ON DERIVATIONS AND ELEMENTARY OPERATORS ON C^*-ALGEBRAS

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Abstract. Let A be a unital C^*-algebra with the canonical (H) C^*-bundle $\mathfrak{A}$ over the maximal ideal space of the center of A, and let $\mathrm{E}(A)$ be the set of all elementary operators on A. We consider derivations on A which lie in the completely bounded norm closure of $\mathrm{E}(A)$, and show that such derivations are necessarily inner in the case when each fibre of $\mathfrak{A}$ is a prime C^*-algebra. We also consider separable C^*-algebras A for which $\mathfrak{A}$ is an (F) bundle. For these C^*-algebras we show that the following conditions are equivalent: (a) $\mathrm{E}(A)$ is closed in the operator norm; (b) A as a Banach module over its center is topologically finitely generated; (c) fibres of $\mathfrak{A}$ have uniformly finite dimensions, and each restriction bundle of $\mathfrak{A}$ over a set where its fibres are of constant dimension is of finite type as a vector bundle.

1. Introduction

Let A be a unital C^*-algebra. An *elementary operator* on A is a map $T : A \to A$ which can be expressed as a finite sum

$$T = \sum_{i=1}^{n} M_{a_i,b_i}$$

of two-sided multiplication operators $M_{a,b} : x \mapsto axb$ $(a, b \in A)$. The set of all elementary operators on A is denoted by $\mathrm{E}(A)$ and its operator norm closure (resp. completely bounded norm closure) is denoted by $\overline{\mathrm{E}(A)}$ (resp. $\overline{\mathrm{E}(A)}_{cb}$).

A *derivation* on A is a linear map $\delta : A \to A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Each element $a \in A$ induces the *inner derivation* δ_a given by

$$\delta_a(x) := ax - xa$$

for all $x \in A$. By $\mathrm{Der}(A)$ (resp. $\mathrm{Inn}(A)$) we denote the set of all derivations (resp. inner derivations) on A. It is well known that each derivation δ on A is completely bounded with $\|\delta\|_{cb} = \|\delta\|$.

In our previous papers [18, 19, 20] we considered variants of the following two problems:

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Problem 1.1. Characterize all unital C^*-algebras A with the property

$$\operatorname{Der}(A) \cap \overline{\overline{\mathrm{E}(A)}}_{cb} = \operatorname{Inn}(A). \tag{1.1}$$

Problem 1.2. Characterize all unital C^*-algebras A with the property

$$\overline{\overline{\mathrm{E}(A)}} = \mathrm{E}(A). \tag{1.2}$$

The motivation for considering these problems comes from understanding the operator (or completely bounded) norm closure of $\mathrm{E}(A)$. One can also consider the dual problems:

Problem 1.3. Characterize all unital C^*-algebras A with the property

$$\operatorname{Der}(A) \cap \overline{\overline{\mathrm{E}(A)}}_{cb} = \operatorname{Der}(A). \tag{1.3}$$

Problem 1.4. Characterize all unital C^*-algebras A with the property

$$\overline{\overline{\mathrm{E}(A)}} = \mathrm{IB}(A), \tag{1.4}$$

where $\mathrm{IB}(A)$ is the set of all bounded linear maps $\phi : A \to A$ which preserve closed two-sided ideals of A.

Note that these (dual) problems are already solved in a separable case. More precisely, Magajna in [24] showed that a unital separable C^*-algebra A satisfies (1.4) if and only if A is ($*$-isomorphic to) a finite direct sum of (unital separable) homogeneous C^*-algebras, which solves Problem 1.4. On the other hand, if a separable A satisfies (1.3), then $\operatorname{Der}(A)$ must be separable (since $\mathrm{E}(A)$ is separable whenever A is separable). By a result of Elliott [14, Theorem 1], $\operatorname{Der}(A)$ is separable if and only if all derivations on A are inner. Furthermore, Akemann and Pedersen in [1, Corollary 3.10] characterized the unital separable C^*-algebras admitting only inner derivations, as those C^*-algebras which are ($*$-isomorphic to) a finite direct sum of (unital separable) simple and homogeneous C^*-algebras. This solves Problem 1.3 in the separable case as well. It would be also interesting to see what happens in the inseparable case.

Returning to Problem 1.1, we proved that A satisfies (1.1) if A is prime [18, Theorem 4.3] or if A has a Hausdorff primitive spectrum [18, Theorem 5.6]. On the other hand, we exhibited an example of a C^*-algebra admitting an outer derivation lying in $\mathrm{E}(A)$ [18, Example 6.1], which makes Problem 1.1 non-trivial (and also interesting). Of course, if a C^*-algebra A is prime or if it has a Hausdorff primitive spectrum, then each Glimm ideal of A is prime, so the next natural step would be to consider this more general case. It turns out that such a class of C^*-algebras indeed satisfies (1.1):

Theorem 1.5. *Let A be a unital C^*-algebra. If every Glimm ideal of A is prime, then A satisfies* (1.1).

This result will be proved in Section 3. Furthermore, note that Theorem 1.5 is indeed a strict generalization of [18, Theorem 4.3] and [18, Theorem 5.6], since the class of C^*-algebras in which every Glimm ideal is prime includes all standard C^*-algebras (see [3, p. 90] for the definition), in particular all prime C^*-algebras, C^*-algebras with a Hausdorff primitive spectrum, quotients of AW^*-algebras (by [30, Lemma 2.8]) and local multiplier algebras (by [3, Corollary 3.5.11]). At this point, we would also like to mention that (as far as the author knows) it is still uncertain whether there exists a quotient of an AW^*-algebra which admits an outer derivation δ. If such a quotient exists, then by Theorem 1.5, δ cannot lie in $\overline{\overline{\mathrm{E}(A)}}_{cb}$.

On the other hand, in [19, Theorem 2.6] we showed that if a unital separable C^*-algebra A satisfies (1.2) then A is necessarily subhomogeneous of finite type. This means that A is subhomogeneous and the C^*-bundles corresponding to the homogeneous subquotients of A must be of finite type (see [19] for a detailed explanation). By [20, Proposition 1.1], such C^*-algebras are characterized with the following (more intrinsic) property: There exists a finite number of elements $a_1, \ldots, a_m \in A$ such that

$$\mathrm{span}\{a_1 + P, \ldots, a_m + P\} = A/P \quad \text{for all } P \in \mathrm{Prim}(A). \tag{1.5}$$

Moreover, if A satisfies (1.2) then by [20, Theorem 2.3], $\mathrm{Prim}(A)$ in (1.5) can be replaced by the larger set $\mathrm{Primal}_2(\mathrm{A})$ of 2-primal ideals of A, so (1.2) also implies

$$\mathrm{span}\{a_1 + Q, \ldots, a_m + Q\} = A/Q \quad \text{for all } Q \in \mathrm{Primal}_2(A). \tag{1.6}$$

If in addition $\mathrm{Prim}(A)$ is Hausdorff, then the conditions (1.5) and (1.6) are equivalent (since every proper 2-primal ideal of A primitive). In this (Hausdorff) case we showed that the condition (1.6)($\Leftrightarrow$ (1.5)) in fact characterizes unital separable C^*-algebras satisfying (1.2) [20, Theorem 3.9]. However, in a general case the condition (1.6) is stronger than (1.5), and the problem of whether (1.6) implies (1.2) remained open in [20].

To obtain a larger class of C^*-algebras satisfying (1.2), we shall consider the canonical (H) C^*-bundle $\mathfrak{A}$ of A over $\mathrm{Max}(Z(A))$ (the maximal ideal space of the center of A). If $\mathrm{Prim}(A)$ is Hausdorff, then the map $\mathrm{Max}(Z(A)) \to \mathrm{Prim}(A)$, given by $x \mapsto xA$ ($x \in \mathrm{Max}(Z(A))$), is a homeomorphism, and since the norm functions $P \mapsto \|a + P\|$ ($a \in A$) are continuous on $\mathrm{Prim}(A)$ (by [9, II.6.5.8]), $\mathfrak{A}$ is an (F) bundle. Then (1.5) is equivalent to the fact that all restriction bundles of $\mathfrak{A}$ over a set where its fibres are pairwise $*$-isomorphic are of finite type as vector bundles (and as C^*-bundles, by [27, Proposition 2.9], since all fibres of $\mathfrak{A}$ are simple). It turns out that the continuity of the bundle $\mathfrak{A}$ is the only information needed to prove (1.2). More precisely, we shall obtain the following result.

Theorem 1.6. *Let A be a unital separable C^*-algebra. If the canonical C^*-bundle $\mathfrak{A}$ of A over the space $X = \mathrm{Max}(Z(A))$ is an (F) bundle, then the following conditions are equivalent:*

(i) $\mathrm{E}(A)$ *is closed in the operator norm.*
(ii) *A satisfies* (1.6).
(iii) *Fibres $\mathcal{A}_x$ of $\mathfrak{A}$ have uniformly finite dimensions, and each restriction bundle of $\mathfrak{A}$ over a set where $\dim \mathcal{A}_x$ is constant is of finite type as a vector bundle.*
(iv) *A as a Banach module over $Z(A)$ is topologically finitely generated.*

The condition (iv) of Theorem 1.6 means that there exists a finite number of elements in A whose $Z(A)$-linear span is norm dense in A.

At the end of this introductory, we would like to mention that the main problem which occurs in proving Theorem 1.6 is the fact that we do not know whether (1.2) implies that each restriction bundle of $\mathfrak{A}$ over a set where fibres of $\mathfrak{A}$ are pairwise $*$-isomorphic is of finite type as a C^*-bundle (some fibres $\mathcal{A}_x$ are no longer simple, unless $\mathrm{Prim}(A)$ is Hausdorff, so [27, Proposition 2.9] cannot be applied). Our technique of proving this theorem is essentially based on the existence of a $C_0(X_i)$-valued inner product $\langle\cdot,\cdot\rangle_i$ on each subquotient $\Gamma_0(\mathfrak{A}|_{X_i})$, where $X_i := \{x \in X : \dim \mathcal{A}_x = i\}$, whose induced norm $a \mapsto \|\langle a,a\rangle_i\|^{\frac{1}{2}}$ is equivalent to the C^*-norm on $\Gamma_0(\mathfrak{A}|_{X_i})$. This will enable us to bypass the mentioned difficulty by using the methods from [21] developed for (F) Hilbert bundles.

2. Preliminaries

Through this paper A will denote a C^*-algebra, and $Z(A)$ its center. By $\hat{A}$ and $\mathrm{Prim}(A)$ we respectively denote the *spectrum* of A (i.e. the set of all classes of irreducible representations of A) and the *primitive spectrum* of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology. If all irreducible representations of A have the same finite dimension n, we say that A is *(n-)homogeneous*, and if

$$n := \sup\{\dim \pi \ : \ [\pi] \in \hat{A}\} < \infty,$$

we say that A is *(n-)subhomogeneous*.

Let $\mathrm{Id}(A)$ be the set of all ideals of A (by an ideal we always mean a closed two-sided ideal). We equip $\mathrm{Id}(A)$ with the *strong topology* τ_s which is by definition the weakest topology making the functions $I \mapsto \|a+I\|$ $(I \in \mathrm{Id}(A))$ continuous. Under this topology $\mathrm{Id}(A)$ becomes a compact Hausdorff space (see [4]).

An ideal $Q \in \mathrm{Id}(A)$ is said to be *n-primal* ($n \in \mathbb{N}$, $n \geq 2$) if whenever $J_1,\dots,J_n$ are n ideals of A with $J_1 \cdots J_n = \{0\}$, then $J_i \subseteq Q$ for at least one value of i. If Q is n-primal for all n, then Q is said to be *primal.* By $\mathrm{Primal}_n(A)$ (resp. $\mathrm{Primal}(A)$) we denote the set of all n-primal (resp. primal) ideals of A.

We now recall some facts about the complete regularization of $\mathrm{Prim}(A)$ (see [5] for further details). For $P,Q \in \mathrm{Prim}(A)$ let

$$P \approx Q \quad \text{if} \quad f(P) = f(Q) \quad \text{for all } f \in C_b(\mathrm{Prim}(A)).$$

Then $\approx$ is an equivalence relation on $\mathrm{Prim}(A)$ and the equivalence classes are closed subsets of $\mathrm{Prim}(A)$. Hence, there is one-to-one correspondence between the quotient set $\mathrm{Prim}(A)/\approx$ and a set of ideals of A given by $[P] \mapsto \bigcap [P]$, where $[P]$ denotes the equivalence class of $P \in \mathrm{Prim}(A)$. The ideals obtained in this way are known as *Glimm ideals*, and by $\mathrm{Glimm}(A)$ we denote the set of all Glimm ideals of A. It is easy to see that every proper 2-primal ideal of A contains a unique Glimm ideal of A (see the proof of [5, Lemma 2.2]). We equip $\mathrm{Glimm}(A)$ with the quotient topology τ_q. Then $\mathrm{Glimm}(A)$ becomes a Hausdorff space, and the quotient map

$$\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$$

is known as the *complete regularization map.* If A is unital, then it follows from the Dauns-Hoffman theorem [28, Theorem A.34] that

$$P \approx Q \quad \Leftrightarrow \quad P \cap Z(A) = Q \cap Z(A).$$

Moreover, in this case the map $\zeta_A : \mathrm{Glimm}(A) \to \mathrm{Max}(Z(A))$, $G \mapsto G \cap Z(A)$ is a homeomorphism, so when A is unital, $\mathrm{Glimm}(A)$ is a compact Hausdorff space, and we may identify $C(\mathrm{Glimm}(A))$ with $Z(A)$. The following facts are well known (see [5]).

Proposition 2.1. *Let A be a C^*-algebra.*

(i) *For all $a \in A$, $\sup\{\|a+G\| \ : \ G \in \mathrm{Glimm}(A)\} = \|a\|$.*

(ii) *The function* $G \mapsto \|a+G\|$ *is upper semicontinuous on* $\mathrm{Glimm}(A)$ *for each* $a \in A$.
(iii) *The function* $G \mapsto \|a+G\|$ *is continuous on* $\mathrm{Glimm}(A)$ *for each* $a \in A$ *if and only if* ϕ_A *is an open map.*

Since the topology τ_q on $\mathrm{Glimm}(A)$ is weaker than the relative τ_s-topology, as a direct consequence of Proposition 2.1 we obtain:

Corollary 2.2. *If* A *is a* C^**-algebra, then* ϕ_A *is open if and only if the topology* τ_q *on* $\mathrm{Glimm}(A)$ *coincides with the relative* τ_s*-topology.*

By a *Hilbert* A*-module* we mean a left A-module V, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ which is A-linear in the first and conjugate linear in the second variable, such that V is a Banach space with the norm $\|v\| := \|\langle v, v\rangle\|^{\frac{1}{2}}$. The basic theory of Hilbert C^*-modules can be found in [23, 28, 34].

Following [12], by an *(H)* C^**-bundle* ((H) stands for Hofmann) we mean a triple $\mathfrak{A} := (p, \mathcal{A}, X)$ where $\mathcal{A}$ and X are topological spaces with a continuous open surjection $p : \mathcal{A} \to X$, together with operations and norms making each *fibre* $\mathcal{A}_x := p^{-1}(x)$ into a C^*-algebra, such that the following conditions are satisfied:

(A1) The maps $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}$ given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous ($\mathcal{A} \times_X \mathcal{A}$ denotes the Whitney sum).
(A2) The map $\mathcal{A} \to \mathbb{R}$, defined by norm on each fibre, is upper semicontinuous.
(A3) If $x \in X$ and if (a_α) is a net in $\mathcal{A}$ such that $\|a_\alpha\| \to 0$ and $p(a_\alpha) \to x$ in X, then $a_\alpha \to 0_x$ in $\mathcal{A}$ (0_x denotes the zero-element of $\mathcal{A}_x$).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that $\mathfrak{A}$ is an *(F)* C^**-bundle* ((F) stands for Fell). If $\mathfrak{A} = (p, \mathcal{A}, X)$ is an (H) C^*-bundle and $Y \subseteq X$ then we denote by

$$\mathfrak{A}|_Y := (p|_{p^{-1}(Y)}, p^{-1}(Y), Y)$$

the *restriction bundle* of $\mathfrak{A}$ to Y. We say that two (H) C^*-bundles $\mathfrak{A} = (p, \mathcal{A}, X)$ and $\mathfrak{A}' = (p', \mathcal{A}', X)$ are *isomorphic* if there exists a homeomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$, such that $\Phi(\mathcal{A}_x) = \mathcal{A}'_x$ and $\Phi|_{\mathcal{A}_x} : \mathcal{A}_x \to \mathcal{A}'_x$ defines a $*$-isomorphism from $\mathcal{A}_x$ onto $\mathcal{A}'_x$. In this case we write $\Phi : \mathfrak{A} \cong \mathfrak{A}'$. By the *product bundle over* X *with fibre* A we mean

$$\epsilon(X, A) := (p_1, X \times A, X),$$

where p_1 is a projection on the first coordinate. An (H) C^*-bundle $\mathfrak{A}$ over X is said to be *trivial* if there exists a C^*-algebra A such that $\mathfrak{A} \cong \epsilon(X, A)$. If there exists a C^*-algebra A and an open cover $\{U_\alpha\}$ of X such that for each α we have $\mathfrak{A}|_{U_\alpha} \cong \epsilon(U_\alpha, A)$, we say that $\mathfrak{A}$ is *locally trivial.* If in addition X admits a finite open cover over which $\mathfrak{A}$ is locally trivial, we say that $\mathfrak{A}$ is of *finite type (as a* C^**-bundle).* Obviously, every locally trivial (H) C^*-bundle is automatically an (F) C^*-bundle. If all fibres of $\mathfrak{A}$ are finite dimensional and pairwise $*$-isomorphic, then $\mathfrak{A}$ is locally trivial by [16, Theorem 3.1]. In this case we can also consider $\mathfrak{A}$ as a vector bundle, by forgetting the additional structure. If the underlying vector bundle of $\mathfrak{A}$ is of finite type, then we say that $\mathfrak{A}$ is *of finite type as a vector bundle.* Note that in the case when $\mathfrak{A}$ is of finite type as vector bundle and all fibres of $\mathfrak{A}$ are $*$-isomorphic to the matrix algebra $\mathrm{M}_n(\mathbb{C})$, then $\mathfrak{A}$ is also of finite type as a C^*-bundle, by [27, Proposition 2.9]. It would be interesting to see an example of an (F) C^*-bundle $\mathfrak{A}$ which is of finite type as a vector bundle, but is not of finite type as a C^*-bundle.

By a *section* of an (H) C^*-bundle $\mathfrak{A} = (p, \mathcal{A}, X)$ we mean a map $s : X \to \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. The set of all continuous sections of $\mathfrak{A}$ is denoted by $\Gamma(\mathfrak{A})$. Then $\Gamma(\mathfrak{A})$ is a $*$-algebra and also a $C(X)$-module, with respect to the natural pointwise operations. If in addition X is locally compact and Hausdorff, by $\Gamma_0(\mathfrak{A})$ we denote the set of all $s \in \Gamma(\mathfrak{A})$ which vanish at infinity (i.e. for which the set $\{x \in X \ : \ \|s(x)\| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$). Then $\Gamma_0(\mathfrak{A})$ becomes a C^*-algebra with respect to the sup-norm.

Given a unital C^*-algebra A, one can construct a canonical (H) C^*-bundle $\mathfrak{A}$ over $X := \mathrm{Max}(Z(A))$ such that $A \cong \Gamma(\mathfrak{A})$, as follows.

For $x \in X$ let $G_x := xA$. Then G_x is a Glimm ideal of A (which is indeed closed by the Hewitt-Cohen factorization theorem [10, Theorem A.6.2]). The quotient $\mathcal{A}_x := A/G_x$ is called *the fibre of A over x*. If $a \in A$, then we write $a(x)$ for the canonical image of a in $\mathcal{A}_x$. Set

$$\mathcal{A} := \bigsqcup_{x \in X} \mathcal{A}_x,$$

and let $p : \mathcal{A} \to X$ be the canonical map. For $a \in A$ we define the map $\hat{a} : X \to \mathcal{A}$, $\hat{a}(x) := a(x)$, and set $\Omega := \{\hat{a} \ : \ a \in A\}$. By Fell's theorem [35, Theorem C.25] there exists a unique topology on $\mathcal{A}$ making $\mathfrak{A} := (p, \mathcal{A}, X)$ into an (H) C^*-bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee's theorem [35, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the map $\Gamma : A \to \Gamma(\mathfrak{A})$, given by $\Gamma : a \mapsto \hat{a}$, becomes a $C(X)$-linear $*$-isomorphism of A onto $\Gamma(\mathfrak{A})$ (the $C(X)$-action on A is defined by $\varphi a := \mathcal{G}^{-1}(\varphi)a$, where $\mathcal{G} : Z(A) \to C(X)$ is the Gelfand transform). Furthermore, $\mathfrak{A}$ is an (F) bundle if and only if ϕ_A is an open map.

At the and of this section, let us briefly recall some facts about the canonical contraction θ_A from the Haagerup tensor product $A \otimes_h A$ into the set $\mathrm{CB}(A)$ of all completely bounded maps on A, where A is a unital C^*-algebra. On elementary tensors, θ_A is given by

$$\theta_A(a \otimes b) := M_{a,b}.$$

It is easy to see that θ_A is contractive, and Mathieu showed that θ_A is isometric if and only if A is prime (see [3, Proposition 5.4.11]). If A is not prime, one considers the central Haagerup tensor product $A \otimes_{Z,h} A$ and the induced contraction $\theta_A^Z : A \otimes_{Z,h} A \to \mathrm{CB}(A)$ (see [31]). The problem of when θ_A^Z is isometric has been recently completely solved by Archbold, Somerset and Timoney in [32, Theorem 4] and [8, Theorem 7] (see also [7]); θ_A^Z is isometric if and only if each Glimm ideal of A is primal. As an easy consequence of this result, we obtain:

Proposition 2.3. *If each Glimm ideal of a unital C^*-algebra A is primal, then $\overline{\mathrm{E}(A)}_{cb} = \mathrm{Im}\,\theta_A$, where $\mathrm{Im}\,\theta_A$ denotes the image of θ_A .*

3. Derivations on C^*-algebras in which every Glimm ideal is prime

We start with the proof of Theorem 1.5. First recall, if A is a C^*-algebra and $I, J \in \mathrm{Id}(A)$ with the associated quotient maps $q_I : A \to A/I$ and $q_J : A \to A/J$, then by [2, Corollary 2.6] the induced map $q_I \otimes q_J : A \otimes_h A \to (A/I) \otimes_h (A/J)$ is also a quotient map, and

$$\ker(q_I \otimes q_J) = I \otimes_h A + A \otimes_h J,$$

so that $(A \otimes_h A)/(I \otimes_h A + A \otimes_h J)$ is isometrically isomorphic to $(A/I) \otimes_h (A/J)$.

The next fact can be deduced from Proposition 2.1 (ii) and [6, Lemma 3.1] (see also [6, Remark 3.2]).

Lemma 3.1. *If A is a C^*-algebra, then the map*

$$G \mapsto \|(q_G \otimes q_G)(t)\|_h = \|t + (G \otimes_h A + A \otimes_h G)\|$$

is upper semicontinuous on $\mathrm{Glimm}(A)$ *for each $t \in A \otimes_h A$.*

Proof of Theorem 1.5. Let $\delta \in \mathrm{Der}(A) \cap \overline{\overline{\mathrm{E}(A)}}_{cb}$. Since each Glimm ideal of A is prime (hence primal), by Proposition 2.3, there exists $t \in A \otimes_h A$ such that $\delta = \theta_A(t)$. For $G \in \mathrm{Glimm}(A)$ let δ_G be the induced derivation on A/G, $\delta_G(x+G) = \delta(x) + G$. By [18, Remark 5.4] we have

$$\delta_G = \theta_{A/G}((q_G \otimes q_G)(t))$$

Since every Glimm quotient A/G is a prime C^*-algebra, by [3, Proposition 5.4.11], $\theta_{A/G}$ is isometric. Hence,

$$\|\delta_G\| = \|\delta_G\|_{cb} = \|\theta_{A/G}((q_G \otimes q_G)(t))\|_{cb} = \|(q_G \otimes q_G)(t)\|_h,$$

for all $G \in \mathrm{Glimm}(A)$. Let us fix $G_0 \in \mathrm{Glimm}(A)$. Since A/G_0 is a prime C^*-algebra, by [18, Theorem 4.3] δ_{G_0} is inner in A/G_0, and choose $a \in A$ such that $\delta_G = (\delta_a)_G$. Let $\varepsilon > 0$ be given. By Lemma 3.1, the function

$$G \mapsto \|\delta_G\| = \|(q_G \otimes q_G)(t)\|_h$$

is upper semicontinuous on $\mathrm{Glimm}(A)$, so there exists an open neighborhood U of G_0 in $\mathrm{Glimm}(A)$ such that

$$\|\delta_G - (\delta_a)_G\| = \|(\delta - \delta_a)_G\| < \varepsilon$$

for all $G \in U$. Since $\mathrm{Glimm}(A)$ is compact, we can find a finite open cover $\{U_j\}_{1 \le j \le m}$ of $\mathrm{Glimm}(A)$ and elements $a_1, \ldots, a_m \in A$ such that

$$\|\delta_G - (\delta_{a_j})_G\| < \varepsilon$$

for all $G \in U_j$. Choose a partition of unity $\{f_j\}_{1 \le j \le m}$ subordinated to the cover $\{U_j\}_{1 \le j \le m}$, and define $z_j := \Psi_A^{-1}(f_j)$, where $\Psi_A : Z(A) \to C(\mathrm{Prim}(A)) = C(\mathrm{Glimm}(A))$ is the Dauns-Hofmann isomorphism (see [28, Theorem A.34]). If

$$a := \sum_{j=1}^m z_j a_j \in A,$$

then for $G \in \mathrm{Glimm}(A)$ and $x \in A$, $\|x\| \le 1$ we have

$$\begin{aligned}
\|(\delta - \delta_a)_G(x+G)\| &= \|(\delta(x) - \delta_a(x)) + G\| \\
&= \left\| \sum_{j=1}^m [z_j(\delta(x) - \delta_{a_j}(x)) + G] \right\| \\
&= \left\| \sum_{j=1}^m f_j(G)(\delta_G - (\delta_{a_j})_G)(x+G) \right\| \\
&\le \sum_{j=1}^m f_j(G)\|\delta_G - (\delta_{a_j})_G\| \\
&< \varepsilon.
\end{aligned}$$

It follows that $\|(\delta-\delta_a)_G\| \le \varepsilon$ for all $G \in \mathrm{Glimm}(A)$. Hence, by Proposition 2.1 (i),

$$\|\delta-\delta_a\| = \sup\{\|(\delta-\delta_a)_G\| \ : \ G \in \mathrm{Glimm}(A)\} \le \varepsilon.$$

This means that δ lies in the operator norm closure of $\mathrm{Inn}(A)$. Since all Glimm ideals of A are prime, they are also primal, so by [30, Theorem 2.7] (or [32, Corollary 4.6]), $\mathrm{Inn}(A)$ is closed in the operator norm. Hence, $\delta \in \mathrm{Inn}(A)$. □

In [26] Pedersen proved that every derivation on a separable C^*-algebra A becomes inner in its local multiplier algebra $M_{\mathrm{loc}}(A)$ (see [3] for definition and properties of $M_{\mathrm{loc}}(A)$). However, in the inseparable case the problem is still open. On the other hand, every Glimm ideal of $M_{\mathrm{loc}}(A)$ (where A is a general C^*-algebra) is prime [3, Corollary 3.5.10]. Therefore, as a direct consequence of Theorem 1.5 we obtain the following result.

Corollary 3.2. *Let A be a C^*-algebra, let δ be a derivation on A, and let $\tilde{\delta}$ denote the unique extension of δ to a derivation on $M_{\mathrm{loc}}(A)$. The following conditions are equivalent:*

(i) *$\tilde{\delta}$ is inner (in $M_{\mathrm{loc}}(A)$).*
(ii) $\tilde{\delta} \in \overline{\overline{\mathrm{E}(M_{\mathrm{loc}}(A))}}_{cb}$.

In particulary, every derivation $\delta \in \mathrm{Der}(A) \cap \overline{\overline{\mathrm{E}(A)}}_{cb}$ is implemented by a local multiplier (if A is non-unital, we can assume that the coefficients of elementary operators on A lie in the multiplier algebra $M(A)$ of A).

Remark 3.3. In [18] we conjectured that the class of all unital C^*-algebras A in which every Glimm ideal is primal also satisfies (1.1). Unfortunately, there are two main obstacles why the proof of Theorem 1.5 cannot be generalized for such class of C^*-algebras. The first one is that we do not know whether each Glimm quotient A/G admits only inner derivations lying in $\mathrm{Im}\,\theta_{A/G}$ (see [5, Proposition 3.6] and [18, Example 6.1]). The second difficulty is that for $\delta \in \mathrm{Der}(A) \cap \overline{\overline{\mathrm{E}(A)}}_{cb}$, the function $G \mapsto \|\delta_G\|$ does not have to be upper semicontinuous on $\mathrm{Glimm}(A)$ (even if δ is inner), as the next example shows.

Example 3.4. Let $\beta\mathbb{N}$ denote the Stone-Čech compactification of $\mathbb{N}$, and choose an arbitrary point $x_0 \in \beta\mathbb{N} \setminus \mathbb{N}$. We define A to be a C^*-algebra consisting of all functions $a \in C(\beta\mathbb{N}, \mathrm{M}_2(\mathbb{C}))$ with the property that $a(x_0)$ is a diagonal matrix. Note that A is unital, $\mathrm{Glimm}(A)$ is canonically homeomorphic to $\beta\mathbb{N}$ (we denote this correspondence by $x \leftrightarrow G(x)$), and each Glimm ideal of A is primal (the Glimm quotient $A/G(x)$ is isomorphic to $\mathrm{M}_2(\mathbb{C})$ if $x \neq x_0$, and $\mathbb{C}\oplus\mathbb{C}$ if $x = x_0$). Therefore by [29, Theorem 2.8], A admits only inner derivations (so, in particulary, A satisfies (1.1)). On the other hand, let a be an element of A defined by $a(x) := e_{1,1}$ for all $x \in \beta\mathbb{N}$ (where $e_{1,1}$ is the matrix unit which has a non-zero entry 1 at $(1,1)$-position), and let $\delta := \delta_a$. One can easily check that $\|\delta_{G(x)}\| = 1$ if $x \neq x_0$ and $\|\delta_{G(x_0)}\| = 0$. Therefore, the function $G \mapsto \|\delta_G\|$ is not upper semicontinuous on $\mathrm{Glimm}(A)$.

4. Elementary operators and (F) C^*-bundles

In order to prove Theorem 1.6, we shall first need some auxiliary results.

Proposition 4.1. *Let A be an n-subhomogeneous C^*-algebra. If the complete regularization map $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is open, then every Glimm ideal of A is primal. In particular,*

$$\sup\{|\mathrm{Prim}(A/G)| \ : \ G \in \mathrm{Glimm}(A)\} \leq n, \tag{4.1}$$

where $|\mathrm{Prim}(A/G)|$ is the cardinality of $\mathrm{Prim}(A/G)$.

Proof. Suppose that the degree of subhomogenity of A equals n and let J be the n-homogeneous ideal of A (i.e. J is the intersection of the kernels of all irreducible representations of A whose dimension is at most $n-1$). We claim that ϕ_A is invariant under $\mathrm{Prim}(J)$ (i.e. $\phi_A(P) = P$ for all $P \in \mathrm{Prim}(J)$). Indeed let $P \in \mathrm{Prim}(J)$ and $Q \in \mathrm{Prim}(A)$. First suppose that $Q \in \mathrm{Prim}(J)$ and that $P \neq Q$. Since J, as a C^*-algebra, is n-homogeneous, J is central (in a sense of [18, Definition 3.10]), there exists a function $f \in C_0(\mathrm{Prim}(J)) \subseteq C_b(\mathrm{Prim}(A))$ such that $f(P) = 1$ and $f(Q) = 0$. In particular, $f(P) \neq f(Q)$, so $P \not\approx Q$ in this case. If $Q \in \mathrm{Prim}(A/J)$, then for any function $f \in C_0(\mathrm{Prim}(J)) \subseteq C_b(\mathrm{Prim}(A))$ we have $f(Q) = 0$, so $P \not\approx Q$ in this case as well. Since ϕ_A is open, the τ_q-topology on $\mathrm{Glimm}(A)$ coincides with the relative τ_s-topology, by Corollary 2.2. Hence, by [4, Corollary 4.3], those Glimm ideals which belong to the τ_s-closure of $\mathrm{Prim}(J)$ are primal. Let U be the complement in $\mathrm{Prim}(A)$ of the closure of $\mathrm{Prim}(J)$. Then U is open in $\mathrm{Prim}(A)$. Suppose that U is non-empty and let K be the ideal of A such that $\mathrm{Prim}(K) = U$. Then K is k-subhomogeneous for some $k < n$ and let I be the k-homogeneous ideal of K. Using the same arguments as before, we conclude that $\phi_A(\mathrm{Prim}(I)) = \mathrm{Prim}(I)$ is a new family of primitive Glimm ideals of A, and that each Glimm ideal which belongs to the closure of $\mathrm{Prim}(I)$ in $\mathrm{Glimm}(A)$ is primal. Proceeding by induction, we conclude that every Glimm ideal of A is primal.

To prove (4.1), let $G \in \mathrm{Glimm}(A)$. Since G is primal, by [4, Proposition 3.2] there is a net in $\mathrm{Prim}(A)$ which converges to every point of $\mathrm{Prim}(A/G)$. Since A is liminal, by [13, Theorem 4.3.7] we can identify $\mathrm{Prim}(A)$ with the spectrum $\hat{A}$ of A, and thus by [15, Corollary 1, p. 388], $|\mathrm{Prim}(A/G)| \leq n$. □

Remark 4.2. (i). Note that the proof given above shows that for a subhomogeneous C^*-algebra A, $\mathrm{Glimm}(A)$ contains a dense open subset of primitive ideals.

(ii). In [5], Archbold and Somerset introduced the class of *quasi-standard* C^*-algebras. By [5, Theorem 3.3], quasi-standard C^*-algebras are precisely those C^*-algebras A satisfying the following two conditions:

(a) the complete regularization map ϕ_A is open,
(b) each Glimm ideal of A is primal.

Note that Proposition 4.1 implies that the condition (b) is superfluous in a subhomogeneous case. Hence, a subhomogeneous C^*-algebra A is quasi-standard if and only if ϕ_A is open. Furthermore, if A is unital, note that in this case (4.1) implies that the dimensions of fibres of the canonical (F) C^*-bundle $\mathfrak{A}$ of A over $\mathrm{Max}(Z(A))$ are automatically finite and uniformly bounded.

On the other hand, if A is a subhomogeneous C^*-algebra, one may wonder if the conditions (a) and (b) from Remark 4.2 are in fact equivalent. D. Somerset informed us that this is not true in general, as the next example shows.

Example 4.3. Let B be a C^*-subalgebra of $C([0,1],\mathrm{M}_2(\mathbb{C}))$ consisting of all elements $b \in C([0,1],\mathrm{M}_2(\mathbb{C}))$ such that

$$b(1) = \begin{bmatrix} \lambda(b) & 0 \\ 0 & \mu(b) \end{bmatrix}$$

for some $\lambda(b), \mu(b) \in \mathbb{C}$. If $C := C([1,2])$, set $D := B \oplus C$ and define

$$A := \{(b,\varphi) \in D \ : \ \lambda(b) = \varphi(1)\}.$$

Then A is a C^*-algebra which is obviously 2-subhomogeneous. Furthermore, it is easy to see that $\mathrm{Glimm}(A)$ is canonically homeomorphic to $[0,2]$ and that every Glimm ideal of A is primal. On the other hand, let $a = (b,\varphi) \in A$, where $b(x) := e_{2,2}$ (the matrix unit which has a non-zero entry 1 at $(2,2)$-position) for all $x \in [0,1]$ and $\varphi := 0$. If $G(x)$ denotes the Glimm ideal of A corresponding to $x \in [0,2]$, we have

$$\|a + G(1)\| = 1 \quad \text{and} \quad \lim_{x \to 1+} \|a + G(x)\| = 0,$$

so the norm function $G \mapsto \|a+G\|$ is not continuous on $\mathrm{Glimm}(A)$. By Proposition 2.1 (iii), ϕ_A is not open.

Suppose that V is a non-degenerate Banach $C_0(X)$-module, where X is a locally compact Hausdorff space. In our previous paper [21] we introduced a notion of a $C_0(X)$-projective rank, denoted by $\mathrm{rank}_X^{\pi}(V)$, as the smallest natural number N (if such exists) with the following property: For every Banach $C_0(X)$-module W, each tensor t in the $C_0(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C_0(X)} W$ can be written in a form

$$t = \sum_{i=1}^{n} v_i \otimes_X w_i$$

for some $v_i \in V$ and $w_i \in W$, where $n \le N$ (see [21] for details). If such N does not exist, we define $\mathrm{rank}_X^{\pi}(V) := \infty$. The next fact is useful for proving that V is of finite $C_0(X)$-projective rank (see the proof of [21, Proposition 3.4]).

Proposition 4.4. *Let V be a non-degenerate Banach $C_0(X)$-module, where X is a locally compact Hausdorff space. Let us say that V satisfies the condition* $(\mathbf{P})$ *if there exists $N \in \mathbb{N}$ such that for every sequence $(a_i) \in \ell^1(V)$ there exist $n \le N$, elements $v_1, \ldots, v_n \in V$ and sequences $(\varphi_{i,1})_i, \ldots, (\varphi_{i,n})_i \in \ell^1(C_0(X))$ such that*

$$a_i = \sum_{j=1}^{n} \varphi_{i,j} v_j \tag{4.2}$$

for all $i \in \mathbb{N}$. If V satisfies $(\mathbf{P})$*, then* $\mathrm{rank}_X^{\pi}(V) \le N$.

In the same paper [21] we showed that if $\mathfrak{H} = (p, \mathcal{H}, X)$ is an (F) Hilbert bundle over a compact metrizable space X, then $V := \Gamma(\mathfrak{H})$ satisfies $(\mathbf{P})$ if and only if fibres $\mathcal{H}_x$ of $\mathfrak{H}$ have uniformly finite dimensions, and each restriction bundle of $\mathfrak{H}$ over a set where $\dim \mathcal{H}_x$ is constant is of finite type (as a vector bundle). Now we shall prove the same result for a similar class of C^*-algebras.

Lemma 4.5. *Let B be a unital C^*-algebra with the unit 1_B. Then B is finite dimensional if and only if there exists a state ω on B with a constant $0 < C \le 1$ such that*

$$\omega(b^*b)1_B \ge C \cdot b^*b \quad \text{for all } b \in B. \tag{4.3}$$

Moreover, if B is finite dimensional, then every faithful tracial state ω on B satisfies (4.3) *for some constant* $0 < C \leq 1$.

Proof. Suppose that B admits a state ω satisfying (4.3). Obviously, ω is faithful and

$$\langle b_1, b_2 \rangle_\omega := \omega(b_1 b_2^*) \quad (b_1, b_2 \in B)$$

defines a (definite) complex-valued inner product on B. Moreover, (4.3) implies that its norm

$$\|b\|_\omega := \langle b, b \rangle_\omega^{\frac{1}{2}} = \omega(bb^*)^{\frac{1}{2}}$$

is equivalent to the C^*-norm on B, so that $(B, \langle \cdot, \cdot \rangle_\omega)$ is a (complete) Hilbert space. In particular, this implies that B (as a C^*-algebra) is reflexive. Hence, by [33, p. 54, Exercise 2], B must be finite dimensional.

To prove the converse, first suppose that B is a full matrix algebra $\mathrm{M}_n(\mathbb{C})$. By [25, Example 6.2.1], there exists a unique faithful tracial state ω on B, which is given by

$$\omega(b) = \frac{1}{n} \operatorname{tr}(b),$$

where $\operatorname{tr}(\cdot)$ is a standard trace on $\mathrm{M}_n(\mathbb{C})$. If $b \in B$, let $u \in \mathrm{M}_n(\mathbb{C})$ be a unitary matrix such that $ub^*bu^* = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ (where $\lambda_i \geq 0$ are eigenvalues of b^*b). Then

$$\begin{aligned} \omega(b^*b)1_B &= \frac{1}{n} \operatorname{tr}(b^*b)1_B = \frac{1}{n} \operatorname{tr}(ub^*bu^*)1_B = \frac{1}{n} \left(\sum_{i=1}^n \lambda_i \right) 1_B \\ &\geq \frac{1}{n} \operatorname{diag}(\lambda_1, \ldots, \lambda_n). \end{aligned}$$

Hence,

$$\omega(b^*b)1_B = u^*(\omega(b^*b)1_B)u \geq \frac{1}{n} u^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n)u = \frac{1}{n} b^*b,$$

so we may take $C = \frac{1}{n}$ in this case.

Now suppose that B is an arbitrary finite dimensional C^*-algebra. By [33, Theorem I.11.9], there is a finite number of central pairwise orthogonal projections $p_1, \ldots, p_m \in Z(B)$ with $\sum_{i=1}^m p_i = 1_B$, such that

$$B = p_1 B \oplus \cdots \oplus p_m B, \tag{4.4}$$

and each $p_i B$ is $*$-isomorphic to a full matrix algebra $\mathrm{M}_{n_i}(\mathbb{C})$. Choose an arbitrary faithful tracial state ω on B and let $k_i := \omega(p_i) > 0$. Then

$$\omega_i : p_i b \mapsto \frac{1}{k_i} \omega(p_i b) \tag{4.5}$$

defines a faithful tracial state on $p_i B \cong \mathrm{M}_{n_i}(\mathbb{C})$, so by the first part of the proof

$$\omega_i(p_i b^* b)p_i \geq \frac{1}{n_i} p_i b^* b \tag{4.6}$$

for all $b \in B$. Hence, by (4.5) and (4.6) for $b \in B$ we have

$$\begin{aligned}\omega(b^*b)1_B &= \sum_{i=1}^m \omega(p_i b^* b)1_B \geq \sum_{i=1}^m \omega(p_i b^* b)p_i \\ &= \sum_{i=1}^m k_i \omega_i(p_i b^* b)p_i \geq \sum_{i=1}^m \frac{k_i}{n_i} p_i b^* b \\ &\geq C \cdot b^* b,\end{aligned}$$

where $C := \min\{\frac{k_i}{n_i} : 1 \leq i \leq m\}$. This completes the proof. $\square$

Lemma 4.6. *Let $\mathfrak{A}$ be an (F) C^*-bundle over a locally compact Hausdorff space X, whose all fibres have the same finite dimension n, and set $A := \Gamma_0(\mathfrak{A})$.*

(i) *There exists a finite number of clopen pairwise disjoint subsets $\{U_j\}$ of X which cover X such that all fibres of $\mathfrak{A}|_{U_j}$ are pairwise $*$-isomorphic. Moreover, each restriction bundle $\mathfrak{A}|_{U_j}$ is locally trivial as a C^*-bundle.*

(ii) *There exists a $C_0(X)$-valued inner product $\langle \cdot, \cdot \rangle$ on A, such that $(A, \langle \cdot, \cdot \rangle)$ becomes a Hilbert $C_0(X)$-module, whose norm $a \mapsto \|\langle a, a\rangle\|^{\frac{1}{2}}$ is equivalent to the C^*-norm on A.*

(iii) *$\mathfrak{A}$ is of finite type as a vector bundle if and only if A satisfies the condition* (**P**) *of Proposition* 4.4.

Proof. (i). Every finite dimensional C^*-algebra is ($*$-isomorphic to) a finite direct sum of full matrix algebras. In particular, two finite dimensional C^*-algebras are $*$-isomorphic if and only if they have the same matrix decomposition (up to a permutation). The claim now follows from [16, Theorem 3.1].

(ii). Using (i), it is sufficient to prove the assertion in the case when all fibres of $\mathfrak{A}$ are $*$-isomorphic to a fixed finite dimensional C^*-algebra B. Let us decompose B as in (4.4). On each $p_i B$ choose a unique faithful tracial state ω_i, and define a state ω on B by

$$\omega(b) := \left(\sum_{i=1}^m \dim \pi_i\right)^{-1} \sum_{i=1}^m \dim \pi_i \cdot \omega_i(p_i b),$$

where π_i denotes the irreducible representation $\pi_i : b \mapsto p_i b$ (if $p_i B \cong \mathrm{M}_{n_i}(\mathbb{C})$, then $\dim \pi_i = n_i$). Obviously, ω is a faithful tracial state on B. Moreover, it is easy to see that ω is invariant under the group $\mathrm{Aut}(B)$ of all $*$-automorphisms of B, that is

$$\omega(\Phi(b)) = \omega(b) \quad \text{for all } b \in B \text{ and } \Phi \in \mathrm{Aut}(B). \tag{4.7}$$

For example, if $B = \mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$, then each $\Phi \in \mathrm{Aut}(B)$ is in the form

$$\Phi(b_1, b_2, b_3) = (u_1^* b_1 u_1, u_2^* b_2 u_2, u_3^* b_3 u_3) \text{ or } \Phi(b_1, b_2, b_3) = (u_1^* b_1 u_1, u_2^* b_3 u_2, u_3^* b_2 u_3),$$

for some unitary matrices $u_1 \in \mathrm{M}_3(\mathbb{C})$, $u_2, u_3 \in \mathrm{M}_2(\mathbb{C})$, so in both cases we see that ω satisfies (4.7). Since $\mathfrak{A}$ is locally trivial (by [16, Theorem 3.1]), there exists an open cover $\{U_\alpha\}$ of X such that $\Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong \epsilon(U_\alpha, B)$, where Φ_α is an isomorphism of C^*-bundles. Let $a \in A$. For $x \in X$ choose an index α such that $x \in U_\alpha$, and define

$$E(a)(x) := \omega(\Phi_\alpha(a(x))). \tag{4.8}$$

By (4.7), the value $E(a)(x)$ is well defined, and the local triviality of $\mathfrak{A}$ implies that $E(a) : x \mapsto E(a)(x)$ is a continuous function on X. Moreover,

$$|E(a)(x)| = |\omega(\Phi_\alpha(a(x)))| \leq \|\Phi_\alpha(a(x))\| = \|a(x)\|, \tag{4.9}$$

so $E(a)$ lies in $C_0(X)$, and the map $E : A \to C_0(X)$, $E : a \to E(a)$ is a positive $C_0(X)$-linear contraction. By Lemma 4.5, there exists a constant $0 < C \leq 1$ such that (4.3) holds. Then for $a \in A$ and $x \in X$ we have

$$|E(a^*a)(x)| = \|\omega(\Phi_\alpha(a(x)^*a(x)))1_B\| \geq C\|\Phi_\alpha(a(x)^*a(x))\| = C\|a(x)\|^2. \tag{4.10}$$

Hence, (4.9) and (4.10) imply

$$C\|a\|^2 \leq \|E(a^*a)\| \ \leq \|a\|^2 \quad \text{for all } a \in A. \tag{4.11}$$

Now for $a_1, a_2 \in A$ we define

$$\langle a_1, a_2\rangle := E(a_1 a_2^*).$$

Then $\langle\cdot,\cdot\rangle$ is a $C_0(X)$-valued inner product (which is $C_0(X)$-linear in the first and conjugate linear in the second variable). Moreover, $(A, \langle\cdot,\cdot\rangle)$ is a (complete) Hilbert $C_0(X)$-module, since (4.11) implies that its norm $a \mapsto \|E(aa^*)\|^{\frac{1}{2}}$ is equivalent to the C^*-norm on A.

(iii). By (ii), we can equip A with a $C_0(X)$-valued inner product $\langle\cdot,\cdot\rangle$ in such way that A becomes a Hilbert $C_0(X)$-module, whose norm is equivalent to the C^*-norm on A. If on each fibre $\mathcal{A}_x$ we suppress the C^*-norm and endow it with the Hilbert space norm induced by this inner product, we obtain a new bundle; call it $\mathfrak{H}$. Obviously, $\mathfrak{H}$ is an (F) Hilbert bundle. By [21, Theorem 3.6] and [21, Theorem 1.1], $\Gamma_0(\mathfrak{H})$ satisfies $(\mathbf{P})$ if and only if $\mathfrak{H}$ is of finite type as a vector bundle. Since the underlying vector bundle of $\mathfrak{H}$ coincides with the underlying vector bundle of $\mathfrak{A}$, we see that $\Gamma_0(\mathfrak{H})$ satisfies $(\mathbf{P})$ if and only if $\mathfrak{A}$ is of finite type as a vector bundle. Furthermore, since the C^*-norm on A is equivalent to this Hilbert module norm, it follows that the (formal) identity map $\mathrm{id} : \Gamma_0(\mathfrak{H}) \to \Gamma_0(\mathfrak{A}) = A$ defines a $C_0(X)$-linear isomorphism of Banach $C_0(X)$-modules. In particular, A satisfies $(\mathbf{P})$ if and only if $\Gamma_0(\mathfrak{H})$ satisfies $(\mathbf{P})$, so the proof is now finished. $\square$

Problem 4.7. If $\mathfrak{B}$ is an (F) Banach bundle over a locally compact Hausdorff space X, whose fibers are of the same finite dimension n, is it possible to find a $C_0(X)$-valued inner product $\langle\cdot,\cdot\rangle$ on $\Gamma_0(\mathfrak{B})$, whose norm $a \mapsto \|\langle a, a\rangle\|^{\frac{1}{2}}$ is equivalent to the standard sup norm on $\Gamma_0(\mathfrak{B})$?

Remark 4.8. Note that the map E defined by (4.8) plays the role of a conditional expectation of finite index (in a sense of [17]) from A into $C_0(X)$.

Problem 4.9. Let $\mathfrak{A}$ be an (F) C^*-bundle over a locally compact Hausdorff space X, such that $\sup_{x\in X} \dim \mathcal{A}_x < \infty$. Does there exist a constant $0 < C \leq 1$ and a positive $C_0(X)$-linear contraction $E : \Gamma_0(\mathfrak{A}) \to C_0(X)$ such that

$$E(a^*a)(x)1_x \geq C \cdot a(x)^*a(x)$$

for all $a \in \Gamma_0(\mathfrak{A})$ and $x \in X$ (where 1_x is the unit of $\mathcal{A}_x$)? In particular, whether every unital subhomogeneous quasi-standard C^*-algebra A admits a conditional expectation $E : A \to Z(A)$ of finite index?

Lemma 4.10. *Let $\mathfrak{A}$ be an (F) C^*-bundle over a compact metrizable space X such that $n := \sup_{x\in X} \dim \mathcal{A}_x < \infty$. The following conditions are equivalent:*

(i) *Each restriction bundle of* $\mathfrak{A}$ *over a set where* $\dim \mathcal{A}_x$ *is constant is of finite type as a vector bundle.*

(ii) *There exists a finite number of sections* $a_1, \ldots, a_m \in \Gamma(\mathfrak{A})$ *such that*

$$\operatorname{span}\{a_1(x), \ldots, a_m(x)\} = \mathcal{A}_x \quad \text{for all } x \in X. \tag{4.12}$$

Proof. Let $X_0, \ldots, X_k$ be pairwise disjoint non-empty subsets of X covering X and let $0 \leq n_0 < \cdots < n_k = n$ be integers such that all fibres of $\mathfrak{A}|_{X_i}$ are n_i-dimensional. By [11, Proposition 1.6], X_0, $X_0 \cup X_1$,…,$X_0 \cup X_1 \cup \cdots \cup X_{k-1}$ are closed subsets of X. By Lemma 4.6 (ii), for $0 \leq i \leq k$ there exists a $C_0(X_i)$-valued inner product $\langle \cdot, \cdot \rangle_i$ on $\Gamma_0(\mathfrak{A}|_{X_i})$, whose norm $a \mapsto \|\langle a, a\rangle\|_i^{\frac{1}{2}}$ is equivalent to the C^*-norm on $\Gamma_0(\mathfrak{A}|_{X_i})$. Now one can substitute $\mathfrak{A}|_{X_i}$ by the corresponding (F) Hilbert bundle $\mathfrak{H}_i$ over X_i (as in the proof of Lemma 4.6 (iii)), so that $\Gamma_0(\mathfrak{A}|_{X_i}) = \Gamma_0(\mathfrak{H}_i)$, and proceed by using the same arguments as in the proof of [21, Proposition 3.2]. □

Proposition 4.11. *Let* $\mathfrak{A}$ *be an (F)* C^**-bundle over a compact metrizable space* X *and let* $A := \Gamma(\mathfrak{A})$*. The following conditions are equivalent:*

(i) *Fibres* $\mathcal{A}_x$ *of* $\mathfrak{A}$ *have uniformly finite dimensions, and each restriction bundle of* $\mathfrak{A}$ *over a set where* $\dim \mathcal{A}_x$ *is constant is of finite type as a vector bundle.*

(ii) A *as a Banach* $C(X)$*-module is topologically finitely generated.*

(iii) A *satisfies the condition* (**P**) *of Proposition* 4.4.

Proof. (i) ⇔ (ii). By Lemma 4.10, $\mathfrak{A}$ satisfies (i) if and only if there are sections $a_1, \ldots, a_m \in A$ satisfying (4.12). Now one can proceed by using the same arguments as in the proof of [21, Theorem 1.1].

(i) ⇒ (iii). Suppose that

$$n := \sup\{\dim \mathcal{A}_x \ : \ x \in X\} < \infty,$$

and let $U := \{x \in X \ : \ \dim \mathcal{A}_x = n\}$. By [11, Proposition 1.6], U is open, so that $Y := X \setminus U$ is closed, hence compact. Analyzing the proof of (i) ⇒ (ii) [21, Theorem 3.6] we see that, in order to prove that A satisfies (**P**), it is sufficient to prove that $J := \Gamma_0(\mathfrak{A}|_U)$ (as a $C_0(U)$-module) and $B := \Gamma(\mathfrak{A}|_Y)$ (as a $C(Y)$-module) satisfy (**P**). By Lemma 4.6 (iii), J indeed satisfies (**P**), and let

$$n' := \sup\{\dim \mathcal{A}_y \ : \ y \in Y\}.$$

Then $n' < n$, and let $U' := \{y \in Y \ : \ \dim \mathcal{A}_y = n'\}$. Then U' is open in Y (by [11, Proposition 1.6]), so Lemma 4.6 (iii) implies that $J := \Gamma_0(\mathfrak{A}|_{U'})$ (as a $C_0(U')$-module) satisfies (**P**). Proceeding by induction, we conclude that B satisfies (**P**).

(iii) ⇒ (i). Note that the condition (**P**) in particularly implies that there exists $N \in \mathbb{N}$ such that every algebraically finitely generated $C(X)$-submodule of A can be generated with $k \leq N$ generators. The assertion can be now proved by using Lemma 4.6 (ii) together with the same arguments as in the proof of (iv) ⇒ (i) [21, Theorem 3.6]. □

Proof of Theorem 1.6. Let us identify A with $\Gamma(\mathfrak{A})$, using the $*$-isomorphism $\Gamma : A \to \Gamma(\mathfrak{A})$, $\Gamma : a \mapsto \hat{a}$ (see Section 2).

(i) ⇒ (ii). If $\mathrm{E}(A)$ is closed in the operator norm, then obviously $\operatorname{Im} \theta_A = \mathrm{E}(A)$. The claim now follows from [20, Theorem 2.3].

(ii) ⇒ (iii). Obviously, (1.6) implies that A is subhomogeneous (since $\mathrm{Prim}(A) \subseteq \mathrm{Primal}_2(A)$). Since $\mathfrak{A}$ is an (F) bundle, the complete regularization map ϕ_A is open (see Section 2). By Proposition 4.1, every Glimm ideal of A is primal, so it is in particularly 2-primal. Hence, (1.6) implies

$$\operatorname{span}\{a_1+G,\dots,a_m+G\}=A/G \quad \text{for all } G\in \operatorname{Glimm}(A),$$

which can be rewritten as

$$\operatorname{span}\{a_1(x),\dots,a_m(x)\}=\mathcal{A}_x \quad \text{for all } x\in X=\operatorname{Max}(Z(A)).$$

Thus, $\sup_{x\in X}\dim\mathcal{A}_x<\infty$ and by Lemma 4.10, each restriction bundle of $\mathfrak{A}$ over a set where $\dim\mathcal{A}_x$ is constant is of finite type as a vector bundle.

(iii) $\Leftrightarrow$ (iv). This follows directly from Proposition 4.11.

(iii) $\Rightarrow$ (i). Since A is obviously subhomogeneous, the completely bounded norm and the operator norm on $\mathrm{E}(A)$ are equivalent (for example, see [18, Remark 6.2]), so $\overline{\overline{\mathrm{E}(A)}}=\overline{\overline{\mathrm{E}(A)}}_{cb}$. Moreover, since by Proposition 4.1 each Glimm ideal of A is primal, using Proposition 2.3 and [31, Theorem 4], we can identify

$$\overline{\overline{\mathrm{E}(A)}}=\overline{\overline{\mathrm{E}(A)}}_{cb}=\operatorname{Im}\theta_A=A\otimes_{Z,h}A. \tag{4.13}$$

On the other hand, by [22, Theorem 6.1], the projective norm $\|\cdot\|_\pi$ and the Haagerup norm $\|\cdot\|_h$ are equivalent on $A\otimes A$. This implies that the (formal) identity map

$$\mathrm{id}:(A\otimes A,\|\cdot\|_\pi)\to(A\otimes A,\|\cdot\|_h)$$

defines an isomorphism of normed spaces, so its extension on the completed tensor products $\mathrm{id}:A\overset{\pi}{\otimes}A\to A\otimes_h A$ defines an isomorphism of Banach spaces. Of course, the same conclusion holds for the (formal) identity map

$$\mathrm{id}:A\overset{\pi}{\otimes}_{Z(A)}A\cong A\otimes_{Z,h}A. \tag{4.14}$$

By Proposition 4.11, A satisfies the condition $(\mathbf{P})$, so by Proposition 4.4, there exists $N\in\mathbb{N}$ such that each tensor $t\in A\overset{\pi}{\otimes}_{Z(A)}A$ can be written in a form $t=\sum_{i=1}^k a_i\otimes_Z b_i$, for some $a_i,b_i\in A$ and $k\le N$. Applying (4.14), we see that the same conclusion holds for the tensors in $A\otimes_{Z,h}A$. Finally, (4.13) yields $\overline{\overline{\mathrm{E}(A)}}=\mathrm{E}(A)$. □

Remark 4.12. Suppose that A is a unital separable C^*-algebra in which every Glimm ideal is 2-primal. If A satisfies (1.2), then A is topologically finitely generated over $Z(A)$ by [20, Theorem 2.3] and the Stone Weierstrass-theorem for (H) C^*-bundles [35, Proposition C.24]. We do not know whether the converse is true in general, although we conjecture that the answer is affirmative.

Remark 4.13. If A is a unital C^*-algebra which is algebraically finitely generated over $Z(A)$, then A is ($*$-isomorphic to) a finite direct sum of unital homogeneous C^*-algebras by [19, Theorem 2.4]. In particular, the canonical C^*-bundle $\mathfrak{A}$ of A over $\operatorname{Max}(Z(A))$ is an (F) bundle. The next example shows that this is not true in general for unital C^*-algebras A which are topologically finitely generated over $Z(A)$.

Example 4.14. Let A be a C^*-algebra from Example 4.3 and let $(e_{i,j})$ be the standard matrix units of $\mathrm{M}_2(\mathbb{C})$, considered as constant elements of $C([0,1],\mathrm{M}_2(\mathbb{C}))$. If $\varphi\in C_0([0,1))$ is a strictly positive function, one can easily check (for example, by applying [35, Proposition C.24]) that the $Z(A)$-submodule of A generated by the set

$$\{(1_B,1_C),(e_{1,1},1_C),(\varphi e_{1,2},0),(\varphi e_{2,1},0),(\varphi e_{2,2},0)\}$$

is norm dense in A. On the other hand, as noted in Example 4.3, ϕ_A is not open, so the canonical C^*-bundle $\mathfrak{A}$ of A over $\mathrm{Max}(Z(A))$ is not an (F) bundle.

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