DERIVATIONS WHICH ARE INNER AS COMPLETELY BOUNDED MAPS

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Abstract. We consider derivations in the image of the canonical contraction θ_A from the Haagerup tensor product of a C^*-algebra A with itself to the space of completely bounded maps on A. We show that such derivations are necessarily inner if A is prime or if A is central. We also provide an example of a C^*-algebra which has an outer derivation implemented by an elementary operator.

1. Introduction

Let A be a C^*-algebra and let $\mathrm{ICB}(A)$ be the space of all completely bounded maps $T : A \to A$ such that $T(J) \subseteq J$, for every closed two-sided ideal J of A. If $A \otimes_h A$ denotes the Haagerup tensor product of A with itself, there is a canonical contraction $\theta_A : A \otimes_h A \to \mathrm{ICB}(A)$ given on elementary tensors $a \otimes b \in A \otimes A$ by

$$\theta_A(a \otimes b)(x) := axb, \quad \text{for all } x \in A.$$

Mathieu showed that θ_A is isometric if and only if A is a prime C^*-algebra (see [3, 5.4.11]). If A is not prime then θ_A is not even injective, and then it is natural to consider the central Haagerup tensor product $A \otimes_{Z,h} A$, and the induced contraction $\theta_A^Z : A \otimes_{Z,h} A \to \mathrm{ICB}(A)$ (see [22], [8] and [7] for the further details and results in this subject).

Since every derivation on a C^*-algebra A is an operator in $\mathrm{ICB}(A)$, it is natural to study how large can the set $\mathrm{Der}(A) \cap \mathrm{Im}\,\theta_A$ be (where $\mathrm{Der}(A)$ denotes the space of all derivations on A and $\mathrm{Im}\,\theta_A$ denotes the image of θ_A). To ensure that at least all the inner derivations on A are in $\mathrm{Im}\,\theta_A$ (A is not assumed to be unital), we shall require that A is quasicentral (see section 3). In this paper we shall be mainly interested in the question when is the set $\mathrm{Der}(A) \cap \mathrm{Im}\,\theta_A$ as small as possible, and hence (in the quasicentral case) equal to the set $\mathrm{Inn}(A)$ of all inner derivations on A. This is certainly true for all von Neumann algebras (since by the Kadison-Sakai theorem [20, 4.1.6], every derivation on a von Neumann algebra is inner). As we shall see, this property is also satisfied for the class of all unital prime C^*-algebras and for the class of all central C^*-algebras. We also conjecture that this property holds for the larger class of all quasicentral C^*-algebras in which every Glimm ideal is primal, but we were not able to prove this.

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The paper is organized as follows. In Section 3 we provide some basic facts about quasicentral and central C^*-algebras.

In Section 4, we concentrate on prime C^*-algebras. We show that every derivation $\delta \in \operatorname{Im}\theta_A$ on a unital prime C^*-algebra A is necessarily inner in A. If a prime C^*-algebra A is non-unital (and hence non-quasicentral) we show that the only derivation $\delta \in \operatorname{Im}\theta_A$ is in fact the zero-derivation.

In Section 5, we concentrate on C^*-algebras with Hausdorff primitive spectrum. We show that every derivation $\delta \in \operatorname{Im}\theta_A$ is smooth (see Definition 5.1) and hence inner in its multiplier algebra $M(A)$. Moreover, if A is central, we prove that every derivation $\delta \in \operatorname{Im}\theta_A$ is in fact inner in A. We also show that a quasicentral C^*-algebra A is central if and only if every inner derivation on A is smooth.

In Section 6, we give an example of a unital separable 2-subhomogeneous C^*-algebra A for which the space of elementary operators $\mathrm{E}(A)$ is a (cb-)closed subspace of $\mathrm{ICB}(A)$ (and hence $\operatorname{Im}\theta_A = \mathrm{E}(A)$), but for which the space of inner derivations is not closed in $\mathrm{Der}(A)$. It follows that such C^*-algebra must have an outer derivation which is implemented by an elementary operator.

2. Notation and Preliminaries

Through this paper A will denote a C^*-algebra, A_+ the positive part and A_h the self-adjoint part of A. By $Z(A)$ we denote the center of A. By an ideal of A we shall always mean a closed two-sided ideal. The set of all ideals of A is denoted by $\mathrm{Id}(A)$. By $\hat{A}$ we shall denote the spectrum of A (i.e. the set of all equivalence classes of irreducible representations of A) and by $\mathrm{Prim}(A)$ the primitive spectrum of A (i.e. the set of all primitive ideals of A), both equipped with the Jacobson topology. By $M(A)$ we denote the multiplier algebra of A and by $\tilde{A}$ we denote the minimal unitization of A.

We now recall the definition of the complete regularization of $\mathrm{Prim}(A)$ (see [6] for further details). For $P, Q \in \mathrm{Prim}(A)$ let

$$P \approx Q \quad \text{if} \quad f(P) = f(Q), \quad \text{for all } f \in C_b(\mathrm{Prim}(A)). \tag{2.1}$$

Then $\approx$ is an equivalence relation on $\mathrm{Prim}(A)$ and the equivalence classes are closed subsets of $\mathrm{Prim}(A)$. It follows that there is one-to-one correspondence between the quotient set $\mathrm{Prim}(A)/\approx$ and the set of ideals of A given by

$$[P]_\approx \leftrightarrow \bigcap [P]_\approx \quad (P \in \mathrm{Prim}(A)),$$

where $[P]_\approx$ denotes the equivalence class of P. The set of ideals obtained in this way is denoted by $\mathrm{Glimm}(A)$, and its elements are called *Glimm ideals* of A. The quotient map $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is known as the *complete regularization map.*

For $f \in C_b(\mathrm{Prim}(A))$ let $f_\approx : \mathrm{Glimm}(A) \to \mathbb{C}$ be a (bounded) function defined by $f_\approx(G) := f(P)$, where $P \in \mathrm{Prim}(A/G)$ (of course, $f_\approx$ is well defined).

There are two natural topologies on $\mathrm{Glimm}(A)$:

- the quotient topology τ_q, for which the space $(\mathrm{Glimm}(A), \tau_q)$ is Hausdorff;
- the completely regular topology τ_{cr}, which is the weakest topology for which all the functions $f_\approx$ ($f \in C_b(\mathrm{Prim}(A))$) are continuous. Of course, $(\mathrm{Glimm}(A), \tau_{cr})$ is a Tychonoff space.

Note that τ_q is stronger than τ_{cr} and that

$$\begin{aligned} C_b(\mathrm{Glimm}(A)) &:= C_b(\mathrm{Glimm}(A), \tau_q) = C_b(\mathrm{Glimm}(A), \tau_{cr}) \\ &= \{f_{\approx} : f \in C_b(\mathrm{Prim}(A))\}. \end{aligned}$$

In many cases we have $\tau_q = \tau_{cr}$ (for example, if A is unital or if ϕ_A is τ_q-open or τ_{cr}-open, see [6]). We also note that if A is unital, then by [6] for $P, Q \in \mathrm{Prim}(A)$

$$P \approx Q \quad \Leftrightarrow \quad P \cap Z(A) = Q \cap Z(A), \tag{2.2}$$

and

$$\mathrm{Glimm}(A) = \{JA : J \in \mathrm{Max}(Z(A))\}, \tag{2.3}$$

where $\mathrm{Max}(Z(A))$ denotes the maximal ideal space of $Z(A)$ (for $J \in \mathrm{Max}(Z(A))$, JA is closed ideal by Cohen's factorization theorem [10, A.6.2]).

A *derivation* on a C^*-algebra A is a linear map $\delta : A \to A$ satisfying the *Leibniz rule*

$$\delta(xy) = \delta(x)y + x\delta(y), \quad \text{for all } x, y \in A. \tag{2.4}$$

The *inner derivation* implemented by the element $a \in A$ is a map $\delta_a : A \to A$, given by

$$\delta_a(x) := ax - xa, \quad \text{for all } x \in A.$$

If a derivation $\delta \in \mathrm{Der}(A)$ is not inner, we say that δ is *outer*. By $\mathrm{Der}(A)$ and $\mathrm{Inn}(A)$ we denote, respectively, the set of all derivations on A and the set of all inner derivations on A. It is well known that $\mathrm{Der}(A) \subseteq \mathrm{ICB}(A)$, and that for $\delta \in \mathrm{Der}(A)$ we have

$$\|\delta\|_{cb} = \|\delta\| = \sup\{\|\delta_P\| : P \in \mathrm{Prim}(A)\},$$

where δ_J ($J \in \mathrm{Id}(A)$) denotes the induced derivation on A/J;

$$\delta_J(x + J) = \delta(x) + J \quad (x \in A).$$

When A is a primitive and unital C^*-algebra, $a \in A$, and $\lambda(a)$ the nearest scalar to a (i.e. $\|a - \lambda(a)\| = d(a, \mathbb{C})$), by Stampfli's formula [3, 4.1.17] we have

$$\|\delta_a\|_{cb} = \|\delta_a\| = 2\|a - \lambda(a)\|. \tag{2.5}$$

3. Quasicentral and central C^*-algebras

Definition 3.1. [12, Def. 1] A C^*-algebra A is said to be *quasicentral* if no primitive ideal of A contains $Z(A)$ (or equivalently, if no Glimm ideal of A contains $Z(A)$).

The next proposition gives a useful characterization of quasicentral C^*-algebras:

Proposition 3.2. *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) *A is quasicentral;*
(ii) *A has a central approximate unit (that is, there exists an approximate unit (e_α) of A such that $e_\alpha \in Z(A)$ for each α);*
(iii) *$A = Z(A)A$;*
(iv) *A is unital or $A \in \mathrm{Glimm}(\tilde{A})$.*

Proof. If A is unital, we have nothing to prove, so assume that A is non-unital.

(i)$\Leftrightarrow$(ii). This follows from [4, Thm. 1].

(ii)$\Rightarrow$(iii). This follows directly from Cohen's factorization theorem [10, A.6.2], since A is a nondegenerate Banach $Z(A)$-module.

(iii)$\Rightarrow$(iv). Since A is non-unital, the equality $Z(A)A = A$ implies that $Z(A) \neq \{0\}$, so $Z(A)$ is a maximal ideal of $Z(\tilde{A})$ and $Z(A)\tilde{A} = A$. By (2.3) $A \in \mathrm{Glimm}(\tilde{A})$.

(iv)$\Rightarrow$(i). Suppose that A is non-quasicentral. If $Z(A) = \{0\}$, then $Z(\tilde{A}) = \mathbb{C}1$. It follows that $\mathrm{Glimm}(\tilde{A}) = \{0\}$, so $A \notin \mathrm{Glimm}(\tilde{A})$. If $Z(A) \neq \{0\}$, then $Z(A)$ is a maximal ideal of $Z(\tilde{A})$. Since A is non-quasicentral, there exists $P \in \mathrm{Prim}(A)$ such that $Z(A) \subseteq P$. Then $P \in \mathrm{Prim}(\tilde{A})$, and since A is a maximal (primitive) ideal of $\tilde{A}$ and $Z(A) \subseteq A$ (trivially), (2.2) implies that $P \approx A$ in $\tilde{A}$. Hence,

$$\bigcap [A]_{\approx} \subseteq P \subsetneq A,$$

so $A \notin \mathrm{Glimm}(A)$. □

Lemma 3.3. *Let A be a quasicentral C^*-algebra. Then* $\mathrm{Inn}(A) \subseteq \mathrm{Im}\,\theta_A$.

Proof. By Proposition 3.2, each $a \in A$ can be written in the form $a = zb$, for some $z \in Z(A)$ and $b \in A$. It follows that $\delta_a = \theta_A(z \otimes b - b \otimes z)$. □

Question 3.4. *If A is a C^*-algebra with the property that* $\mathrm{Inn}(A) \subseteq \mathrm{Im}\,\theta_A$*, is A necessarily quasicentral?*

Let A be a C^*-algebra. By Dauns-Hofmann theorem [19, A.34], there exists an isomorphism $\Psi_A : Z(M(A)) \to C_b(\mathrm{Prim}(A))$ such that

$$za + P = \Psi_A(z)(P)(a + P), \quad \text{for all } z \in Z(M(A)),\ a \in A \text{ and } P \in \mathrm{Prim}(A).$$

Since the norm functions $P \mapsto \|a + P\|$ $(a \in A)$, $\mathrm{Prim}(A) \to \mathbb{R}_+$ vanish at infinity (see [18, 4.4.4]), we have $\Psi_A(Z(A)) \subseteq C_0(\mathrm{Prim}(A))$. If A is quasicentral then it follows from [11, Prop. 1] (see also [4]) that

$$\Psi_A(Z(A)) = C_0(\mathrm{Prim}(A)). \tag{3.1}$$

Using (3.1) it is easy to prove the following fact:

Proposition 3.5. *Let A be a quasicentral C^*-algebra. The following conditions are equivalent:*

(i) *A is unital;*
(ii) $\mathrm{Prim}(A)$ *is compact.*

Proof. Implication (i)$\Rightarrow$(ii) follows from [13, 3.1.8].

(ii)$\Rightarrow$(i). If $\mathrm{Prim}(A)$ is compact, then by (3.1) we have $Z(A) \cong C_0(\mathrm{Prim}(A)) = C(\mathrm{Prim}(A))$. Hence, $Z(A)$ is unital. By Proposition 3.2 (iii) it follows that the unit of $Z(A)$ must also be the unit of A. □

Remark 3.6. If A is a quasicentral C^*-algebra, it follows that for each $P \in \mathrm{Prim}(A)$ there exists a positive element $z_P \in Z(A)_+$ such that $\|z_P\| = 1$ and $\|z_P + P\| = \Psi_A(z_P)(P) = 1$. Hence, each primitive quotient A/P is unital with the unit $z_P + P$. Moreover, using the Gelfand transform of $Z(A)$, it can be easily seen (like in the proof of [4, Thm. 5]) that for each compact subset $K \subseteq \mathrm{Prim}(A)$ there exists $z \in Z(A)_+$ such that $\|z\| = 1$ and $\|z + P\| = \Psi_A(z)(P) = 1$, for each $P \in K$.

Lemma 3.7. *Let A be a quasicentral C^*-algebra and let $P, Q \in \mathrm{Prim}(A)$. The following conditions are equivalent:*

(i) $P \approx Q$ *(in the sense of (2.1));*
(ii) $f(P) = f(Q)$, *for all* $f \in C_0(\mathrm{Prim}(A))$*;*
(iii) $P \cap Z(A) = Q \cap Z(A)$.

Proof. Implications (i)⇒(ii) and (i)⇒(iii) follow immediately.

(ii)⇒(i). Let $g \in C_b(\mathrm{Prim}(A))$ and let $f := \Psi_A(z_P)$, where $z_P \in Z(A)_+$ is as in Remark 3.6. Then $f \in C_0(\mathrm{Prim}(A))$ and $f(P) = 1$. By the assumption, we have $f(Q) = 1$ and $(fg)(P) = (fg)(Q)$ (since $fg \in C_0(\mathrm{Prim}(A))$). Hence

$$g(P) = f(P)g(P) = (fg)(P) = (fg)(Q) = f(Q)g(Q) = g(Q).$$

(iii)⇒(ii). Let $f \in C_0(\mathrm{Prim}(A))$. By (3.1) there exists $z \in Z(A)$ such that $\Psi_A(z) = f$. Let $z_P, z_Q \in Z(A)_+$ be as in Remark 3.6, and let $u := \max\{z_P, z_Q\}$. Then for $v := z - f(P)u$ we have $v \in P \cap Z(A) = Q \cap Z(A)$, and so

$$0 = \Psi_A(v)(Q) = f(Q) - f(P).$$

□

If A is unital, it follows from [6] that $\tau_q = \tau_{cr}$ and that $\mathrm{Glimm}(A)$ is a compact Hausdorff space. Also, the map $\zeta_A : G \mapsto G \cap Z(A)$, from $\mathrm{Glimm}(A)$ onto $\mathrm{Max}(Z(A))$ is a homeomorphism with the inverse $\zeta_A^{-1}(J) = JA$ $(J \in \mathrm{Max}(Z(A)))$. The next proposition gives a generalization of this result for quasicentral C^*-algebras.

Proposition 3.8. *Let A be a quasicentral C^*-algebra. Then $\tau_q = \tau_{cr}$,* $\mathrm{Glimm}(A)$ *is a locally compact Hausdorff space and the map*

$$\zeta_A : \mathrm{Glimm}(A) \to \mathrm{Max}(Z(A)), \quad \zeta_A : G \mapsto G \cap Z(A)$$

is a homeomorphism with the inverse $\zeta_A^{-1}(J) = JA$ $(J \in \mathrm{Max}(Z(A)))$.

Proof. Let $G \in \mathrm{Glimm}(A)$ be fixed. Since A is quasicentral, there exists $z \in Z(A)_+$ such that $\|z + G\| > 0$. By Dauns-Hofmann theorem, $P \mapsto \|z + P\| = \Psi_A(z)(P)$ is a continuous function on $\mathrm{Prim}(A)$. Let $P \in \mathrm{Prim}(A/G)$. If $Q \in \mathrm{Prim}(A/G)$, then $Q \approx P$, so $\|z + Q\| = \|z + P\|$. It follows that

$$\|z + G\| = \sup\{\|z + Q\| : \ Q \in \mathrm{Prim}(A/G)\} = \|z + P\|.$$

Hence, the function $H \mapsto \|z + H\|$ $(H \in \mathrm{Glimm}(A))$ coincides with the function $\Psi_A(z)_{\approx}$. Let

$$\mathcal{U} := \left\{ H \in \mathrm{Glimm}(A) : \ \|z + H\| \geq \frac{1}{2}\|z + G\| \right\}.$$

We claim that $\mathcal{U}$ is a τ_q-compact neighborhood of G in $\mathrm{Glimm}(A)$. Indeed, since $[H \mapsto \|z + H\|] \in C_b(\mathrm{Glimm}(A))$, $\mathcal{U}$ is a τ_q-neighborhood of G. To show that $\mathcal{U}$ is τ_q-compact, note that $\mathcal{U} = \phi_A(\mathcal{O})$, where

$$\mathcal{O} := \left\{ P \in \mathrm{Prim}(A) : \ \|z + P\| \geq \frac{1}{2}\|z + G\| \right\}$$

is a compact subset of $\mathrm{Prim}(A)$ (by (3.1)). It follows that $(\mathrm{Glimm}(A), \tau_q)$ is locally compact Hausdorff space, and hence τ_q coincides with the weak topology induced by $C_0(\mathrm{Glimm}(A), \tau_q) \subseteq C_b(\mathrm{Glimm}(A))$. Thus, $\tau_q = \tau_{cr}$.

We now prove that ζ_A is a homeomorphism. Since each irreducible representation of $Z(A)$ can be lifted to the irreducible representation of A (see [9, II.6.4.11]), ζ_A is surjective. That ζ_A is also injective follows from Lemma 3.7 (iii). Since the topology of (the locally compact Hausdorff space) Glimm(A) coincides with the weak topology induced by $C_0(\mathrm{Glimm}(A))_+$ and since

$$C_0(\mathrm{Glimm}(A))_+ = \{f_\approx :\ f \in C_0(\mathrm{Prim}(A))_+\} = \{\Psi_A(z)_\approx :\ z \in Z(A)_+\},$$

for a net (G_α) in Glimm(A) and $G \in \mathrm{Glimm}(A)$ we have

$$\begin{aligned} G_\alpha \to G \quad &\iff \quad \Psi_A(z)_\approx(G_\alpha) \to \Psi_A(z)_\approx(G), \text{ for all } z \in Z(A)_+ \\ &\iff \quad \|z + G_\alpha\| \to \|z + G\|, \text{ for all } z \in Z(A)_+ \\ &\iff \quad \|z + G_\alpha \cap Z(A)\| \to \|z + G \cap Z(A)\|, \text{ for all } z \in Z(A)_+ \\ &\iff \quad G_\alpha \cap Z(A) \to G \cap Z(A). \end{aligned}$$

It follows that ζ_A is a homeomorphism.

Finally, if $J \in \mathrm{Max}(Z(A))$, note that JA is a proper ideal of A (which is closed by Cohen's factorization theorem) and $JA \cap Z(A) = J$. Then for $P \in \mathrm{Prim}(A)$ we have $P \cap Z(A) = J$ if and only if $JA \subseteq P$. Hence, $JA \in \mathrm{Glimm}(A)$ and $\zeta_A^{-1}(J) = JA$. □

Remark 3.9. If A is a non-unital quasicentral C^*-algebra, then by Proposition 3.5 Prim(A) and (hence) Glimm(A) are non-compact spaces. For $J \in \mathrm{Id}(A)$ let $J_\sim$ be the unique ideal of $\tilde{A}$ such that $A \cap J_\sim = J$. By Proposition 3.2 (iv) and Proposition 3.8 it follows that the map $G \mapsto G_\sim$ is a homeomorphism from Glimm(A) onto its image $\mathrm{Glimm}(\tilde{A}) \setminus \{A\}$ in Glimm$(\tilde{A})$. Since $\tilde{A}$ is unital, Glimm$(\tilde{A})$ is a compact Hausdorff space, and hence Glimm$(\tilde{A})$ is the Alexandroff compactification of Glimm(A). Since $\zeta_{\tilde{A}}(A) = Z(A)$, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Prim}(A) & \xrightarrow{\phi_A} & \mathrm{Glimm}(A) & \xrightarrow{\zeta_A} & \mathrm{Max}(Z(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Prim}(\tilde{A}) & \xrightarrow{\phi_{\tilde{A}}} & \mathrm{Glimm}(\tilde{A}) & \xrightarrow{\zeta_{\tilde{A}}} & \mathrm{Max}(Z(\tilde{A})), \end{array}$$

where the vertical maps denote the canonical embeddings.

Definition 3.10. [15, §9] A C^*-algebra A is said to be *central* if it satisfies the following two conditions:

(i) A is quasicentral;
(ii) If $P, Q \in \mathrm{Prim}(A)$ and $P \cap Z(A) = Q \cap Z(A)$, then $P = Q$.

Remark 3.11. By [11, Prop. 3] (see also [15, 9.1]) a quasicentral C^*-algebra A is central if and only if Prim(A) is Hausdorff. Note that this fact follows immediately from Lemma 3.7, since a locally compact space Prim(A) is Hausdorff if and only $C_0(\mathrm{Prim}(A))$ is a separating family for Prim(A). In this case $\mathrm{Glimm}(A) = \mathrm{Prim}(A)$, so by Proposition 3.8 $\zeta_A : P \mapsto P \cap Z(A)$ is a homeomorphism from Prim(A) onto Max$(Z(A))$.

The proof of the next fact can be found in [11, Prop. 3], but let us nevertheless present the short argument for completeness.

Proposition 3.12. *Let A be a C^*-algebra. Then A is central if and only if $\tilde{A}$ is central.*

Proof. If A is unital, we have nothing to prove, so assume that A is non-unital.

Suppose that A is central and let $P, Q \in \operatorname{Prim}(\tilde{A})$ such that $P \neq Q$. Then $P \cap A$ and $Q \cap A$ are distinct elements of $\operatorname{Prim}(A) \cup \{A\}$. Since A is central, it follows that they have distinct intersection with $Z(A) \subseteq Z(\tilde{A})$.

Conversely, suppose that A is central. By Remark 3.11 $\operatorname{Prim}(\tilde{A})$ is Hausdorff. Then $\operatorname{Glimm}(\tilde{A}) = \operatorname{Prim}(\tilde{A})$, so $A \in \operatorname{Glimm}(\tilde{A})$. By Proposition 3.2 A is quasicentral. Since $\operatorname{Prim}(A)$ is homoeomorphic to the (open) subset $\operatorname{Prim}(\tilde{A}) \setminus \{A\}$ of $\operatorname{Prim}(\tilde{A})$, $\operatorname{Prim}(A)$ is also Hausdorff. By Remark 3.11 A is central. □

4. Derivations in $\operatorname{Im}\theta_A$ on Prime C^*-algebras

Recall that a C^*-algebra A is called *prime* if the zero ideal $\{0\}$ is a prime ideal of A. Since by [3, 1.2.47] the center $Z(A)$ of a prime C^*-algebra A is either zero (if A is non-unital) or isomorphic to $\mathbb{C}$ (if A is unital), it follows from Proposition 3.5 that A is unital if and only if it is quasicentral.

Remark 4.1. Mathieu showed that the canonical contraction θ_A is an isometry if and only if A is prime C^*-algebra (see [3, 5.4.11]). Since by [3, 1.1.7] A is prime if and only if $M(A)$ is prime, it follows (using the Kaplansky's density theorem) that in this case the map

$$\Theta_A : M(A) \otimes_h M(A) \to \operatorname{ICB}(A), \quad \Theta_A(t) := \theta_{M(A)}(t)|_A$$

is also an isometry.

Recall from [21, 3.2] that a subset $\{a_n\}$ of a C^*-algebra A such that the series $\sum_{n=1}^{\infty} a_n^* a_n$ is norm convergent is said to be *strongly independent* if whenever $(\alpha_n) \in \ell^2$ is a square summable sequence of complex numbers such that $\sum_{n=1}^{\infty} \alpha_n a_n = 0$, we have $\alpha_n = 0$, for all $n \in \mathbb{N}$.

The next lemma is a combination of [10, 1.5.6], [21, 4.1] and [2, 2.3].

Lemma 4.2. *Let A be a C^*-algebra.*

(i) *Every tensor $t \in A \otimes_h A$ has a representation as a convergent series $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where (a_n) and (b_n) are sequences of A such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent. Moreover, $\{b_n\}$ can be chosen to be strongly independent.*

(ii) *If $t = \sum_{n=1}^{\infty} a_n \otimes b_n$ is a representation of t as above, with $\{b_n\}$ strongly independent, then $t = 0$ if and only if $a_n = 0$, for all $n \in \mathbb{N}$.*

Theorem 4.3. *Let A be a prime C^*-algebra. Every derivation $\delta \in \operatorname{Der}(A) \cap \operatorname{Im}\theta_A$ is inner in A. If A is non-unital, then $\operatorname{Der}(A) \cap \operatorname{Im}\theta_A = \{0\}$.*

Proof. Let Θ_A be the map as in Remark 4.1 and let $t \in A \otimes_h A$ be a tensor such that $\Theta_A(t) = \delta$ (we assume that $A \otimes_h A \subseteq M(A) \otimes_h M(A)$, by the injectivity of the Haagerup tensor product). Suppose that $t = \sum_{n=1}^{\infty} a_n \otimes b_n$ is a representation of t as in Lemma 4.2 (i), with $\{b_n\}$ strongly independent. Since δ is a derivation on A, Leibniz rule (2.4) implies that

$$\delta(x)y = \sum_{n=1}^{\infty} (a_n x - x a_n) y b_n, \quad \text{for all } x, y \in A,$$

or equivalently

$$\Theta_A(\delta(x) \otimes 1) = \Theta_A\Big(\sum_{n=1}^{\infty} (a_n x - x a_n) \otimes b_n\Big), \quad \text{for all } x \in A. \tag{4.1}$$

By remark 4.1 Θ_A is an isometry (and hence injective), so the equality (4.1) is equivalent to the equality

$$\delta(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n x - x a_n) \otimes b_n, \quad \text{for all } x \in A, \tag{4.2}$$

of tensors in $M(A) \otimes_h M(A)$. Suppose that $\delta \neq 0$. Then (4.2) implies that A must be unital, so $A = M(A)$. Indeed, choose $x_0 \in A$ such that $\delta(x_0) \neq 0$, and let $\varphi \in M(A)^*$ be an arbitrary bounded linear functional such that $\varphi(\delta(x_0)) \neq 0$. If we act on the equality (4.2) (for $x = x_0$) with the right slice map R_φ (recall that for a C^*-algebra B and $\psi \in B^*$, the right slice map R_ψ is a unique bounded map $B \otimes_h B \to B$ given on elementary tensors by $R_\psi(a \otimes b) = \psi(a)b$, see [21, Section 4]), we obtain

$$1 = \frac{1}{\varphi(\delta(x_0))} \sum_{n=1}^{\infty} \varphi(a_n x_0 - x_0 a_n) b_n, \tag{4.3}$$

and hence $1 \in A$. Let

$$\alpha_n := \frac{\varphi(a_n x_0 - x_0 a_n)}{\varphi(\delta(x_0))} \quad (n \in \mathbb{N}).$$

Since each bounded functional on a C^*-algebra is completely bounded (see [17, 3.8]), and since the series $\sum_{n=1}^{\infty} (a_n x_0 - x_0 a_n)(a_n x_0 - x_0 a_n)^*$ is norm convergent, we have $(\alpha_n) \in \ell^2$, and (4.3) implies that $\sum_{n=1}^{\infty} \alpha_n b_n = 1$. Then it follows from (4.2) that

$$\sum_{n=1}^{\infty} (\alpha_n \delta(x) - a_n x + x a_n) \otimes b_n = 0, \quad \text{for all } x \in A,$$

and consequently, since $\{b_n\}$ is strongly independent, Lemma 4.2 (ii) implies that

$$\alpha_n \delta(x) = a_n x - x a_n \quad \text{for all } x \in A \text{ and } n \in \mathbb{N}. \tag{4.4}$$

Since $\sum_{n=1}^{\infty} \alpha_n b_n = 1$, there is some $k \in \mathbb{N}$ such that $\alpha_k \neq 0$. If $a := \frac{a_k}{\alpha_k}$, then the equality (4.4) implies that $\delta = \delta_a \in \operatorname{Inn}(A)$. $\square$

5. Derivations in $\operatorname{Im} \theta_A$ on C^*-algebras with Hausdorff primitive spectrum

Definition 5.1. Let A be a C^*-algebra, and let δ be a derivation on A. We define a bounded function

$$|\delta| : \operatorname{Prim}(A) \to \mathbb{R}_+ \quad \text{by} \quad |\delta|(P) := \|\delta_P\| \quad (P \in \operatorname{Prim}(A)).$$

By [1, 2.2] $|\delta|$ is a lower semi-continuous function on $\operatorname{Prim}(A)$. If $|\delta|$ is continuous on $\operatorname{Prim}(A)$, we say that δ is *smooth*.

Remark 5.2. The function $|\delta|$ is usually defined on the spectrum $\hat{A}$ of A, by $|\delta|([\pi]) := \|\delta_\pi\|$ ($[\pi] \in \hat{A}$), where $\pi \in [\pi]$, and δ_π denotes the induced derivation on $\pi(A)$ ($\delta_\pi(\pi(a)) = \pi(\delta(a))$ $(a \in A)$). In this case δ is said to be smooth if $|\delta|$, as a function on $\hat{A}$, is continuous (see [1, 2.3] or [3, 4.2.6]). Since $\|\delta_\pi\| = \|\delta_P\|$, where $P := \ker \pi$, we note that this two definitions are consistent with each other.

The notion of the smooth derivation is important, since by [1, 2.4] (or [3, 4.2.7]) each smooth derivation on a C^*-algebra A is inner in $M(A)$.

Let A be a C^*-algebra and let $I, J \in \mathrm{Id}(A)$. If $q_I : A \to A/I$ and $q_J : A \to A/J$ denote the quotient maps, it follows from [2, 2.8] that the induced map $q_I \otimes q_J : A \otimes_h A \to (A/I) \otimes_h (A/J)$ is also quotient and that

$$\ker(q_I \otimes q_J) = I \otimes_h A + A \otimes_h J.$$

Hence, we have

$$(A \otimes_h A)/(I \otimes_h A + A \otimes_h J) \cong (A/I) \otimes_h (A/J),$$

isometrically.

For $t \in A \otimes_h A$ we define a bounded function

$$|t| : \mathrm{Prim}(A) \to \mathbb{R}_+ \quad \text{by} \quad |t|(P) := \|(q_P \otimes q_P)(t)\|_h \quad (P \in \mathrm{Prim}(A)).$$

Recall from [5] that the *strong topology* τ_s on $\mathrm{Id}(A)$ is the weakest topology that makes all norm functions $J \mapsto \|a + J\|$ $(a \in A)$ continuous on $\mathrm{Id}(A)$.

Lemma 5.3. *Let A be a C^*-algebra with Hausdorff primitive spectrum. For each tensor $t \in A \otimes_h A$ the function $|t|$ is continuous on* $\mathrm{Prim}(A)$.

Proof. Since $\mathrm{Prim}(A)$ is Hausdorff, by [18, 4.4.5] the functions $P \mapsto \|a+P\|$ $(a \in A)$ are continuous on $\mathrm{Prim}(A)$. Hence, the Jacobson topology and the τ_s-topology restricted to $\mathrm{Prim}(A)$ coincide. By [22, Prop. 2] for each $t \in A \otimes_h A$ the map

$$\mathrm{Id}(A) \times \mathrm{Id}(A) \to \mathbb{R}_+, \quad (I, J) \mapsto \|t + (I \otimes_h A + A \otimes_h J)\| = \|(q_I \otimes q_J)(t)\|_h$$

is continuous for the product τ_s-topology on $\mathrm{Id}(A) \times \mathrm{Id}(A)$. If D denotes the diagonal of $\mathrm{Prim}(A) \times \mathrm{Prim}(A)$, the map

$$(P, P) \mapsto \|(q_P \otimes q_P)(t)\|_h = |t|(P)$$

is continuous on D, and so the map $|t|$ is continuous on $\mathrm{Prim}(A)$. $\square$

Remark 5.4. Let A be a C^*-algebra. It is easy to check that for all $J \in \mathrm{Id}(A)$ the following diagram

$$\begin{array}{ccc} A \otimes_h A & \xrightarrow{\theta_A} & \mathrm{ICB}(A) \\ {\scriptstyle q_J \otimes q_J}\big\downarrow & & {\scriptstyle Q_J}\big\downarrow \\ (A/J) \otimes_h (A/J) & \xrightarrow{\theta_{A/J}} & \mathrm{ICB}(A/J) \end{array}$$

commutes, where Q_J denotes the induced map $Q_J : \mathrm{ICB}(A) \to \mathrm{ICB}(A/J)$,

$$Q_J(T)(q_J(x)) := q_J(T(x)), \text{ for all } T \in \mathrm{ICB}(A) \text{ and } x \in A. \tag{5.1}$$

Hence, if $\delta \in \mathrm{Der}(A) \cap \mathrm{Im}\,\theta_A$ and $t \in A \otimes_h A$ such that $\delta = \theta_A(t)$, we have

$$\delta_J = Q_J(\theta_A(t)) = \theta_{A/J}((q_J \otimes q_J)(t)). \tag{5.2}$$

Remark 5.5. Let A be a C^*-algebra and let $\delta \in \mathrm{Der}(A) \cap \mathrm{Im}\,\theta_A$, with $\delta = \theta_A(t)$, for some tensor $t \in A \otimes_h A$. If we embed A into its von Neumann envelope A^{**}, then by [3, 4.2.3] δ can be extended (by ultraweak continuity) to the derivation δ^{**} on A^{**}. It follows that $\delta^{**} = \theta_{A^{**}}(t)$ (where $A \otimes_h A \subseteq A^{**} \otimes_h A^{**}$, by the injectivity of the Haagerup tensor product), and hence $\tilde{\delta} = \delta^{**}|_{\tilde{A}} = \theta_{\tilde{A}}(t)$, where $\tilde{\delta}$ denotes the (unique) extension of δ to the derivation on the minimal unitization $\tilde{A}$ of A.

Theorem 5.6. *Let A be a C^*-algebra with Hausdorff primitive spectrum. Every derivation $\delta \in \operatorname{Im}\theta_A$ is smooth and hence inner in $M(A)$. Moreover, if A central, then every derivation $\delta \in \operatorname{Im}\theta_A$ is inner in A.*

Proof. Let $t \in A \otimes_h A$ be a tensor such that $\delta = \theta(t)$, and let $P \in \operatorname{Prim}(A)$. By (5.2) we have $\delta_P = \theta_{A/P}((q_P \otimes q_P)(t))$. Since A/P is primitive (simple in fact, since $\operatorname{Prim}(A)$ is Hausdorff), $\theta_{A/P}$ is an isometry, and hence

$$|\delta|(P) = \|\delta_P\| = \|\delta_P\|_{cb} = \|\theta_{A/P}((q_P \otimes q_P)(t))\|_{cb} = \|(q_P \otimes q_P)(t)\|_h = |t|(P).$$

Since $P \in \operatorname{Prim}(A)$ was arbitrary, Lemma 5.3 implies that $|\delta| = |t|$ is a continuous function on $\operatorname{Prim}(A)$, and hence, δ is smooth. By [1, 2.4] (or [3, 4.2.7]) there exists an element $b \in M(A)$ such that $\delta = \delta_b$.

Now suppose that A is central, and let $\tilde{\delta}$ be the (unique) extension of δ to the derivation on $\tilde{A}$. By Remark 5.5 we have $\theta_{\tilde{A}}(t) = \tilde{\delta}$. Since $\tilde{A}$ is also central (Proposition 3.12), by Remark 3.11 $\operatorname{Prim}(\tilde{A})$ is Hausdorff. Hence, by the first part of the proof, there exists $b \in \tilde{A}$ which implements $\tilde{\delta}$. If we choose $\alpha \in \mathbb{C}$ such that $a := b - \alpha 1 \in A$, then obviously a also implements $\tilde{\delta}$. It follows that $\delta = \tilde{\delta}|_A$ is inner in A. □

Question 5.7. *Can one always (without the assumption of quasicentrality) conclude that* $\operatorname{Der}(A) \cap \operatorname{Im}\theta_A \subseteq \operatorname{Inn}(A)$, *when* $\operatorname{Prim}(A)$ *is Hausdorff?*

Corollary 5.8. *Let A be a C^*-algebra.*

(i) *If A is central then each inner derivation on A is smooth.*
(ii) *If each inner derivation on A is smooth then* $\operatorname{Prim}(A)$ *is Hasudorff.*

Hence, a quasicentral C^*-algebra A is central if and only if each inner derivation on A is smooth.

Proof. (i). Since A is central, by Lemma 3.3 $\operatorname{Inn}(A) \subseteq \operatorname{Im}\theta_A$, so by Theorem 5.6 each inner derivation on A is smooth.

(ii). Let $a \in A_h$. Since δ_a is smooth, by [1, 2.10] the function $P \mapsto \|(a+z)+P^\sim\|$ is continuous on $\operatorname{Prim}(A)$, for each $z \in Z(M(A))_h$, where $P^\sim$ (for $P \in \operatorname{Prim}(A)$) denotes the unique primitive ideal of $M(A)$ such that $A \cap P^\sim = P$. Hence, for $z = 0$, the function $P \mapsto \|a + P^\sim\| = \|a + P\|$ is continuous on $\operatorname{Prim}(A)$, and since $a \in A_h$ was arbitrary, by [18, 4.4.5] $\operatorname{Prim}(A)$ is Hausdorff. □

The result of Corollary 5.8 is not true in general for non-central C^*-algebras, even if $\operatorname{Prim}(A)$ is Hausdorff and every primitive quotient of A is unital.

Example 5.9. Let A be a C^*-algebra consisting of all continuous functions $a : [0,1] \to \mathrm{M}_2(\mathbb{C})$ such that

$$a(1) = \begin{pmatrix} \lambda(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for some } \lambda(a) \in \mathbb{C}.$$

It is easy to check that every irreducible representation of A is equivalent to some representation π_t ($t \in [0,1]$), where $\pi_t : a \mapsto a(t)$, for $t \in [0,1)$, and $\pi_1 : a \mapsto \lambda(a)$, and that the map $t \mapsto P_t := \ker \pi_t$ is a homeomorphism from $[0,1]$ onto $\operatorname{Prim}(A)$. Since

$$Z(A) = \left\{ \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : \ f \in C_0([0,1)) \right\} \subseteq P_1,$$

A is not quasicentral. Let a be an element of A such that

$$a(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } t \in [0,1],$$

and let $\delta := \delta_a$. By Stampfli's formula (2.5) we have

$$\|\delta_{P_t}\| = 2d(a + P_t, \mathbb{C}) = \begin{cases} 1, & \text{if } 0 \le t < 1, \\ 0, & \text{if } t = 1 \end{cases}$$

and hence, δ is not smooth.

6. An example of a C^*-algebra with outer elementary derivations

In this section we shall give an example of a unital C^*-algebra A which has an outer elementary derivation (that is, an outer derivation $\delta \in \mathrm{E}(A)$). For this C^*-algebra A the space $\mathrm{Inn}(A)$ is not closed in the space $\mathrm{Der}(A)$. By [23, 4.6] this happens if and only if $\mathrm{Orc}(A) = \infty$, where $\mathrm{Orc}(A)$ is a constant arising from a certain graph structure on $\mathrm{Prim}(A)$ which is defined as follows.

We say that two primitive ideals $P, Q \in \mathrm{Prim}(A)$ are *adjacent* (and write $P \sim Q$) if P and Q cannot be separated by disjoint open subsets of $\mathrm{Prim}(A)$. A *path* of length n from P to Q is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1} \sim P_i$, for all $1 \le i \le n$. The *distance* $d(P,Q)$ from P to Q is defined as follows:

- If $P = Q$, $d(P,Q) = d(P,P) := 1$,
- If $P \ne Q$ and there exists a path from P to Q, then $d(P,Q)$ is equal to the minimal length of a path from P to Q.
- If there is no path from P to Q, $d(P,Q) := \infty$.

The *connecting order* $\mathrm{Orc}(A)$ of A is defined by

$$\mathrm{Orc}(A) := \sup\{d(P,Q) : \ P, Q \in \mathrm{Prim}(A) \text{ such that } d(P,Q) < \infty\}.$$

Note that $\mathrm{Orc}(A) = 1$ if $\mathrm{Prim}(A)$ is Hausdorff, but that the converse does not hold in general (as noted in [22], $\mathrm{Orc}(A) = 1$ if and only if every Glimm ideal of A is 2-primal).

We shall also use the following notation. Let B be a unital C^*-algebra and let $A \subseteq B$ be a C^*-subalgebra of B. An *elementary operator* on B *with the coefficients* in A is a map $T : B \to B$ which can be expressed in the form

$$T = \sum_{k=1}^{d} a_k \odot b_k, \quad \text{for some } a_k, b_k \in A \ (1 \le k \le d),$$

where

$$\left(\sum_{k=1}^{d} a_k \odot b_k\right)(x) := \theta_B \left(\sum_{k=1}^{d} a_k \otimes b_k\right)(x) = \sum_{k=1}^{d} a_k x b_k \quad (x \in B).$$

The space of all elementary operators on B with the coefficients in A is denoted by $\mathrm{E}_A(B)$. If $A = B$ then (as usual) we write $\mathrm{E}(B)$ for $\mathrm{E}_B(B)$; the set of all elementary operators on B. We also denote by $\mathrm{E}(B \to A)$ the subspace of all $T \in \mathrm{E}(B)$ such that $T(B) \subseteq A$.

Example 6.1. Let $\tilde{X} := [1,\infty]$ be the Alexandroff compactification of the interval $X := [1,\infty)$, let $B := C(\tilde{X}, \mathrm{M}_2(\mathbb{C}))$, and let A be a C^*-subalgebra of B which consists of all $a \in B$ such that

$$a(n) = \begin{pmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{pmatrix} \ (n \in \mathbb{N}) \ \text{ and } \ a(\infty) = \begin{pmatrix} \lambda(a) & 0 \\ 0 & \lambda(a) \end{pmatrix},$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers with $\lim_n \lambda_n(a) = \lambda(a)$. Then $\mathrm{Orc}(A) = \infty$ and $\mathrm{E}(A)$ is a cb-closed subspace of $\mathrm{ICB}(A)$. Consequently, A has an outer elementary derivation.

This example is just a slightly modified version of the C^*-algebra $A(\infty)$ in [23, 2.8]. We indicate that the justification of the example will occupy most of this section.

First recall, that a primitive ideal $P \in \mathrm{Prim}(A)$ is said to be *separated* in $\mathrm{Prim}(A)$ if whenever $Q \in \mathrm{Prim}(A)$ and $P \not\subseteq Q$ then there exist disjoint open neighborhoods of P and Q in $\mathrm{Prim}(A)$. In our example it is easy to check that

$$\mathrm{Prim}(A) = \{P_t : \ t \in X \setminus \mathbb{N}\} \cup \{Q_n : \ n \in \mathbb{N}\} \cup \{Q\},$$

where P_t $(t \in X \setminus \mathbb{N})$ denotes a kernel of $a \mapsto a(t)$, Q_n $(n \in \mathbb{N})$ denotes a kernel of $a \mapsto \lambda_n(a)$, and Q denotes the kernel of $a \mapsto \lambda(a)$. Also note that the points P_t $(t \in X \setminus \mathbb{N})$ and Q are separated in $\mathrm{Prim}(A)$, while $Q_i \sim Q_j$ if and only if $|i-j| \leq 1$. It follows that $d(Q_1, Q_{n+1}) = n$, for all $n \in \mathbb{N}$, and hence $\mathrm{Orc}(A) = \infty$. By [23, 4.6] $\mathrm{Inn}(A)$ is not closed in $\mathrm{Der}(A)$. One can also check this by direct calculations. For example, it is not difficult to see that for each function $f \in C_0(X)$ such that the series $\sum_{n=1}^{\infty} f(n)$ does not converge, the element

$$b = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in B$$

derives A (that is $bx - xb \in A$, for all $x \in A$) and the induced derivation (which is obviously not inner in A) is in the closure of $\mathrm{Inn}(A)$.

To prove that $\mathrm{E}(A)$ is closed in $\mathrm{ICB}(A)$ we shall first need some additional technical results which will be stated in a more general setting.

Let A be a C^*-algebra. Recall that A is called n-homogeneous $(n \in \mathbb{N})$ if $\dim \pi = n$, for all $[\pi] \in \hat{A}$. Then by [14, 3.2] $\Delta := \mathrm{Prim}(A)$ is a (locally compact) Hausdorff space and A is isomorphic to the C^*-algebra $\Gamma_0(E)$ of all continuous sections vanishing at infinity of a locally trivial C^*-bundle E over Δ with fibres isomorphic to $\mathrm{M}_n(\mathbb{C})$. If the base space Δ of E admits a finite open covering $\{U_j\}$ such that each $E|_{U_j}$ is trivial (as a C^*-bundle) we say that E (and hence A) is of *finite type*.

If

$$\sup\{\dim \pi : \ [\pi] \in \hat{A}\} = n$$

then we say that A is *n-subhomogeneous*. In this case

$$J := \bigcap \{\ker \pi : \ [\pi] \in \hat{A} \text{ such that } \dim \pi < n\}$$

is called *n-homogeneous ideal* of A, and is the largest ideal of A which is n-homogeneous, as a C^*-algebra.

Remark 6.2. If A is n-subhomogeneous C^*-algebra, note that for each operator $T \in \mathrm{Im}\, \theta_A$ we have

$$\|T\|_{cb} \leq n\|T\|.$$

Indeed, if for $J \in \mathrm{Id}(A)$ we put $T_J := Q_J(T)$ (where Q_J is the map from (5.1)), then this can be easily seen by using the formulas

$$\|T\| = \sup\{\|T_P\| : \ P \in \mathrm{Prim}(A)\} \quad \text{and} \quad \|T\|_{cb} = \sup\{\|T_P\|_{cb} : \ P \in \mathrm{Prim}(A)\},$$

(see [3, 5.3.12]) and noting that each operator $S : \mathrm{M}_m(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C})$ is completely bounded (elementary in fact) with $\|S\|_{cb} \leq m\|S\|$ (see [17, Exercise 3.11]). Hence, if A is subhomogeneous, we do not have to specify which norm do we consider when speaking about closures of $\mathrm{Im}\,\theta_A$ or $\mathrm{E}(A)$.

Lemma 6.3. *Let B be a unital n-homogeneous C^*-algebra and let $J \in \mathrm{Id}(B)$. Then $\mathrm{E}_J(B) = \mathrm{E}(B \to J)$. In particular, $\mathrm{E}_J(B)$ is a closed subspace of $\mathrm{E}(B)$.*

Proof. Let E be a locally trivial C^*-bundle E over $\Delta := \mathrm{Prim}(B)$ (which is compact since B is unital) whose fibres are isomorphic to $\mathrm{M}_n(\mathbb{C})$ such that $B = \Gamma(E)$ (we identify B with $\Gamma(E)$ via the canonical isomorphism). By compactness of Δ and local triviality of E, there exists a finite open cover $\{U_j\}_{1\leq j\leq m}$ of Δ such that each $E|_{\overline{U_j}}$ is trivial. Using a finite partition of unity (subordinated to the cover $\{U_j\}_{1\leq j\leq m}$) one can reduce the proof to the situation when $m = 1$, so we may assume E is trivial. Then $B = C(\Delta, \mathrm{M}_n(\mathbb{C}))$, and since J is an ideal of B, there is a closed subset Y of Δ such that

$$J = \{a \in B : \ a|_Y = 0\}.$$

Let $(E_{i,j})_{1\leq i,j\leq n}$ denote the standard matrix units of $\mathrm{M}_n(\mathbb{C})$ considered as constant elements of $B = C(\Delta, \mathrm{M}_n(\mathbb{C}))$, and let $T \in \mathrm{E}(B \to J)$. Then T can be written in the form

$$T = \sum_{i,j,p,q=1}^{n} f_{i,j,p,q} E_{i,j} \odot E_{p,q}, \tag{6.1}$$

for some functions $f_{i,j,p,q} \in C(\Delta) \cong Z(B)$. Let $1 \leq r, s \leq n$ be the fixed numbers. Since $T(B) \subseteq J$, we have

$$T(E_{r,s}) = \sum_{i,j,p,q=1}^{n} f_{i,j,p,q} E_{i,j} E_{r,s} E_{p,q} = \sum_{i,q=1}^{n} f_{i,r,s,q} E_{i,q} \in J.$$

Thus, $f_{i,r,s,q}|_Y = 0$, for all $i, q = 1, \ldots, n$. Since r, s were arbitrary, we have

$$f_{i,j,p,q}|_Y = 0, \quad \text{for all } 1 \leq i, j, p, q \leq n$$

Note that every function $f \in C(\Delta)$ with the property $f|_Y = 0$ can be factorized in the form $f = gh$, where $g, h \in C(\Delta)$ such that $g|_Y = 0$ and $h|_Y = 0$ (for example, put $g := \sqrt{|f|}$ i $h := f/\sqrt{|f|}$). If we apply this factorization to the functions $f_{i,j,p,q}$,

$$f_{i,j,p,q} = g_{i,j,p,q} \cdot h_{i,j,p,q},$$

then it follows from (6.1) that

$$T = \sum_{i,j,p,q=1}^{n} f_{i,j,p,q} E_{i,j} \odot E_{p,q} = \sum_{i,j,p,q=1}^{n} g_{i,j,p,q} E_{i,j} \odot h_{i,j,p,q} E_{p,q}.$$

Thus $T \in \mathrm{E}_J(B)$. $\square$

Remark 6.4. Suppose that

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} Z \longrightarrow 0$$

is an exact sequence of normed spaces, where q is a bounded linear map. If q is also open, note that Y is a Banach space if and only if X and Z are Banach spaces. Also note that if $\dot{Y} \subseteq Y$ and $\dot{Z} \subseteq Z$ are (not necessarily closed) subspaces such that $q(\dot{Y}) = \dot{Z}$ and which fit into the exact sequence

$$0 \longrightarrow X \longrightarrow \dot{Y} \xrightarrow{\dot{q}} \dot{Z} \longrightarrow 0,$$

where $\dot{q} := q|_{\dot{Y}}$ (and hence $\dot{Y} = \dot{q}^{-1}(\dot{Z}) = q^{-1}(\dot{Z})$), then $\dot{q}$ is open whenever q is open.

Lemma 6.5. *Suppose that A is a unital n-subhomogeneous C^*-algebra with n-homogeneous ideal J which is of finite type. If B is any unital n-homogeneous C^*-algebra which contains A and such that J is the essential ideal of B, then* $\mathrm{E}(A)$ *is closed subspace of* $\mathrm{ICB}(A)$ *if and only if* $\mathrm{E}_{A/J}(B/J)$ *is a closed subspace of* $\mathrm{ICB}(B/J)$.

Proof. First note that J is also essential in A. Also note that such B exists, since by [16, 3.3] $M(J)$ is n-homogeneous, and $A \subseteq M(J)$, since J is essential in A. By Kaplansky's density theorem the restriction map $T \mapsto T|_A$ is an isometric isomorphism from $\mathrm{E}_A(B)$ onto $\mathrm{E}(A)$. Hence, we may identify $\mathrm{E}(A)$ with $\mathrm{E}_A(B)$. Let $q_J : B \to B/J$ be a quotient map, and let $\dot{Q}_J$ be the restriction of the induced contraction Q_J to $\mathrm{E}(B)$ (see (5.1)). Obviously $\dot{Q}_J(\mathrm{E}(B)) = \mathrm{E}(B/J)$ and the kernel of $\dot{Q}_J$ is the set $\mathrm{E}(B \to J)$, which can be identified with the set $\mathrm{E}_J(B)$, by Lemma 6.3. Since B and B/J are unital homogeneous C^*-algebras, by [16, 1.1] we have equalities $\mathrm{ICB}(B) = \mathrm{E}(B)$ and $\mathrm{ICB}(B/J) = \mathrm{E}(B/J)$. Thus $\mathrm{E}(B)$ and $\mathrm{E}(B/J)$ are Banach spaces, and by the open mapping theorem, $\dot{Q}_J$ is an open map. Since $\dot{Q}_J(\mathrm{E}_A(B)) = \mathrm{E}_{A/J}(B/J)$, note that the exact sequence

$$0 \longrightarrow \mathrm{E}_J(B) \longrightarrow \mathrm{E}(B) \xrightarrow{\dot{Q}_J} \mathrm{E}(B/J) \longrightarrow 0$$

of Banach spaces induces the exact sequence of normed spaces

$$0 \longrightarrow \mathrm{E}_J(B) \longrightarrow \mathrm{E}_A(B) \xrightarrow{\ddot{Q}_J} \mathrm{E}_{A/J}(B/J) \longrightarrow 0,$$

where $\ddot{Q}_J$ denotes a restriction of $\dot{Q}_J$ to the set $\mathrm{E}_A(B)$, since $\ker \ddot{Q}_J = \ker \dot{Q}_J = \mathrm{E}_J(B)$. By Remark 6.4, $\ddot{Q}_J$ is also an open map, and since $\mathrm{E}_J(B)$ is a Banach space (Lemma 6.3), $\mathrm{E}_A(B)$ is a Banach space if and only if $\mathrm{E}_{A/J}(B/J)$ is a Banach space. □

Now we prove the second claim of the Example 6.1.

Lemma 6.6. *Let A and B be the C^*-algebras from the Example 6.1. Then* $\mathrm{E}(A)$ *is a closed subspace of* $\mathrm{ICB}(A)$.

Proof. Let

$$J := \{a \in A : \ a(n) = 0, \text{ for all } n \in \mathbb{N}\}$$

be the 2-homogeneous (Glimm) ideal of A. Then J is an essential ideal of A and B, and it follows from Lemma 6.5 that it is sufficient to show that $\mathrm{E}_{A/J}(B/J)$ is a closed subspace of $\mathrm{ICB}(B/J)$ which is equal to $\mathrm{E}(B/J)$, by [16, 1.1]. Let

$$\dot{B} := C(\tilde{\mathbb{N}}, \mathrm{M}_2(\mathbb{C})) \quad \text{and} \quad \dot{A} := \left\{ \begin{pmatrix} f & 0 \\ 0 & \tilde{f} \end{pmatrix} : \ f \in C(\tilde{\mathbb{N}}) \right\},$$

where $\tilde{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ denotes the Alexandroff compactifcation of $\mathbb{N}$, and for $f \in C(\tilde{\mathbb{N}})$, $\tilde{f}$ is a function defined by $\tilde{f}(n) := f(n+1)$ $(n \in \mathbb{N})$. Obviously $B/J \cong \dot{B}$ and $A/J \cong \dot{A}$, and in the following, we shall identify this C^*-algebras. If $(E_{i,j})_{1\le i,j\le 2}$ denote the standard matrix units of $\mathrm{M}_2(\mathbb{C})$ considered as constant elements of $\dot{B}$, we claim that the set $\mathrm{E}_{\dot{A}}(\dot{B})$ can be identified with the set of all operators $T \in \mathrm{E}(\dot{B})$ which can be written in the form

$$T = fE_{1,1} \odot E_{1,1} + gE_{1,1} \odot E_{2,2} + hE_{2,2} \odot E_{1,1} + \tilde{f}E_{2,2} \odot E_{2,2}, \tag{6.2}$$

where $f, g, h \in C(\tilde{\mathbb{N}})$ are functions such that

$$L(T) := f(\infty) = g(\infty) = h(\infty).$$

One can easily show that every $T \in \mathrm{E}_{\dot{A}}(\dot{B})$ can be written in the form (6.2). Conversely, if $T \in \mathrm{E}(\dot{B})$ is of the form (6.2), then

$$\begin{aligned} T \ &= \ (f - L(T))E_{1,1} \odot E_{1,1} + (g - L(T))E_{1,1} \odot E_{2,2} \\ &+ \ (h - L(T))E_{2,2} \odot E_{1,1} + (\tilde{f} - L(T))E_{2,2} \odot E_{2,2} + L(T)\mathrm{Id}, \end{aligned}$$

where Id denotes the identity map on $\dot{B}$. Hence, to prove that $T \in \mathrm{E}_{\dot{A}}(\dot{B})$, it is sufficient to prove that for arbitrary functions $f, g, h \in C_0(\mathbb{N})$ all operators T_1, T_2 and T_3 are the elements of $\mathrm{E}_{\dot{A}}(\dot{B})$, where

$$T_1 := fE_{1,1} \odot E_{1,1} + \tilde{f}E_{2,2} \odot E_{2,2}, \quad T_2 := gE_{1,1} \odot E_{2,2} \quad \text{and} \quad T_3 := hE_{2,2} \odot E_{1,1}.$$

Claim 1. T_1 can be written in the form

$$T_1 = a_1 \odot b_1 + a_2 \odot b_2, \quad \text{for some } a_i, b_i \in \dot{A}.$$

To prove this, by looking at the entries of the corresponding decomposition of T_1, it is sufficient to find two sequences of vectors $(\vec{v}_n)$ and $(\vec{w}_n)$ in $\mathbb{C}^2$ such that $\lim_n \vec{v}_n = \lim_n \vec{w}_n = (0, 0)$, and

$$\vec{v}_n \cdot \vec{w}_n^* = f(n), \quad \vec{v}_n \cdot \vec{w}_{n+1}^* = \vec{v}_{n+1} \cdot \vec{w}_n^* = 0, \quad \text{for all } n \in \mathbb{N}, \tag{6.3}$$

where $\cdot$ denotes a standard inner product of $\mathbb{C}^2$, and for $\vec{v} = (\alpha, \beta) \in \mathbb{C}^2$, $\vec{v}^* := (\bar{\alpha}, \bar{\beta})$. Let $\varphi, \psi \in C_0(\mathbb{N})$ be any functions such that $f = \varphi\psi$. Then we can achieve (6.3) by putting

$$\vec{v}_n = ([n+1]\varphi(n), [n]\varphi(n)) \quad \text{and} \quad \vec{w}_n = ([n+1]\psi(n), [n]\psi(n)) \ (n \in \mathbb{N})$$

where $[n] = 1$ if n is even and $[n] = 0$ if n is odd.

Claim 2. T_2 can be written in the form

$$T_2 = a_1 \odot b_1 + a_2 \odot b_2 + a_3 \odot b_3, \quad \text{for some } a_i, b_i \in \dot{A}.$$

To prove this, like in the proof of Claim 1, it is sufficient to find two sequences of vectors $(\vec{v}_n)$ and $(\vec{w}_n)$ in $\mathbb{C}^3$ such that $\lim_n \vec{v}_n = \lim_n \vec{w}_n = (0, 0, 0)$, and

$$\vec{v}_n \cdot \vec{w}_n^* = \vec{v}_{n+1} \cdot \vec{w}_n^* = 0, \quad \vec{v}_n \cdot \vec{w}_{n+1}^* = g(n), \quad \text{for all } n \in \mathbb{N}. \tag{6.4}$$

Let $\varphi, \psi \in C_0(\mathbb{N})$ be any functions such that $g = \varphi\psi$. If $(\vec{e}_i)_{1\le i\le 3}$ denote the canonical basis of $\mathbb{C}^3$, we can achieve (6.4) by putting

$$\vec{v}_n = \varphi(n)\vec{e}_{\langle n\rangle} \quad \text{and} \quad \vec{w}_i = \psi(n-1)\vec{e}_{\langle n+2\rangle} \ (n \in \mathbb{N}),$$

where $\psi(0) := 1$, and for $n = 3k + l$, $\langle n \rangle = l$ if $l = 1, 2$ and $\langle n \rangle = 3$ if $l = 0$.

Claim 3. T_3 can be written in the form

$$T_3 = a_1 \odot b_1 + a_2 \odot b_2 + a_3 \odot b_3, \quad \text{for some } a_i, b_i \in \dot{A}.$$

This can be proved like Claim 2.

Using (6.2) it is now easy to vertify that $\mathrm{E}_{\dot{A}}(\dot{B})$ is closed in $\mathrm{ICB}(\dot{B}) = \mathrm{E}(\dot{B})$. □

Question 6.7. *Does every unital C^*-algebra A with $\mathrm{Orc}(A) = \infty$ have an outer elementary derivation, or at least an outer derivation $\delta \in \mathrm{Im}\,\theta_A$?*

Let A be a separable C^*-algebra, and let $J \in \mathrm{Id}(A)$. By [18, 8.6.15] we know that each derivation $\dot{\delta} \in \mathrm{Der}(A/J)$ can be lifted to the derivation $\delta \in \mathrm{Der}(A)$. Obviously, each operator $\dot{T} \in \mathrm{Im}\,\theta_{A/J}$ can also be lifted to an operator $T \in \mathrm{Im}\,\theta_A$. The next example shows that in general we cannot expect that a derivation $\dot{\delta} \in \mathrm{Der}(A/J) \cap \mathrm{Im}\,\theta_{A/J}$ has a lift to a derivation $\delta \in \mathrm{Der}(A) \cap \mathrm{Im}\,\theta_A$.

Example 6.8. Let A be the C^*-algebra from the Example 6.1 and choose any faithful unital representation $\pi : A \to \mathrm{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap \mathrm{K}(\mathcal{H}) = \{0\}$, where $\mathrm{K}(\mathcal{H})$ denotes the C^*-algebra of all compact operators on $\mathcal{H}$. To justify the existence of such π, we may may first choose a faithful representation ρ of A on a separable Hilbert space $\mathcal{H}_\rho$ (such ρ exists since A is separable), and then we may put $\mathcal{H} := \mathcal{H}_\rho^{(\infty)}$ and $\pi := \rho^{(\infty)}$, where $\rho^{(\infty)}$ denotes the corresponding amplification of ρ. Let $B := \pi(A) + \mathrm{K}(\mathcal{H})$. Obviously B is a unital, separable and primitive C^*-algebra and hence, by Theorem 4.3, we have $\mathrm{Der}(B) \cap \mathrm{Im}\,\theta_B = \mathrm{Inn}(B)$. On the other hand, since

$$B/\mathrm{K}(\mathcal{H}) \cong \pi(A)/(\pi(A) \cap \mathrm{K}(\mathcal{H})) \cong \pi(A) \cong A,$$

by Example 6.1 there exists an outer derivation $\dot{\delta} \in \mathrm{Im}\,\theta_{B/\mathrm{K}(\mathcal{H})}$. It follows that such derivation cannot be lifted to a (necessarily inner) derivation $\delta \in \mathrm{Im}\,\theta_B$.

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