(i) *Θ_X is isometric.*
(ii) *Θ_X is injective.*
(iii) *A is a prime C^*-algebra.*

Proof. (i) $\Longrightarrow$ (ii). This is trivial.

(ii)$\Longrightarrow$ (iii). Assume that A is not prime. Then by Proposition 3.4 there are non-zero elements $u \in \mathbb{K}(X)$ and $a \in A$ such that $uxa = 0$ for all $x \in X$. Then $u \otimes a$ is a non-zero tensor in $\mathbb{K}(X) \otimes A \subseteq \mathbb{B}(X) \otimes M(A)$ but

$$\Theta_X(u \otimes a)(x) = uxa = 0$$

for all $x \in X$.

(iii) $\Longrightarrow$ (i). Since the canonical embeddings $\iota_{\mathbb{B}(X)} : \mathbb{B}(X) \hookrightarrow \mathcal{L}_{\mathrm{ext}}(X)$ and $\iota_{M(A)} : M(A) \hookrightarrow \mathcal{L}_{\mathrm{ext}}(X)$ are completely isometric, the injectivity of the Haagerup tensor product implies

$$\|(\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)\|_h = \|t\|_h \qquad \forall t \in \mathbb{B}(X) \otimes_h M(A)$$

(see e.g. [4, Section 1.5.5]). If A is a prime C^*-algebra, then by Proposition 3.4 $\mathcal{L}_{\mathrm{ext}}(X)$ is also prime, so Theorem 1.1 implies

$$\|\Theta_{\mathcal{L}_{\mathrm{ext}}(X)}(t')\|_{cb} = \|t'\|_h \qquad \forall t' \in \mathcal{L}_{\mathrm{ext}}(X) \otimes_h \mathcal{L}_{\mathrm{ext}}(X).$$

Then using Lemmas 3.7 and 3.11 we see that for all $t \in \mathbb{B}(X) \otimes_h M(A)$ we have

$$\begin{aligned} \|\Theta_X(t)\|_{cb} &= \|\widetilde{\Theta_X(t)}\|_{cb} = \|\Theta_{\mathcal{L}_{\mathrm{ext}}(X)}((\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t))\|_{cb} \\ &= \|(\iota_{\mathbb{B}(X)} \otimes \iota_{M(A)})(t)\|_h = \|t\|_h. \end{aligned}$$

Thus, Θ_X is isometric. □

Acknowledgements

We thank Professor Michael Frank for reading the manuscript and for his comments. We also thank the anonymous referee for useful suggestions that helped us to improve the presentation of our results.

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