CHARACTERIZING JORDAN EMBEDDINGS BETWEEN PARABOLIC SUBALGEBRAS VIA PRESERVING PROPERTIES

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Abstract. We consider arbitrary parabolic subalgebras $\mathcal{A} \subseteq M_n$ (i.e. subalgebras of M_n which contain the algebra of upper-triangular matrices) and their Jordan embeddings. We first describe Jordan embeddings $\phi : \mathcal{A} \to M_n$ as maps of the form

$$\phi(X) = TXT^{-1} \qquad \text{or} \qquad \phi(X) = TX^tT^{-1},$$

where $T \in M_n$ is an invertible matrix, and then we obtain a simple criteria of when one parabolic subalgebra Jordan-embeds into another (and in that case we describe the form of such embeddings). As a main result, we characterize Jordan embeddings $\phi : \mathcal{A} \to M_n$ (when $n \geq 3$) as continuous injective maps which preserve commutativity and spectrum. We show by counterexamples that all these assumptions are indispensable (unless $\mathcal{A} = M_n$ when injectivity is superfluous).

1. Introduction

Jordan algebras were first introduced by Pascual Jordan in 1933 in the context of quantum mechanics [19]. The majority of the practically relevant Jordan algebras naturally arise as subalgebras of an associative algebra $\mathcal{A}$ under a symmetric product given by

$$x \circ y = xy + yx.$$

This gave rise to the study of Jordan homomorphisms in the context of associative rings and algebras. Namely, recall that an additive (linear) map $\phi : \mathcal{A} \to \mathcal{B}$ between rings (algebras) $\mathcal{A}$ and $\mathcal{B}$ is a *Jordan homomorphism* if

$$\phi(a \circ b) = \phi(a) \circ \phi(b), \qquad \text{for all } a, b \in \mathcal{A}.$$

When the rings (algebras) are 2-torsion-free, this is equivalent to

$$\phi(a^2) = \phi(a)^2, \qquad \text{for all } a \in \mathcal{A}.$$

Clearly, multiplicative and antimultiplicative maps are immediate examples of such maps. One of the main problems in the context of Jordan homomorphisms is under which assumptions on rings (algebras) $\mathcal{A}$ and $\mathcal{B}$, usually without 2-torsion, can we conclude that every Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ (possibly satisfying some extra conditions such as surjectivity) is either multiplicative or antimultiplicative. More generally, the question is whether one can express all such Jordan homomorphisms as a suitable combination of ring (algebra) homomorphisms and antihomomorphisms. This question goes a long way back. Namely, in 1950 Jacobson and Rickart [18] proved that a Jordan homomorphism from an arbitrary ring into an integral domain is either a homomorphism or an antihomomorphism. This paper is particularly relevant for our discussion as it also proves that a Jordan homomorphism from

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the ring of $n \times n$ matrices, $n \geq 2$, over an arbitrary unital ring is the sum of a homomorphism and an antihomomorphism. In the same vein, Herstein [16] concludes that a Jordan homomorphism onto a prime ring is either a homomorphism or an antihomomorphism. The same result was later refined by Smiley [30].

Let M_n be the algebra of $n \times n$ matrices over the field of complex numbers. By combining the aforementioned result of Herstein with the well-known fact that all automorphisms of M_n are inner (for a short and elegant proof see [29]), one obtains that all nonzero Jordan endomorphisms ϕ of M_n are precisely maps of the form

$$\phi(X) = TXT^{-1} \qquad \text{or} \qquad \phi(X) = TX^tT^{-1} \tag{1.1}$$

(globally) for some invertible matrix $T \in M_n^\times$.

There have been many attempts to characterize Jordan homomorphisms, particularly on matrix algebras, using preserver properties. These attempts date back at least to 1970 and Kaplansky's famous problem [20] which asks under which conditions on unital (complex) Banach algebras A and B is a linear unital map $\phi : A \to B$ which preserves invertibility necessarily a Jordan homomorphism. This problem received a lot of attention and progress was made in some special cases, but it is still widely open, even for C^*-algebras (see [7], page 270). For other interesting types of linear preserver problems resulting in more general kind of maps, we refer to the survey paper [21] and references within. We would also like to distinguish the following nonlinear preserver problem which elegantly characterizes Jordan automorphisms of M_n:

Theorem 1.1 (Šemrl)**.** *Let $\phi : M_n \to M_n, n \geq 3$ be a continuous map which preserves commutativity and spectrum. Then there exists $T \in M_n^\times$ such that ϕ is of the form* (1.1)*.*

A precursor to this result was first formulated in [27] and it assumed its current form a decade later in [28]. It also serves as the main motivation for our investigation. Namely, we are interested in the following general problem:

Problem 1.2. Find necessary and sufficient conditions on a (Jordan) subalgebra $\mathcal{A}$ of M_n such that each Jordan automorphism of $\mathcal{A}$, or more generally a Jordan embedding (monomorphism) $\phi : \mathcal{A} \to M_n$, extends to a Jordan automorphism of M_n. Additionally, for such $\mathcal{A}$, characterize all such mappings ϕ via suitable preserving properties, similarly as in Theorem 1.1.

The first natural example to consider in the context of Problem 1.2 is the algebra $\mathcal{T}_n$ of $n \times n$ upper-triangular complex matrices. First of all, it is well-known that all Jordan automorphisms of $\mathcal{T}_n$ are of the form (1.1) for suitable $T \in M_n^\times$ (see e.g. [25, Corollary 4]). The same holds true for all Jordan embeddings $\mathcal{T}_n \to M_n$ (a special case of our first result, Theorem 1.3). Also, Jordan automorphisms of $\mathcal{T}_n$ (as well as more general type of maps on $\mathcal{T}_n$) were characterized via both linear and nonlinear preserving properties by several authors (see e.g. [11, 12, 13, 14, 17, 22, 26, 35]). In particular, following Theorem 1.1, Petek [26] described all Jordan automorphisms of $\mathcal{T}_n$ as continuous spectrum and commutativity preserving surjective mappings $\mathcal{T}_n \to \mathcal{T}_n$. Analyzing the main result of [26] it is easy to verify that the same holds true if instead of surjectivity one assumes injectivity, thus providing a positive solution for (both parts of) Problem 1.2 when $\mathcal{A} = \mathcal{T}_n$.

Continuing in this vein, the next class of algebras we consider are subalgebras $\mathcal{A}$ of M_n which contain $\mathcal{T}_n$. Such algebras are known in the literature as *parabolic subalgebras* of M_n

(see e.g. [1, 32]) and are precisely of the form

$$\mathcal{A}_{k_1,\ldots,k_r} := \begin{bmatrix} M_{k_1,k_1} & M_{k_1,k_2} & \cdots & M_{k_1,k_r} \\ 0 & M_{k_2,k_2} & \cdots & M_{k_2,k_r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_r,k_r} \end{bmatrix} \tag{1.2}$$

for some $r, k_1, \ldots, k_r \in \mathbb{N}$ such that $k_1 + \cdots + k_r = n$ [32]. In other words, these are precisely the algebras of so-called block upper-triangular matrices. Note that the parabolic algebras $\mathcal{A}_{1,n-1}$ and $\mathcal{A}_{n-1,1}$ are exactly (up to similarity) the unital strict subalgebras of M_n of maximal dimension (see [1]). Our first introductory result verifies that parabolic algebras indeed satisfy the desired extension property, providing an affirmative answer for the first part of Problem 1.2:

Theorem 1.3. *Let $\mathcal{A} \subseteq M_n$ be a parabolic subalgebra and let $\phi : \mathcal{A} \to M_n$ be a Jordan embedding. Then there exists $T \in M_n^\times$ such that ϕ is of the form* (1.1)*.*

After providing the short proof of Theorem 1.3 (in Section §3) we also present simple criteria of when one parabolic algebra (Jordan-)embeds into another (see Corollary 3.2 and 3.3). To put Theorem 1.3 in a wider context, there is a number of related (and somewhat more general) results concerning Jordan homomorphisms on certain classes of (block) upper-triangular rings and algebras. For example, an influential result by Beidar, Brešar and Chebotar states that every Jordan isomorphism of the algebra of upper-triangular matrices $\mathcal{T}_n(\mathcal{C}), n \geq 2$ over a 2-torsion-free commutative unital ring $\mathcal{C}$ without nontrivial idempotents onto an arbitrary $\mathcal{C}$-algebra is necessarily multiplicative or antimultiplicative [4]. A generalization was given in [23] by removing the assumption that $\mathcal{C}$ has no nontrivial idempotents. These results were further developed by Benkovič in [5]; any Jordan homomorphism from $\mathcal{T}_n(\mathcal{C}), n \geq 2$ into an algebra $\mathcal{B}$ is a (so-called) near-sum of a homomorphism and an antihomomorphism. We also mention papers [6, 15, 33, 34] which treat similar problems for Jordan homomorphims between more general types of (block) triangular matrix rings.

Concerning parabolic subalgebras in particular, the papers [3, 8, 9] (all sequels of the aforementioned paper [5]) describe Jordan homomorphisms of certain classes of more general algebras. As parabolic algebras are (up to isomorphism) precisely finite-dimensional instances of nest algebras, papers [10, 24] are also relevant. Note that many of the mentioned results above assume that the image of Jordan homomorphisms are rings or algebras. In Theorem 1.3 we make no such assumption but this is obviously compensated by the fact that the codomain is restricted to M_n, which incidentally also enables us to state the explicit form (1.1) of such maps.

After verifying that parabolic algebras $\mathcal{A}$ satisfy the first part of Problem 1.2, the next step is to characterize Jordan embeddings $\phi : \mathcal{A} \to M_n$ via suitable preserver properties. Building upon both Theorem 1.1 and [26, Corollary 3], we arrived at the following theorem, which is also the main result of our paper:

Theorem 1.4. *Let $\mathcal{A} \subseteq M_n, n \geq 3$ be a parabolic subalgebra and let $\phi : \mathcal{A} \to M_n$ be a continuous injective map which preserves commutativity and spectrum. Then ϕ is a Jordan embedding and hence of the form* (1.1) *for some $T \in M_n^\times$.*

Moreover, if we additionally assume that the image of ϕ is contained in $\mathcal{A}$, so that $\phi : \mathcal{A} \to \mathcal{A}$, using the invariance of domain theorem we show that the spectrum preserving assumption can be further relaxed to spectrum shrinking (Corollary 4.2).

This paper is organized as follows. We begin by providing terminology and notation in Section §2 along with some preliminary technical results related to parabolic algebras. Section §3 contains the proof of Theorem 1.3 and its consequences regarding (Jordan) embeddings between two parabolic subalgebras of M_n. Section §4 is the core of the paper. It contains the proof of Theorem 1.4, which is nontrivial and contains most of the actual work. The proof is carried out by induction on n. The base step verifies the result on parabolic subalgebras of M_3 for which the statement is proved by direct technical computations. The inductive step consists of reducing the order of the parabolic algebra by passing to a map obtained by erasing a suitable row and column, which satisfies the same assumptions. Then we notice that the problem can be simplified by assuming that our map is the identity on certain carefully selected sets (like diagonal matrices, matrix units and rank one matrices), a fact which we iteratively exploit until we reach the entire parabolic algebra. After proving Theorem 1.4 a few direct consequences are stated. Finally, in Section §5, we demonstrate the necessity of all assumptions of Theorem 1.4 via counterexamples.

2. Preliminaries

We begin this section by introducing some notation and terminology. Let $n \in \mathbb{N}$.

- $M_n := M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices.
- $\mathcal{T}_n$ and $\mathcal{D}_n$ denote the sets of all upper-triangular and diagonal matrices of M_n, respectively.
- $\mathcal{A}_{k_1,\ldots,k_r}$ denotes the parabolic algebra (1.2) for some $r, k_1, \ldots, k_r \in \mathbb{N}$ such that $k_1 + \cdots + k_r = n$. Additionally, in this context we allow $k_j = 0$ for some $1 \le j \le r$ in the sense $\mathcal{A}_{k_1,\ldots,k_{j-1},0,k_{j+1},\ldots,k_r} = \mathcal{A}_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r}$. We also formally define $k_0 = 0$.
- For $A, B \in M_n$ we denote by $A \leftrightarrow B$ the fact that A and B commute, i.e. $AB = BA$.
- For $A \in M_n$ by k_A we denote the characteristic polynomial of A.
- For $A \in M_n$ we denote by $R(A)$ the image of A and by $N(A)$ the nullspace of A.

By $\mathcal{A}$ and $\mathcal{B}$ we usually denote complex algebras. For a unital algebra $\mathcal{A}$, by $\mathcal{A}^\times$ we denote the set of all invertible elements in $\mathcal{A}$. Note that when $\mathcal{A}$ is finite-dimensional, the set $\mathcal{A}^\times$ is path-connected in $\mathcal{A}$. Namely, for every $A \in \mathcal{A}^\times$ by finiteness of the spectrum we can take an appropriate branch of the logarithm to conclude that $A = \exp B$ for some $B \in \mathcal{A}$. Then $t \mapsto \exp(tB)$ is a (continuous) path from I to A within $\mathcal{A}$.

As usual, we will frequently identify vectors $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ as column-matrices

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and $x^t = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ as row-matrices.

Any matrix $A = (A_{ij})_{ij} \in M_n$ can be understood as a map $\{1, \ldots, n\}^2 \to \mathbb{C}, (i,j) \mapsto A_{ij}$ so we can consider its support $\operatorname{supp} A$ as the set of all indices $(i,j) \in \{1, \ldots, n\}^2$ such that $A_{ij} \neq 0$.

Lemma 2.1. *Let*
$$J := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in M_n.$$
Then the map $\mathcal{A}_{k_1,\ldots,k_r} \to \mathcal{A}_{k_r,\ldots,k_1}, X \mapsto JX^tJ$ is an algebra antiisomorphism.

Proof. It suffices to show that $JX^tJ \in \mathcal{A}_{k_r,\ldots,k_1}$ for all $X \in \mathcal{A}_{k_1,\ldots,k_r}$. Indeed, for each $0 \le s \le r-1$, $1 \le i \le k_1 + \cdots + k_{s+1}$ and $k_1 + \cdots + k_s + 1 \le j \le n$ we have
$$JE_{ij}^tJ = E_{n+1-j,n+1-i} \in \mathcal{A}_{k_r,\ldots,k_1}$$
since $k_r + \cdots + k_s + 1 \le n+1-i \le n$ and $1 \le n+1-j \le k_r + \cdots + k_{s+1}$. $\square$

This map is so ubiquitous that we will introduce a notation
$$X^{\odot} := JX^tJ.$$
Note that it actually corresponds to mirroring the matrix X along its secondary diagonal. Also, for a parabolic algebra $\mathcal{A} \subseteq M_n$ we denote by $\mathcal{A}^{\odot} \subseteq M_n$ the image of $\mathcal{A}$ under the map $X \mapsto X^{\odot}$. Then $\mathcal{A}^{\odot}$ is the parabolic algebra obtained from $\mathcal{A}$ by reversing the sizes of the diagonal blocks.

For $1 \le i \le n$ denote
$$\mathcal{A}_{k_1,\ldots,k_r}^{\leftrightarrow i} := \{X \in \mathcal{A}_{k_1,\ldots,k_r} : XE_{ii} = E_{ii}X = 0\}.$$
It is easy to see that this is a subalgebra of $\mathcal{A}_{k_1,\ldots,k_r}^{\leftrightarrow i}$.

Denote by $\mathcal{A}_{k_1,\ldots,k_r}^{\setminus i}$ the parabolic algebra obtained from $\mathcal{A}_{k_1,\ldots,k_r}$ by removing i-th row and column.

For $X \in \mathcal{A}_{k_1,\ldots,k_r}^{\leftrightarrow i}$ denote by $X^{\flat i} \in \mathcal{A}_{k_1,\ldots,k_r}^{\setminus i}$ the matrix obtained from X by removing i-th row and column.
$$\begin{bmatrix} X^{1,1}_{(i-1)\times(i-1)} & 0 & X^{1,2}_{(i-1)\times(n-i)} \\ 0 & 0 & 0 \\ X^{2,1}_{(n-i)\times(i-1)} & 0 & X^{2,2}_{(n-i)\times(n-i)} \end{bmatrix}^{\flat i} = \begin{bmatrix} X^{1,1}_{(i-1)\times(i-1)} & X^{1,2}_{(i-1)\times(n-i)} \\ X^{2,1}_{(n-i)\times(i-1)} & X^{2,2}_{(n-i)\times(n-i)} \end{bmatrix}.$$
Conversely, for $Y \in \mathcal{A}_{k_1,\ldots,k_r}^{\setminus i}$ and $z \in \mathbb{C}$ denote by $Y^{\sharp(i,z)} \in \mathcal{A}_{k_1,\ldots,k_r}$ the matrix obtained from Y by adding i-th row and column as zeroes, except at the position (i,i) which now contains z.
$$\begin{bmatrix} Y^{1,1}_{(i-1)\times(i-1)} & Y^{1,2}_{(i-1)\times(n-i)} \\ Y^{2,1}_{(n-i)\times(i-1)} & Y^{2,2}_{(n-i)\times(n-i)} \end{bmatrix}^{\sharp(i,z)} = \begin{bmatrix} Y^{1,1}_{(i-1)\times(i-1)} & 0 & Y^{1,2}_{(i-1)\times(n-i)} \\ 0 & z & 0 \\ Y^{2,1}_{(n-i)\times(i-1)} & 0 & Y^{2,2}_{(n-i)\times(n-i)} \end{bmatrix}.$$

Claim 2.1.1. For each $1 \le i \le n$ the maps
$$.^{\sharp(i,0)} : \mathcal{A}_{k_1,\ldots,k_r}^{\setminus i} \to \mathcal{A}_{k_1,\ldots,k_r}^{\leftrightarrow i}$$
and
$$.^{\flat i} : \mathcal{A}_{k_1,\ldots,k_r}^{\leftrightarrow i} \to \mathcal{A}_{k_1,\ldots,k_r}^{\setminus i}$$
are algebra isomorphisms.

Proof. It is clear that the maps are linear and inverses of each other. It suffices to show that $.^{\sharp(i,0)}$ is multiplicative, which follows directly from block-matrix multiplication. $\square$

Denote $\Delta = \operatorname{diag}(1, \dots, n) \in \mathcal{A}_{k_1,\dots,k_r}$. We also introduce this notation: for $A \in \mathcal{A}_{k_1,\dots,k_r}$ of the form (1.2), we write $\operatorname{di}(A) = (X_{11}, \dots, X_{rr})$ extracting the diagonal blocks of A.

Lemma 2.2. *Let $A \in M_n$ be a matrix satisfying $A \leftrightarrow E_{ss}$ for some $1 \le s \le n$ and denote its eigenvalues by $\lambda_1, \dots, \lambda_{s-1}, a_{ss}, \lambda_{s+1}, \dots, \lambda_n$. Then there exists a unitary matrix $U \in M_n$ and $\Theta \in \mathcal{T}_n$ such that $U \leftrightarrow E_{ss}$, $U_{ss} = 1$, $A = U\Theta U^*$ and the diagonal of Θ is precisely $\lambda_1, \dots, \lambda_{s-1}, a_{ss}, \lambda_{s+1}, \dots, \lambda_n$.*

Proof. A can be written as

$$A = \begin{bmatrix} A_{(s-1)\times(s-1)} & 0_{(s-1)\times 1} & A_{(s-1)\times(n-s)} \\ 0_{1\times(s-1)} & a_{ss} & 0_{1\times(n-s)} \\ A_{(n-s)\times(s-1)} & 0_{(n-s)\times 1} & A_{(n-s)\times(n-s)} \end{bmatrix}$$

so by the Schur decomposition there exist a unitary matrix $U \in M_{n-1}$ and an upper-triangular matrix $\Theta \in \mathcal{T}_{n-1}$ with the diagonal $\lambda_1, \dots, \lambda_{s-1}, \lambda_{s+1}, \dots, \lambda_n$ such that

$$A^{\flat s} = U\Theta U^*.$$

By block-matrix multiplication we see that the desired decomposition is achieved as

$$A = U^{\sharp(s,1)}\Theta^{\sharp(s,a_{ss})}(U^*)^{\sharp(s,1)} = U^{\sharp(s,1)}\Theta^{\sharp(s,a_{ss})}(U^{\sharp(s,1)})^*.$$

$\square$

Lemma 2.3. *Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ such that* $\operatorname{di}(A) = (X_1, \dots, X_r)$*. Suppose that A has eigenvalues $\lambda_1, \dots, \lambda_n$ listed such that $\lambda_{k_1+\dots+k_{j-1}+1}, \dots, \lambda_{k_1+\dots+k_j}$ are eigenvalues of X_j for all $1 \le j \le n$. Then there exists $T \in \mathcal{A}^\times_{k_1,\dots,k_r}$ and $\Theta \in \mathcal{T}_n$ such that $A = T\Theta T^{-1}$ and the diagonal of Θ is precisely $\lambda_1, \dots, \lambda_n$.*
Furthermore, if $A \leftrightarrow E_{ss}$ for some $1 \le s \le n$, then T can be chosen to satisfy $T \leftrightarrow E_{ss}$ and $T_{ss} = 1$.

Proof. Suppose $\operatorname{di}(A) = (X_1, \dots, X_r)$. By Claim 2.2, for every $1 \le j \le r$ we can pick a (unitary) matrix $S_j \in M^\times_{k_j}$ such that $S_j X_j S_j^{-1} \in \mathcal{T}_{k_j}$. Then

$$\operatorname{diag}(S_1, \dots, S_r) A \operatorname{diag}(S_1, \dots, S_r)^{-1} \in \mathcal{T}_n$$

since the blocks on the diagonal of this matrix are all upper triangular.
The second claim follows from second statement of Claim 2.2 applied on X_k. $\square$

Lemma 2.4. *Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ such that* $\operatorname{di}(A) = (X_1, \dots, X_r)$*. Suppose that A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ listed such that $\lambda_{k_1+\dots+k_{j-1}+1}, \dots, \lambda_{k_1+\dots+k_j}$ are eigenvalues of X_j for all $1 \le j \le n$. Then there exists $T \in \mathcal{A}^\times_{k_1,\dots,k_r}$ such that $TAT^{-1} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.*
Furthermore, if $A \leftrightarrow E_{ss}$ for some $1 \le s \le n$, then T can be chosen to satisfy $T \leftrightarrow E_{ss}$ and $T_{ss} = 1$.

Proof. We prove the claim by induction on n. For $n = 1$ the statement is clear so suppose that the claim is true for $n - 1$. First by Claim 2.3, there exists $T \in \mathcal{A}^\times_{k_1,\dots,k_r}$ such that $TAT^{-1} \in \mathcal{T}_n$. We can write

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & B \\ 0 & C \end{bmatrix}$$

for some matrices $B \in M_{1,n-1}$ and $C \in \mathcal{T}_{n-1}$. By the induction hypothesis, there exists $Q \in \mathcal{T}^\times_{n-1}$ such that $QCQ^{-1} = \operatorname{diag}(\lambda_2, \dots, \lambda_n)$. Now we have

$$\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} TAT^{-1} \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & BQ^{-1} \\ 0 & \operatorname{diag}(\lambda_2, \dots, \lambda_n) \end{bmatrix}.$$

It remains to diagonalize the latter matrix. It is not hard to verify that

$$R = I + \sum_{j=2}^{n} \frac{(BQ^{-1})_{1(j-1)}}{\lambda_1 - \lambda_j} E_{1j} \in \mathcal{T}_n^\times$$

satisfies

$$R \begin{bmatrix} \lambda_1 & BQ^{-1} \\ 0 & \operatorname{diag}(\lambda_2, \ldots, \lambda_n) \end{bmatrix} R^{-1} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Therefore the matrix $R\operatorname{diag}(1, Q)T \in \mathcal{A}_{k_1,\ldots,k_r}^\times$ satisfies

$$(R\operatorname{diag}(1, Q)T)A(R\operatorname{diag}(1, Q)T)^{-1} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Now we prove the second claim. First assume that $2 \le s \le n$. Firstly, T can be chosen to satisfy $T \leftrightarrow E_{ss}$ and $T_{ss} = 1$ by the second claim of Claim 2.3. Then $TAT^{-1} \leftrightarrow E_{ss}$ as well. By the induction hypothesis, the matrix Q above can by chosen such that $Q \leftrightarrow E_{(s-1)(s-1)}$ and $Q_{(s-1)(s-1)} = 1$. Then Q^{-1} satisfies the same properties so it is easy to see that $(BQ^{-1})_{1(s-1)} = B_{1(s-1)} = 0$. Therefore $R \leftrightarrow E_{ss}$ and $R_{ss} = 1$ as well. We conclude that the matrix $R\operatorname{diag}(1, Q)T$ satisfies the two required properties as a product of three such matrices.
Now assume $s = 1$. Then $\operatorname{diag}(1, Q) \leftrightarrow E_{11}$ and $\operatorname{diag}(1, Q)_{11} = 1$ while the rest of the argument is the same. □

3. Parabolic subalgebras and their Jordan embeddings

First we state a few basic properties of Jordan homomorphisms, proofs of which are elementary and can be found in [18]. For an algebra A, as usual, we denote the commutator of $a, b \in A$ as $[a, b] = ab - ba$.

Lemma 3.1. *Let $\phi : A \to B$ be a Jordan homomorphism between algebras A and B. We have:*

(a) $\phi(aba) = \phi(a)\phi(b)\phi(a)$ for all $a, b \in A$.
(b) $\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a)$ for all $a, b, c \in A$.
(c) $\phi([[a, b], c]) = [[\phi(a), \phi(b)], \phi(c)]$, for all $a, b, c \in A$.
(d) $\phi([a, b]^2) = [\phi(a), \phi(b)]^2$ for all $a, b \in A$.
(e) $\phi(a^k) = \phi(a)^k$ for all $a \in A$ and $k \in \mathbb{N}$. In particular, $\phi(p(a)) = p(\phi(a))$ for all $a \in A$ and polynomials $p \in \mathbb{C}[x]$ such that $p(0) = 0$.
(f) For every idempotent $p \in A$ and $a \in A$ which satisfies $[p, a] = 0$ we have $\phi(pa) = \phi(p)\phi(a) = \phi(a)\phi(p)$.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3.

Claim 1.3.1. Without loss of generality we can assume that $\phi(D) = D$ for all $D \in \mathcal{D}_n$.

Proof. For $1 \le i \ne j \le n$ we have $E_{ii} \leftrightarrow E_{jj}$ so by Lemma 3.1 (g) it follows

$$0 = \phi(E_{ii}E_{jj}) = \phi(E_{ii})\phi(E_{jj}) = \phi(E_{jj})\phi(E_{ii}).$$

We conclude that $\phi(E_{11}), \ldots, \phi(E_{nn})$ is a family of mutually orthogonal nonzero idempotents. Therefore, there exists $S \in M_n^\times$ such that

$$\phi(E_{kk}) = SE_{kk}S^{-1}, \qquad 1 \le k \le n.$$

By passing to the Jordan monomorphism $S\phi(\cdot)S^{-1}$, without loss of generality we can assume that

$$\phi(E_{kk}) = E_{kk}, \qquad 1 \leq k \leq n.$$

The claim follows by linearity. □

Claim 1.3.2. For every non-diagonal matrix unit $E_{ij} \in \mathcal{A}$ there exist scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$, exactly one of which is zero, such that $\phi(E_{ij}) = \alpha_{ij}E_{ij} + \beta_{ij}E_{ji}$.

Proof. By Lemma 3.1 (b), we have

$$\begin{aligned}\phi(E_{ij}) &= \phi(E_{ii}E_{ij}E_{jj} + E_{jj}E_{ij}E_{ii}) = E_{ii}\phi(E_{ij})E_{jj} + E_{jj}\phi(E_{ij})E_{ii} \\ &= \phi(E_{ij})_{ij}E_{ij} + \phi(E_{ij})_{ji}E_{ji}.\end{aligned}$$

Therefore, the only possible nonzero elements are on the positions (i, j) and (j, i) so there exist scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ such that

$$\phi(E_{ij}) = \alpha_{ij}E_{ij} + \beta_{ij}E_{ji}.$$

Furthermore, we have

$$0 = \phi(E_{ij}^2) = \phi(E_{ij})^2 = \alpha_{ij}\beta_{ij}(E_{ii} + E_{jj})$$

so exactly one of α_{ij} and β_{ij} is equal to zero 0 (not both because of injectivity). □

Claim 1.3.3. (a) If $E_{ij}, E_{ik} \in \mathcal{A}$ for some distinct indices $1 \leq i, j, k \leq n$, then either $\alpha_{ij} = \alpha_{ik} = 0$, or $\beta_{ij} = \beta_{ik} = 0$.

(b) If $E_{ij}, E_{kj} \in \mathcal{A}$ for some distinct indices $1 \leq i, j, k \leq n$, then either $\alpha_{ij} = \alpha_{kj} = 0$, or $\beta_{ij} = \beta_{kj} = 0$.

Proof. (a) Suppose that $\phi(E_{ij}) = \alpha_{ij}E_{ij}$ and $\phi(E_{ik}) = \beta_{ik}E_{ki}$. We have

$$\begin{aligned}0 &= \phi(E_{ij}E_{ik} + E_{ik}E_{ij}) = \phi(E_{ij})\phi(E_{ik}) + \phi(E_{ik})\phi(E_{ij}) = \alpha_{ij}\beta_{ik}(E_{ij}E_{ki} + E_{ki}E_{ij}) \\ &= \underbrace{\alpha_{ij}\beta_{ik}}_{\neq 0}E_{kj},\end{aligned}$$

which is a contradiction.

(b) Analogous as (a).

□

Claim 1.3.4. Either all $\alpha_{ij} \neq 0$, or all $\beta_{ij} \neq 0$.

Proof. By Claim 1.3.3 the same option holds throughout the last column, for example assume $\alpha_{1n}, \ldots, \alpha_{(n-1)n} \neq 0$.
Now suppose $E_{ij} \in \mathcal{A}$ is a non-diagonal matrix unit such that $i \neq n$. Then E_{ij} and $E_{in} \in \mathcal{A}$ are in the i-th row so we conclude $\alpha_{ij} \neq 0$.
Finally, suppose $E_{nj} \in \mathcal{A}$ is a non-diagonal matrix unit. Then $E_{1j} \in \mathcal{A}$ so by the previous argument it follows $\alpha_{nj} \neq 0$. This handles all matrix units in $\mathcal{A}$. □

Without loss of generality we can assume $\phi(E_{ij}) = \alpha_{ij}E_{ij}$ for all matrix units $E_{ij} \in \mathcal{A}$, otherwise we pass to the map $\phi(\cdot)^t$.

Claim 1.3.5. For all $1 \leq i, j, k \leq n$ such that $E_{ij}, E_{jk} \in \mathcal{A}$ we have $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$.

Proof. If $E_{ik} = E_{jj}$, then the equality $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ holds trivially. Otherwise, we have

$$\begin{aligned}\alpha_{ik}E_{ik} + \delta_{ik}E_{jj} &= \phi(E_{ik} + \delta_{ik}E_{jj}) = \phi(E_{ij}E_{jk} + E_{jk}E_{ij}) = \phi(E_{ij})\phi(E_{jk}) + \phi(E_{jk})\phi(E_{ij}) \\ &= \alpha_{ij}\alpha_{jk}E_{ij}E_{jk} + \alpha_{jk}\alpha_{ij}E_{jk}E_{ij} = \alpha_{ij}\alpha_{jk}E_{ik} + \delta_{ik}\alpha_{jk}\alpha_{ij}E_{jj}\end{aligned}$$

so by comparing coefficients of E_{ik} we conclude $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$. □

Claim 1.3.6. There exists $T \in M_n^\times$ such that $\phi(X) = TXT^{-1}$ for all $X \in \mathcal{A}$.

Proof. For all matrix units $E_{ij} \in \mathcal{A}$ we have

$$\phi(E_{ij}) = \alpha_{ij}E_{ij} = \frac{\alpha_{1j}}{\alpha_{1i}} \overset{\text{Claim 1.3.5}}{=} \operatorname{diag}(\alpha_{11}, \ldots, \alpha_{1n})^{-1} E_{ij} \operatorname{diag}(\alpha_{11}, \ldots, \alpha_{1n}).$$

Since $\mathcal{A}$ is spanned by its matrix units, by linearity it follows

$$\phi(X) = T^{-1}XT, \quad \text{for all } X \in \mathcal{A}$$

where $T = \operatorname{diag}(\alpha_{11}, \ldots, \alpha_{1n}) \in M_n^\times$. □

□

Note that $\mathcal{A}_{k_1,\ldots,k_p} \subseteq \mathcal{A}_{l_1,\ldots,l_q}$ if and only if there exist $1 \le r_1 < \ldots < r_q = p$ such that

$$\sum_{j=1}^{r_1} k_j = l_1, \quad \sum_{j=r_1+1}^{r_2} k_j = l_2, \quad \ldots, \quad , \sum_{j=r_{q-1}+1}^{r_q} k_j = l_q.$$

Corollary 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be parabolic subalgebras of M_n. Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan embedding. Then one of the following is true:*
(a) $\mathcal{A} \subseteq \mathcal{B}$ and there exists $T \in \mathcal{B}^\times$ such that $\phi(X) = TXT^{-1}$ for all $X \in \mathcal{A}$,
(b) $\mathcal{A}^\odot \subseteq \mathcal{B}$ and there exists $T \in \mathcal{B}^\times J$ such that $\phi(X) = TX^tT^{-1}$ for all $X \in \mathcal{A}$.

Proof. Denote $\mathcal{A} = \mathcal{A}_{k_1,\ldots,k_p}$ and $\mathcal{B} = \mathcal{A}_{l_1,\ldots,l_q}$. By Theorem 1.3, there exists $T \in M_n$ and $\circ \in \{\mathrm{id}, \cdot^t\}$ such that $\phi(X) = TX^\circ T^{-1}$ for all $X \in \mathcal{A}$.
Suppose first that $\circ = \mathrm{id}$ and let us show $\mathcal{A} \subseteq \mathcal{B}$. We will prove the claim by induction on n.
For $n = 1$, the only possible parabolic algebra is $\mathcal{A}_1 = M_1$ so the statement is clear.
Suppose that $n \ge 2$ and that the statement holds for all parabolic subalgebras of M_j for $1 \le j \le n-1$.

Claim 3.2.1. There exists a a permutation $\sigma \in S_n$ such that $\phi(D) = \sigma(D)$ for all $D \in \mathcal{D}_n$.

Proof. By Theorem 1.3, there exists $T \in M_n^\times$ such that $\phi(\cdot) = T \cdot T^{-1}$.
The matrix $\phi(\Delta) = T\Delta T^{-1} \in \mathcal{A}_{l_1,\ldots,l_q}$ has eigenvalues $1, \ldots, n$ so by Lemma 2.4 there exists $S \in \mathcal{A}_{l_1,\ldots,l_q}^\times$ and a permutation $\sigma \in S_n$ such that $S\phi(\Delta)S^{-1} = \sigma(\Delta)$. We can pass to the map $S\phi(\cdot)S^{-1}$ to assume without loss of generality that $\phi(\Delta) = \sigma(\Delta)$. For every $1 \le i \le n$ the matrix $\phi(E_{ii}) = TE_{ii}T^{-1}$ is a rank-one idempotent which satisfies

$$E_{ii} \leftrightarrow \Delta \implies \phi(E_{ii}) \leftrightarrow \phi(\Delta) = \sigma(\Delta)$$

so it is diagonal. Therefore, by injectivity, there exists a permutation $\tau \in S_n$ such that $\phi(E_{ii}) = TE_{ii}T^{-1} = E_{\tau(i)\tau(i)}$ for all $1 \le i \le n$. But now we have

$$\sum_{k=1}^n kE_{\sigma(k)\sigma(k)} = \sigma(\Delta) = \phi(\Delta) = \phi\left(\sum_{k=1}^n kE_{kk}\right) = \sum_{k=1}^n kE_{\tau(k)\tau(k)}$$

which implies $\tau = \sigma$. The claim now follows by linearity. □

Claim 3.2.2. For each non-diagonal matrix unit $E_{ij} \in \mathcal{A}_{k_1,\ldots,k_p}$ there exists $c_{ij} \in \mathbb{C}^\times$ such that $\phi(E_{ij}) = c_{ij}E_{\sigma(i)\sigma(j)}$.

Proof. We have

$$\phi(E_{ij}) = \phi(E_{ii}E_{ij}E_{jj}) = \phi(E_{ii})\phi(E_{ij})\phi(E_{jj}) = E_{\sigma(i)\sigma(i)}\phi(E_{ij})E_{\sigma(j)\sigma(j)},$$

so $\phi(E_{ij})$ is nonzero only in the position $(\sigma(i), \sigma(j))$ □

Claim 3.2.3. We have $\sigma(\{1,\dots,k_1\}) \subseteq \{1,\dots,l_1\}$. In particular $k_1 \le l_1$.

Proof. By Claim 3.2.2, for each $1 \le i \le k_1$, the images of the matrix units $E_{i1},\dots,E_{in} \in \mathcal{A}_{k_1,\dots,k_p}$ by injectivity occupy n distinct positions in the $\sigma(i)$-th row. In order for them to be in $\mathcal{A}_{l_1,\dots,l_q}$, it has to be $1 \le \sigma(i) \le l_1$. ☐

Claim 3.2.4. For every $1 \le i \le n$ we have

$$\phi(\mathcal{A}_{k_1,\dots,k_p}^{\leftrightarrow i}) \subseteq \mathcal{A}_{l_1,\dots,l_q}^{\leftrightarrow \sigma(i)}.$$

Proof. For each $X \in \mathcal{A}_{k_1,\dots,k_p}^{\leftrightarrow i}$ we have

$$XE_{ii} = E_{ii}X = 0$$

so applying ϕ and (3.2.1) yields

$$\phi(X)E_{\sigma(i)\sigma(i)} = E_{\sigma(i)\sigma(i)}\phi(X) = 0$$

which proves the claim. ☐

Claim 3.2.5. For every $1 \le i \le n$, the map $\phi_i : \mathcal{A}_{k_1,\dots,k_p}^{\setminus i} \to \mathcal{A}_{l_1,\dots,l_q}^{\setminus \sigma(i)}$ defined as

$$\phi_i(X) = \phi(X^{\sharp(i,0)})^{\flat\sigma(i)}$$

is an algebra monomorphism.

Proof. Follows from Claim 2.1.1. ☐

Consider ϕ_1. Since $1 \le \sigma(1) \le l_1$, this is a map

$$\phi_1 : \mathcal{A}^{\setminus 1} \to \mathcal{B}^{\setminus \sigma(1)}$$

which is a monomorphism of parabolic subalgebras of M_{n-1}. By the inductive hypothesis it follows $\mathcal{A}^{\setminus 1} \subseteq \mathcal{B}^{\setminus \sigma(1)}$. Since $1 \le \sigma(1) \le l_1$, we have $\mathcal{B}^{\setminus \sigma(1)} = \mathcal{B}^{\setminus 1}$ and therefore

$$\mathcal{A} = \begin{array}{r} k_1\left\{\vphantom{\begin{array}{c} * \\ \vdots \\ * \end{array}}\right. \\ \vphantom{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}} \end{array}\left[\begin{array}{c|cccc} * & * & * & \cdots & * \\ \cline{2-5} \vdots & & & & \\ * & & & & \\ 0 & & \mathcal{A}^{\setminus 1} & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right] \subseteq \begin{array}{r} l_1\left\{\vphantom{\begin{array}{c} * \\ \vdots \\ * \end{array}}\right. \\ \vphantom{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}} \end{array}\left[\begin{array}{c|cccc} * & * & * & \cdots & * \\ \cline{2-5} \vdots & & & & \\ * & & & & \\ 0 & & \mathcal{B}^{\setminus 1} & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right] = \mathcal{B}.$$

This shows $\mathcal{A} \subseteq \mathcal{B}$.

Claim 3.2.6. We have $T \in \mathcal{B}^{\times}$.

Proof. Denote $r_1,\dots,r_q$ as in the paragraph preceding this corollary. Suppose that some $0 \le m \le q-1$ satisfies $T \in \mathcal{A}^{\times}_{l_1,\dots,l_m,n-(l_1+\dots+l_m)}$ (for $m=0$ this is merely $\mathcal{A}^{\times}_{0,n} = M_n^{\times}$). We claim that the same holds for $m+1$, i.e. $T \in \mathcal{A}^{\times}_{l_1,\dots,l_{m+1},n-(l_1+\dots+l_{m+1})}$. To be more precise, we need to verify that all the $*$'s exactly below the $l_{m+1} \times l_{m+1}$ block are zero:

$$
\begin{bmatrix}
 & \boxed{l_1 \times l_1} & & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
 & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & & \boxed{l_m \times l_m} & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & 0 & \cdots & 0 & & & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \boxed{l_{m+1} \times l_{m+1}} & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & * & \cdots & * & \ddots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & * & \cdots & * & * & & & \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \boxed{l_q \times l_q} & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & * & \cdots & * & * & & &
\end{bmatrix}
$$

In symbols, this means

(3.1)

$$T_{qt} = 0, \qquad \text{for all} \qquad l_1+\cdots+l_m+l_{m+1}+1 \le q \le n, \qquad l_1+\cdots+l_m+1 \le t \le l_1+\cdots+l_{m+1}.$$

For every

- $l_1 + \cdots + l_m + l_{m+1} + 1 \le q \le n,$
- $r_m \le g \le r_{m+1} - 1,$
- $l_1 + \cdots + l_m + \sum_{u=r_m+1}^{g} k_u + 1 \le j \le n,$
- $l_1 + \cdots + l_m + 1 \le s \le l_1 + \cdots + l_{m+1},$
- $l_1 + \cdots + l_m + \sum_{u=r_m+1}^{g} k_u + 1 \le t \le l_1 + \cdots + l_m + \sum_{u=r_m+1}^{g+1} k_u$

we have

$$E_{tj} \in \mathcal{A}_{k_1,\dots,k_p} \implies TE_{tj}T^{-1} \in \mathcal{A}_{l_1,\dots,l_q} \implies 0 = (TE_{tj}T^{-1})_{qs} = T_{qt}(T^{-1})_{js}.$$

Fix q, t and suppose that $(T^{-1})_{js} = 0$ for all such j, s. Then we would have $\operatorname{di}(T^{-1}) = (X_1, \dots, X_m, X_{m+1})$ where $X_{m+1} \in \mathcal{A}_{l_{m+1}, l_{m+2}+\cdots+l_q}$ singular matrix since its upper left $l_{m+1} \times l_{m+1}$ block's last $k_{g+1} + \cdots + k_{r_{m+1}}$ rows are zero. This would make T^{-1} singular, which is a contradiction.

Therefore, $(T^{-1})_{js} \neq 0$ for at least one such pair of indices j, s. Consequently, $T_{qt} = 0$, verifying (3.1).

The claim follows for $m = q - 1$. $\square$

Suppose now that $\circ = \cdot^t$. We have $(TJ)(JX^tJ)(TJ)^{-1} = TX^tT^{-1} \in \mathcal{B}$ for all $X \in \mathcal{A}$ which implies $(TJ)X(TJ)^{-1} \in \mathcal{B}$ for all $X \in \mathcal{A}^{\odot}$. From the first part of the proof we get $\mathcal{A}^{\odot} \subseteq \mathcal{B}$ and $TJ \in \mathcal{B}^{\times}$, which is equivalent to $T \in \mathcal{B}^{\times} J$.

$\square$

Corollary 3.3. *Let $\mathcal{A}_{k_1,\dots,k_p}$ and $\mathcal{A}_{l_1,\dots,l_q}$ be parabolic subalgebras of M_n.*

(a) $\mathcal{A}_{k_1,\dots,k_p}$ and $\mathcal{A}_{l_1,\dots,l_q}$ are algebra-isomorphic if and only if $(k_1, \dots, k_p) = (l_1, \dots, l_q)$.

(b) $\mathcal{A}_{k_1,\dots,k_p}$ *and* $\mathcal{A}_{l_1,\dots,l_q}$ *are algebra-antiisomorphic if and only if* $(k_1,\dots,k_p) = (l_q,\dots,l_1)$.
(c) $\mathcal{A}_{k_1,\dots,k_p}$ *and* $\mathcal{A}_{l_1,\dots,l_q}$ *are Jordan-isomorphic if and only if* $(k_1,\dots,k_p) \in \{(l_q,\dots,l_1),(l_q,\dots,l_1)\}$.

Proof. Follows directly from Corollary 3.2. □

4. Proof of the main result

We now proceed with the proof of our main result, namely Theorem 1.4. We begin with the following well-known fact:

Remark 4.0.1. Consider the map $k_\cdot : M_n \to \mathbb{C}_{\le n}[x]$ which maps a matrix A to its characteristic polynomial k_A. Then this map is continuous with respect to the standard topologies on M_n and $\mathbb{C}_{\le n}[x]$ as finite-dimensional complex vector spaces. It is not difficult to check that a sequence of polynomials $(p_j)_{j=1}^\infty$ in $\mathbb{C}_{\le n}[x]$ converges to $p \in \mathbb{C}_{\le n}[x]$ (in the standard topology of $\mathbb{C}_{\le n}[x]$) if and only if $p_j \to p$ pointwise.
Suppose $A_j \to A$ in M_n. Then for each fixed $x \in \mathbb{C}$ we have

$$k_{A_j}(x) = \det(A_j - xI) \xrightarrow{j\to\infty} \det(A - xI) = k_A(x)$$

by the continuity of the determinant $\det : M_n \to \mathbb{C}$. It follows $k_{A_j} \to k_A$ pointwise and hence in $\mathbb{C}_{\le n}[x]$ as well.

Proof of Theorem 1.4. Let $\phi : \mathcal{A}_{k_1,\dots,k_r} \to M_n$ be a continuous injective map which preserves commutativity and spectrum.

Claim 1.4.1. ϕ preserves characteristic polynomial.

Proof. ϕ clearly preserves characteristic polynomial on the set of all matrices in $\mathcal{A}_{k_1,\dots,k_r}$ with n distinct eigenvalues. As a consequence of Claim 2.3, this set is dense in $\mathcal{A}_{k_1,\dots,k_r}$. The claim now follows from the continuity of the characteristic polynomial. □

Claim 1.4.2. Without loss of generality we can assume $\phi(\Delta) = \Delta$.

Proof. By Claim 1.4.1, the matrix $\phi(\Delta) \in M_n$ is diagonalizable with eigenvalues $1,\dots,n$ so there exists $S \in M_n^\times$ such that $\phi(\Delta) = S\Delta S^{-1}$. Now we pass to the map $S^{-1}\phi(\cdot)S$. □

1.1. **Diagonal matrices.**

Claim 1.4.3. We have $\phi(\mathcal{D}_n) \subseteq \mathcal{D}_n$, i.e. ϕ maps diagonal matrices to diagonal matrices. Moreover, if $D \in \mathcal{D}_n$, then $\phi(D)$ is a permutation of D.

Proof. For any $D \in \mathcal{D}_n$ we have $D \leftrightarrow \Delta \implies \phi(D) \leftrightarrow \phi(\Delta) = \Delta$ so $\phi(D) \in \mathcal{D}_n$. The second part follows from Claim 1.4.1. □

Claim 1.4.4. We have $\phi(D) = D$ for all $D \in \mathcal{D}_n$.

Proof. This is a standard argument from [28, Lemma 2.1]. For completeness we include the proof. Let $D = \operatorname{diag}(\lambda_1,\dots,\lambda_n) \in \mathcal{D}_n$ be such that all the eigenvalues are distinct. Then we can choose continuous paths $f_k : [0,1] \to \mathbb{C}, 1 \le k \le n$ from k to λ_k such that for all $t \in [0,1]$ the values $f_1(t),\dots,f_n(t)$ are all distinct. Indeed, for $1 \le k \le n$ we can choose a path

$$\alpha_k : [0,1] \to (\mathbb{C} \setminus \{1,\dots,n,\lambda_1,\dots,\lambda_n\}) \cup \{k,\lambda_k\}$$

from k to λ_k and then define

$$f_k(t) = \begin{cases} k, & \text{if } t \in \left[0, \frac{k-1}{n}\right] \\ \alpha_k\left(n\left(t - \frac{k-1}{n}\right)\right), & \text{if } t \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \\ \lambda_k, & \text{if } t \in \left[\frac{k}{n}, 1\right]. \end{cases}$$

Denote

$$d := \min_{t\in[0,1]} \Big\{|f_i(t) - f_j(t)| : 1 \le i, j \le n\Big\} > 0.$$

Notice that the set

$$\begin{aligned} S &= \{t \in [0, 1] : \phi(\operatorname{diag}(f_1(t), \dots, f_n(t))) \ne \operatorname{diag}(f_1(t), \dots, f_n(t))\} \\ &= \{t \in [0, 1] : \|\phi(\operatorname{diag}(f_1(t), \dots, f_n(t))) - \operatorname{diag}(f_1(t), \dots, f_n(t))\|_\infty \ge d\} \end{aligned}$$

is both open and closed in $[0, 1]$. Since $0 \notin S$, by the connectedness of $[0, 1]$ it follows that $S = \emptyset$.
In particular for $t = 1$ we get

$$\begin{aligned} \phi(\operatorname{diag}(\lambda_1, \dots, \lambda_n)) &= \phi(\operatorname{diag}(f_1(t), \dots, f_n(t))) = \operatorname{diag}(f_1(t), \dots, f_n(t)) \\ &= \operatorname{diag}(\lambda_1, \dots, \lambda_n). \end{aligned}$$

The claim follows by density. □

Claim 1.4.5. Let $S \in \mathcal{A}_{k_1,\dots,k_r}^\times$ be arbitrary. Then there exists $T \in M_n^\times$ such that

$$\phi(SDS^{-1}) = TDT^{-1}, \qquad \text{for all diagonal matrices } D \in \mathcal{D}_n.$$

Proof. The matrix $\phi(S\Delta S^{-1})$ is similar to Δ so there exists $T \in M_n^\times$ such that $\phi(S\Delta S^{-1}) = T\Delta T^{-1}$. The map $\mathcal{A}_{k_1,\dots,k_r} \to M_n, T^{-1}\phi(S \cdot S^{-1})T$ now satisfies all the assumptions as ϕ, including $\Delta \mapsto \Delta$ so by Claim 1.4.4 the claim follows. □

Claim 1.4.6. ϕ is a homogeneous map.

Proof. By density it suffices to show that ϕ is homogeneous on the set of all matrices in $\mathcal{A}_{k_1,\dots,k_r}$ with n distinct eigenvalues. This follows directly from Claim 1.4.5. □

1.2. **Base step.** Now we prove Theorem 1.4 completely for $n = 3$. It suffices to prove the theorem for $\mathcal{A} \in \{\mathcal{A}_{1,2}, \mathcal{A}_{1,1,1}\}$. Indeed, the M_3 case is covered by Theorem 1.1, while if $\phi : \mathcal{A}_{2,1} \to M_3$ is a continuous injective commutativity and spectrum preserving map, then

$$X \mapsto \phi(X^\odot) : \mathcal{A}_{1,2} \to M_3$$

is also such a map so the claim follows from the $\mathcal{A}_{1,2}$ case.

Claim 1.4.7. Without loss of generality we can assume that there exist constants $c_{ij} \in \mathbb{C}^\times$ such that

$$\phi(E_{ij}) = c_{ij}E_{ij}, \qquad \text{for all matrix units } E_{ij} \in \mathcal{A}.$$

Proof. We have $E_{12} \leftrightarrow E_{33}$ so by Claim 1.4.4 it follows $\phi(E_{12}) \leftrightarrow \phi(E_{33}) = E_{33}$. By Claim 1.4.1 and injectivity, the matrix $\phi(E_{12})$ is a nonzero nilpotent so we conclude

$$\phi(E_{12}) = \begin{bmatrix} ac & a^2 & 0 \\ -c^2 & -ac & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $a, c \in \mathbb{C}$ not both equal to 0. Analogously we get

$$\phi(E_{13}) = \begin{bmatrix} bd & 0 & b^2 \\ 0 & 0 & 0 \\ -d^2 & 0 & -bd \end{bmatrix}$$

for some $b, d \in \mathbb{C}$ not both equal to 0. Since $E_{12} \leftrightarrow E_{13}$ we have

$$\begin{bmatrix} abcd & 0 & ab^2c \\ -bc^2d & 0 & -b^2c^2 \\ 0 & 0 & 0 \end{bmatrix} = \phi(E_{12})\phi(E_{13}) = \phi(E_{13})\phi(E_{12}) = \begin{bmatrix} abcd & a^2bd & 0 \\ 0 & 0 & 0 \\ -acd^2 & -a^2d^2 & 0 \end{bmatrix}.$$

If $a \neq 0$, then $-a^2d^2 = 0$ implies $d = 0$ so $b \neq 0$. Now $ab^2c = 0$ implies $c = 0$ and therefore

$$\phi(E_{12}) = a^2E_{12}, \quad \phi(E_{13}) = b^2E_{13}.$$

If however $c \neq 0$, then $-b^2c^2 = 0$ implies $b = 0$ so $d \neq 0$. Now $a^2bd = 0$ implies $a = 0$ and therefore

$$\phi(E_{12}) = -c^2E_{21}, \quad \phi(E_{13}) = -d^2E_{31}.$$

Without loss of generality we can suppose that we are in the first case, as otherwise we can pass to the map $\phi(\cdot)^t$.
As above we obtain

$$\phi(E_{23}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ef & e^2 \\ 0 & -f^2 & -ef \end{bmatrix}$$

for some $e, f \in \mathbb{C}$ not both equal to 0. By using $E_{23} \leftrightarrow E_{13}$ it follows that $f = 0$ and therefore $\phi(E_{23}) = e^2E_{23}$.
This settles the matter for $\mathcal{A} = \mathcal{T}_3$. In the case of $\mathcal{A} = \mathcal{A}_{1,3}$, similarly as above using $E_{32} \leftrightarrow E_{11}, E_{12}$ we obtain $\phi(E_{32}) = c_{32}E_{32}$ for some $c_{32} \in \mathbb{C}^\times$. □

Claim 1.4.8. Without loss of generality we can assume that $c_{12} = c_{13} = 1$.

Proof. We can consider the map

$$\operatorname{diag}(1, c_{12}, c_{13})\phi(\cdot)\operatorname{diag}(1, c_{12}, c_{13})^{-1}.$$

Now we have

$$\operatorname{diag}(1, c_{12}, c_{13})\phi(E_{1j})\operatorname{diag}(1, c_{12}, c_{13})^{-1} = E_{1j}, \quad 2 \leq j \leq 3.$$

□

Claim 1.4.9. ϕ acts as identity on all matrices of the form

$$\begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. We introduce

$$R(\alpha, x, y) = \begin{bmatrix} \alpha & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If we denote

$$S(u, v) = \begin{bmatrix} 1 & -u & -v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have $S(u, v)^{-1} = S(-u, -v)$ so by assuming $\alpha \neq 0$ we obtain

$$R(\alpha, x, y) = \alpha S\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) E_{11} S\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right)^{-1},$$

$$A_1(\alpha,x) := \begin{bmatrix} 0 & -x & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} = \alpha S\left(\frac{x}{\alpha},\frac{y}{\alpha}\right) E_{22} S\left(\frac{x}{\alpha},\frac{y}{\alpha}\right)^{-1},$$

$$A_2(\alpha,y) := \begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \alpha S\left(\frac{x}{\alpha},\frac{y}{\alpha}\right) E_{33} S\left(\frac{x}{\alpha},\frac{y}{\alpha}\right)^{-1}.$$

We have

$$A_1(\alpha,x) \leftrightarrow E_{13}, E_{33}, \quad A_2(\alpha,y) \leftrightarrow E_{12}, E_{22}$$

and therefore

$$\phi(A_1(\alpha,x)) \leftrightarrow E_{13}, E_{33}, \quad \phi(A_2(\alpha,y)) \leftrightarrow E_{12}, E_{22}$$

which implies

$$\phi(A_1(\alpha,x)) = \begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{bmatrix}, \quad \phi(A_2(\alpha,y)) = \begin{bmatrix} e & 0 & g \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix}.$$

By Claim 1.4.1, both diagonals consist of $\alpha, 0, 0$ in some order so there exists continuous functions $a_{12}, a_{13} : \mathbb{C}^\times \times \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned}\phi(A_1(\alpha,x)) &= \begin{bmatrix} 0 & -a_{12}(\alpha,x) & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} = A_1(\alpha, a_{12}(\alpha,x)) \\ &= \alpha S\left(\frac{a_{12}(\alpha,x)}{\alpha}, \frac{a_{13}(\alpha,y)}{\alpha}\right) E_{22} S\left(\frac{a_{12}(\alpha,x)}{\alpha}, \frac{a_{13}(\alpha,y)}{\alpha}\right)^{-1},\end{aligned}$$

$$\begin{aligned}\phi(A_2(\alpha,y)) &= \begin{bmatrix} 0 & 0 & -a_{13}(\alpha,y) \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = A_2(\alpha, a_{13}(\alpha,y)) \\ &= \alpha S\left(\frac{a_{12}(\alpha,x)}{\alpha}, \frac{a_{13}(\alpha,y)}{\alpha}\right) E_{33} S\left(\frac{a_{12}(\alpha,x)}{\alpha}, \frac{a_{13}(\alpha,y)}{\alpha}\right)^{-1}.\end{aligned}$$

Furthermore, we have

$$A_1(\alpha,x) \xrightarrow{\alpha\to 0} A_1(0,x) \implies \phi(A_1(\alpha,x)) \xrightarrow{\alpha\to 0} \phi(A_1(0,x)) = \phi(-xE_{12}) = -xE_{12}$$

but simultaneously also

$$\phi(A_1(\alpha,x)) = A_1(\alpha, a_{12}(\alpha,x)) = \alpha E_{22} - a_{12}(\alpha,x)E_{12}$$

which implies $a_{12}(\alpha,x) \xrightarrow{\alpha\to 0} x$. Therefore, we can define $a_{12}(0,x) = x$ to obtain a continuous function $a_{12} : \mathbb{C}^2 \to \mathbb{C}$ which satisfies $\phi(A_1(\alpha,x)) = \alpha E_{22} - a_{12}(\alpha,x)E_{12}$ for all $\alpha, x \in \mathbb{C}$. Analogously we define $a_{13}(0,y) = y$ and by doing so obtain a continuous function $a_{13} : \mathbb{C}^2 \to \mathbb{C}$ which for all $\alpha, y \in \mathbb{C}$ satisfies $\phi(A_2(\alpha,y)) = \alpha E_{33} - a_{13}(\alpha,y)E_{13}$.
By $E_{11} \leftrightarrow E_{22}, E_{33}$, when $\alpha \neq 0$ we also have

$$R(\alpha,x,y) \leftrightarrow A_1(\alpha,x), A_2(\alpha,y) \implies \phi(R(\alpha,x,y)) \leftrightarrow A_1(\alpha, a_{12}(\alpha,x)), A_2(\alpha, a_{13}(\alpha,y))$$

which implies

$$S\left(\frac{a_{12}(\alpha,x)}{\alpha}, \frac{a_{13}(\alpha,y)}{\alpha}\right)^{-1} \phi(R(\alpha,x,y)) S\left(\frac{a_{12}(\alpha,x)}{\alpha}, \frac{a_{13}(\alpha,y)}{\alpha}\right) \leftrightarrow E_{22}, E_{33}$$

so on the left hand side is a diagonal matrix $D(\alpha, x, y)$. Therefore we have

$$\phi(R(\alpha, x, y)) = S\left(\frac{a_{12}(\alpha, x)}{\alpha}, \frac{a_{13}(\alpha, y)}{\alpha}\right) D(\alpha, x, y) S\left(\frac{a_{12}(\alpha, x)}{\alpha}, \frac{a_{13}(\alpha, y)}{\alpha}\right)^{-1}.$$

The diagonal of $D(\alpha, x, y)$ consists of $\alpha, 0, 0$ in some order so the only options are

$$D(\alpha, x, y) \in \{\alpha E_{11}, \alpha E_{22}, \alpha E_{33}\}.$$

Furthermore, the diagonal elements are continuous in α, x, y.
Fix $\alpha \neq 0$. Then $D(\alpha, \cdot, \cdot)_{11}, D(\alpha, \cdot, \cdot)_{22}, D(\alpha, \cdot, \cdot)_{33} : \mathbb{C}^2 \to \{0, \alpha\}$ are continuous functions and hence constant. Therefore, $D(\alpha, x, y)$ is constant with respect to $x, y \in \mathbb{C}$. Supposing $D(\alpha, x, y) = \alpha E_{22}$, in particular for $x = y = 0$ we would obtain

$$\begin{aligned}
\alpha E_{11} &= \phi(\alpha E_{11}) = \phi(R(\alpha, 0, 0)) \\
&= \alpha S\left(\frac{a_{12}(\alpha, 0)}{\alpha}, \frac{a_{13}(\alpha, 0)}{\alpha}\right) E_{22} S\left(\frac{a_{12}(\alpha, 0)}{\alpha}, \frac{a_{13}(\alpha, 0)}{\alpha}\right)^{-1} \\
&= A_1(\alpha, a_{12}(\alpha, 0)) = \phi(A_1(\alpha, 0)) = \phi(\alpha E_{22}) = \alpha E_{22},
\end{aligned}$$

which is a contradiction. Analogously $D(\alpha, x, y) = \alpha E_{33}$ leads to a contradiction so we conclude that $D(\alpha, x, y) = \alpha E_{11}$ for all $x, y \in \mathbb{C}$. It follows

$$\begin{aligned}
\phi(R(\alpha, x, y)) &= \alpha S\left(\frac{a_{12}(\alpha, x)}{\alpha}, \frac{a_{13}(\alpha, y)}{\alpha}\right) E_{11} S\left(\frac{a_{12}(\alpha, x)}{\alpha}, \frac{a_{13}(\alpha, y)}{\alpha}\right)^{-1} \\
&= R(\alpha, a_{12}(\alpha, x), a_{13}(\alpha, y))
\end{aligned}$$

for all $x, y \in \mathbb{C}$ and $\alpha \neq 0$. In particular we have

$$\begin{aligned}
\phi(R(0, x, y)) &= \lim_{\alpha \to 0} \phi(R(\alpha, x, y)) = \lim_{\alpha \to 0} R(\alpha, a_{12}(\alpha, x), a_{13}(\alpha, y)) = R(0, a_{12}(0, x), a_{13}(0, y)) \\
&= R(0, x, y).
\end{aligned}$$

We also consider the matrix

$$A_3(x, y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -y \\ 0 & 0 & x \end{bmatrix}$$

We have $A_3(x, y) \leftrightarrow E_{11}, E_{21}$ and therefore

$$\phi(A_3(x, y)) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

whose diagonal consists of $x, 0, 0$ in some order. We conclude that there exists a continuous function $a_{23} : \mathbb{C}^2 \to \mathbb{C}$ which for all $x, y \in \mathbb{C}$ satisfies

$$\phi(A_3(x, y)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_{23}(x, y) \\ 0 & 0 & x \end{bmatrix} = A_3(x, a_{23}(x, y)).$$

We have $R(\alpha, x, y) \leftrightarrow A_3(x, y)$ and therefore $\phi(R(\alpha, x, y)) \leftrightarrow \phi(A_3(x, y))$ meaning

$$\begin{bmatrix} \alpha & a_{12}(\alpha, x) & a_{13}(\alpha, y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_{23}(x, y) \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_{23}(x, y) \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} \alpha & a_{12}(\alpha, x) & a_{13}(\alpha, y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which immediately yields the following identity:

$$a_{12}(\alpha, x) a_{23}(x, y) - a_{13}(\alpha, y) x = 0.$$

Letting $\alpha \to 0$ it reduces to

$$xa_{23}(x, y) - yx = a_{12}(0, x)a_{23}(x, y) - a_{13}(0, y)x = 0$$

so we conclude $a_{23}(x, y) = y$ for all $x, y \in \mathbb{C}$. Consequently

$$\phi(A_3(x, y)) = A_3(x, y).$$

The above identity now states

$$a_{12}(\alpha, x)y - a_{13}(\alpha, y)x = 0$$

so for all $x, y \in \mathbb{C}^\times$ we have

$$\frac{a_{13}(\alpha, y)}{y} = \frac{a_{12}(\alpha, x)}{x}.$$

The left hand side is independent of x, and the right hand side is independent of y so both sides are in fact equal to some constant $C(\alpha) \in \mathbb{C}$.
For all $\alpha, x, y \in \mathbb{C}^\times$ the homogeneity of ϕ (Claim 1.4.6) implies

$$\begin{aligned}\phi(R(\alpha, x, y)) &= \alpha\phi\left(R\left(1, \frac{x}{\alpha}, \frac{y}{\alpha}\right)\right) = \alpha R\left(1, C(1)\frac{x}{\alpha}, C(1)\frac{y}{\alpha}\right) \\ &= R\left(\alpha, C(1)x, C(1)y\right) = R(\alpha, C(1)x, C(1)y).\end{aligned}$$

We have

$$\phi(R(\alpha, x, y)) \xrightarrow{\alpha\to 0} \phi(R(0, x, y)) = R(0, x, y),$$

and on the other hand

$$\phi(R(\alpha, x, y)) = R(\alpha, C(1)x, C(1)y) \xrightarrow{\alpha\to 0} R(0, C(1)x, C(1)y)$$

which implies $C(1) = 1$.
Therefore, for all $\alpha, x, y \in \mathbb{C}^\times$ we have

$$\phi(R(\alpha, x, y)) = R(\alpha, x, y)$$

so the continuity of ϕ implies that the above identity holds for all $\alpha, x, y \in \mathbb{C}$.
As a byproduct of relations $a_{12}(\alpha, x) = x$ and $a_{13}(\alpha, y) = y$ we also obtain

$$\phi(A_1(\alpha, x)) = A_1(\alpha, x), \qquad \phi(A_2(\alpha, y)) = A_2(\alpha, y).$$

$\square$

Claim 1.4.10. ϕ acts as identity on all matrices of the form

$$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}.$$

Proof. Denote

$$V(x, y, z) = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & z \end{bmatrix}.$$

Consider the map

$$\psi : \mathcal{T}_3 \to M_3, \qquad \psi(X) = \phi(X^\odot)^\odot$$

which, by Claim 2.1, satisfies all assumptions as ϕ, including $\Delta \mapsto \Delta$. Notice that $V(x, y, z) = R(z, y, x)^\odot$ so we have

$$\phi(V(x, y, z))^\odot = \psi(R(z, y, x)) \stackrel{\text{Claim 1.4.9}}{=} R(z, y, x)$$

implying $\phi(V(x, y, z)) = V(x, y, z)$.

$\square$

Claim 1.4.11. We have
$$\phi\left(\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}\right)=\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}.$$

Proof. By commuting with E_{11} it is clear that
$$\phi\left(\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}\right)=\begin{bmatrix}*&0&0\\0&*&*\\0&*&*\end{bmatrix}.$$
Without loss of generality we can assume that the lower right 2×2 block B of the left hand side matrix is diagonalizable with nonzero distinct eigenvalues λ_2,λ_3. By Claim 2.4, there exists $T\in M_2^\times$ or $\mathcal{T}_2^\times$ (for concreteness, we can assume the latter) such that $TBT^{-1}=\operatorname{diag}(\lambda_2,\lambda_3)$. Let $\gamma:[0,1]\to\mathcal{T}_2^\times$ be a path from I to T. Then
$$[0,1]\to\{0,\lambda_2,\lambda_3\}:\qquad t\mapsto\phi(\operatorname{diag}(0,\gamma(t)B\gamma(t)^{-1}))_{11}$$
is continuous and hence constant. But, by Claim 1.4.4, for $t=1$ it is equal to $\phi(\operatorname{diag}(0,\lambda_2,\lambda_3))_{11}=0$ so it is zero for all $t\in[0,1]$. In particular, for $t=0$ the claim follows. □

Claim 1.4.12. ϕ acts as identity on all rank-one matrices in $\mathcal{A}$ of the form
$$\begin{bmatrix}0&*&*\\0&*&*\\0&*&*\end{bmatrix}.$$

Proof. For fixed $x=(x_1,x_2,x_3)\in\mathbb{C}^3$ and $\lambda_2,\lambda_3\in\mathbb{C}$ let $A(x,\lambda_2,\lambda_3)\in\mathcal{A}$ be the matrix
$$A(x,\lambda_2,\lambda_3)=\begin{bmatrix}0&\lambda_2x&\lambda_3x\end{bmatrix}=\begin{bmatrix}0&\lambda_2x_1&\lambda_3x_1\\0&\lambda_2x_2&\lambda_3x_2\\0&\lambda_2x_3&\lambda_3x_3\end{bmatrix}$$
columnwise. For $v=(v_1,v_2,v_3)\in\mathbb{C}^3$ let $R(v)$ be as in Claim 1.4.9. We have
$$A(x,\lambda_2,\lambda_3)R(v)=0,\qquad R(v)A(x,\lambda_2,\lambda_3)=\begin{bmatrix}0&\lambda_2v^tx&\lambda_3v^tx\\0&0&0\\0&0&0\end{bmatrix}.$$
Therefore, assuming $v^tx=0$, we have
$$A(x,\lambda_2,\lambda_3)\leftrightarrow R(v)\implies\phi(A(x,\lambda_2,\lambda_3))\leftrightarrow\phi(R(v))\overset{\text{Claim 1.4.9}}{=}R(v).$$
If we denote
$$\phi(A(x,\lambda_2,\lambda_3))=\begin{bmatrix}S_1&S_2&S_3\end{bmatrix}=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix},$$
we have
$$\begin{bmatrix}v^tS_1&v^tS_2&v^tS_3\\0&0&0\\0&0&0\end{bmatrix}=R(v)\phi(A)=\phi(A)R(v)=\begin{bmatrix}b_{11}v^t\\b_{21}v^t\\b_{31}v^t\end{bmatrix}.$$
We can assume $v\neq 0$ to immediately obtain $b_{21}=b_{31}=0$. Now, b_{11} is an element of the spectrum of $\phi(A(x,\lambda_2,\lambda_3))$, and hence either 0 or $\operatorname{Tr}A(x,\lambda_2,\lambda_3)=\lambda_2x_2+\lambda_3x_3$.
Consider the map
$$\mathbb{C}\to\{0,\lambda_2x_2+\lambda_3x_3\},\qquad t\mapsto\phi(A((t,x_2,x_3),\lambda_2,\lambda_3))_{11}$$

which is continuous and hence constant. By Claim 1.4.11 have $0 \mapsto 0$ and hence $x_1 \mapsto 0$ as well. The latter precisely means $b_{11} = \phi(A(x, \lambda_2, \lambda_3))_{11} = 0$. In particular it follows $S_1 = 0$ and $v^t S_2 = v^t S_3 = 0$. Since v was an arbitrary nonzero vector with the property $v^t x = 0$, it follows that $S_2, S_3 \in \operatorname{span}\{x\}$. Therefore let us write

$$\phi(A(x, \lambda_2, \lambda_3)) = \begin{bmatrix} 0 & \mu_2 x & \mu_3 x \end{bmatrix} = A(x, \mu_2, \mu_3)$$

for some scalars $\mu_2, \mu_3 \in \mathbb{C}$. Since $A(x, \lambda_2, \lambda_3)$ and $\phi(A(x, \lambda_2, \lambda_3))$ are rank-one matrices, by Claim 1.4.1 it follows

$$\lambda_2 x_2 + \lambda_3 x_3 = \operatorname{Tr} A(x, \lambda_2, \lambda_3) = \operatorname{Tr} \phi(A(x, \lambda_2, \lambda_3)) = \mu_2 x_2 + \mu_3 x_3. \tag{1.1}$$

Now we additionally assume $x_1 \neq 0$ and $\lambda_2 x_2 + \lambda_3 x_3 \neq 0$ (we will remove these assumptions at the end of the proof).

Notice that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 x_3 & \lambda_3 x_2 \\ 0 & \lambda_2 x_3 & -\lambda_2 x_2 \end{bmatrix} A(x, \lambda_2, \lambda_3) = A(x, \lambda_2, \lambda_3) \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 x_3 & \lambda_3 x_2 \\ 0 & \lambda_2 x_3 & -\lambda_2 x_2 \end{bmatrix} = 0$$

and therefore

$$\phi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 x_3 & \lambda_3 x_2 \\ 0 & \lambda_2 x_3 & -\lambda_2 x_2 \end{bmatrix}\right) \phi(A(x, \lambda_2, \lambda_3)) = \phi(A(x, \lambda_2, \lambda_3)) \phi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 x_3 & \lambda_3 x_2 \\ 0 & \lambda_2 x_3 & -\lambda_2 x_2 \end{bmatrix}\right). \tag{1.2}$$

By the above part of the proof, there exist $y_2, y_3 \in \mathbb{C}$ such that

$$\begin{aligned} \phi\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 x_3 & \lambda_3 x_2 \\ 0 & \lambda_2 x_3 & -\lambda_2 x_2 \end{bmatrix}\right) &= \phi(A(0, -\lambda_3, \lambda_2), x_3, -x_2)) = A((0, -\lambda_3, \lambda_2), y_3, -y_2) \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 y_3 & \lambda_3 y_2 \\ 0 & \lambda_2 y_3 & -\lambda_2 y_2 \end{bmatrix}. \end{aligned}$$

Moreover, since $-(\lambda_3 x_3 + \lambda_2 x_2) \neq 0$, by injectivity it cannot be $y_2 = y_3 = 0$. Therefore (1.2) reduces to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 y_3 & \lambda_3 y_2 \\ 0 & \lambda_2 y_3 & -\lambda_2 y_2 \end{bmatrix} \begin{bmatrix} 0 & \mu_2 x_1 & \mu_3 x_1 \\ 0 & \mu_2 x_2 & \mu_3 x_2 \\ 0 & \mu_2 x_3 & \mu_3 x_3 \end{bmatrix} = \begin{bmatrix} 0 & \mu_2 x_1 & \mu_3 x_1 \\ 0 & \mu_2 x_2 & \mu_3 x_2 \\ 0 & \mu_2 x_3 & \mu_3 x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3 y_3 & \lambda_3 y_2 \\ 0 & \lambda_2 y_3 & -\lambda_2 y_2 \end{bmatrix}$$

which yields

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} = \begin{bmatrix} 0 & (\lambda_2\mu_3 - \lambda_3\mu_2) x_1 y_3 & -(\lambda_2\mu_3 - \lambda_3\mu_2) x_1 y_2 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

and consequently $\lambda_2 \mu_3 = \lambda_3 \mu_2$. Now using (1.1) we obtain

$$\begin{aligned} (\lambda_2 x_2 + \lambda_3 x_3)\mu_2 &= (\lambda_2\mu_2) x_2 + (\lambda_3\mu_2) x_3 = (\lambda_2\mu_2) x_2 + (\lambda_2\mu_3) x_3 = \lambda_2(\mu_2 x_2 + \mu_3 x_3) \\ &\overset{(1.1)}{=} \lambda_2(\lambda_2 x_2 + \lambda_3 x_3), \end{aligned}$$

$$\begin{aligned} (\lambda_2 x_2 + \lambda_3 x_3)\mu_3 &= (\lambda_2\mu_3) x_2 + (\lambda_3\mu_3) x_3 = (\lambda_3\mu_2) x_2 + (\lambda_3\mu_3) x_3 = \lambda_3(\mu_2 x_2 + \mu_3 x_3) \\ &\overset{(1.1)}{=} \lambda_3(\lambda_2 x_2 + \lambda_3 x_3) \end{aligned}$$

and since $\lambda_2 x_2 + \lambda_3 x_3 \neq 0$ it follows $\mu_2 = \lambda_2$ and $\mu_3 = \lambda_3$. Therefore $\phi(A(x, \lambda_2, \lambda_3)) = A(x, \lambda_2, \lambda_3)$.
This completes the proof for all $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and $(\lambda_2, \lambda_3) \in \mathbb{C}^2$ such that $x_1 \neq 0$ and $\lambda_2 x_2 + \lambda_3 x_3 \neq 0$. Since these form a dense set, the claim follows for all $x_1, x_2, x_3, \lambda_2, \lambda_3 \in \mathbb{C}$. □

Claim 1.4.13. Let $A \in \mathcal{A}$ be a matrix of rank one. Then one of the two options holds:
(a)

$$A = \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(b)

$$A = \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

Proof. Assume that A is not supported only in the first row. Then in the second or third column there exists a nonzero element at the second or third position, and the first column has to be a multiple of that column. Since the first column has zeroes at the second and third positions, it follows that it is the zero-multiple of that column, i.e. it is the zero column. □

Claim 1.4.14. ϕ acts as identity on rank-one matrices.

Proof. Follows from Claims 1.4.13, 1.4.9 and 1.4.12. □

Claim 1.4.15. ϕ is the identity map.

Proof. By density, it suffices to prove that ϕ is the identity map on the set of matrices in $\mathcal{A}$ with 3 distinct eigenvalues. Let $S \in \mathcal{A}^\times$ be arbitrary. By Claim 1.4.5, there exists $T \in M_3^\times$ such that

$$\phi(SDS^{-1}) = TDT^{-1}, \qquad \text{for all diagonal matrices } D \in \mathcal{D}_3.$$

By Claim 1.4.14 we also have

$$SE_{jj}S^{-1} = \phi(SE_{jj}S^{-1}) = TE_{jj}T^{-1}, \qquad 1 \leq j \leq 3$$

so by linearity it must be $TDT^{-1} = SDS^{-1}$ and consequently

$$\phi(SDS^{-1}) = TDT^{-1} = SDS^{-1}$$

for all diagonal matrices $D \in \mathcal{D}_3$. This proves the claim. □

1.3. **Inductive step.**

Claim 1.4.16. Let $1 \leq s \leq n$ and let $A \in \mathcal{A}_{k_1,\ldots,k_r}$ be a matrix with s-th row and s-th column equal to zero, allowing perhaps that $A_{ss} \neq 0$. Then $\phi(A)$ has the same property:

$$\phi\left(\begin{bmatrix} A^I_{(s-1)\times(s-1)} & 0 & A^{II}_{(s-1)\times(n-s)} \\ 0 & *_{1\times 1} & 0 \\ A^{III}_{(n-s)\times(s-1)} & 0 & A^{IV}_{(n-s)\times(n-s)} \end{bmatrix}\right) = \begin{bmatrix} *_{(s-1)\times(s-1)} & 0 & *_{(s-1)\times(n-s)} \\ 0 & *_{1\times 1} & 0 \\ *_{(n-s)\times(s-1)} & 0 & *_{(n-s)\times(n-s)} \end{bmatrix}.$$

Proof. We have

$$A \leftrightarrow E_{ss} \implies \phi(A) \leftrightarrow \phi(E_{ss}) = E_{ss} \implies A = \begin{bmatrix} *_{(s-1)\times(s-1)} & 0 & *_{(s-1)\times(n-s)} \\ 0 & *_{1\times 1} & 0 \\ *_{(n-s)\times(s-1)} & 0 & *_{(n-s)\times(n-s)} \end{bmatrix}.$$

□

Claim 1.4.17. Let $1 \le s \le n$ and let $D = \operatorname{diag}(d_1, \dots, d_n) \in \mathcal{D}_n$ be a diagonal matrix. Then for all $R \in \mathcal{A}_{k_1,\dots,k_r}^{\times}$, $R \leftrightarrow E_{ss}$, $R_{ss} = 1$ we have

$$\phi(RDR^{-1}) = \begin{bmatrix} *_{(s-1)\times(s-1)} & 0 & *_{(s-1)\times(n-s)} \\ 0 & d_s & 0 \\ *_{(n-s)\times(s-1)} & 0 & *_{(n-s)\times(n-s)} \end{bmatrix}.$$

Proof. By Claim 1.4.16 it only remains to prove that the element at the position (s, s) is equal to d_s, independently of R. To this end, note that the map given by

$$(\mathcal{A}_{k_1,\dots,k_r}^{\setminus s})^{\times} \to \sigma(D), \qquad R \mapsto \phi\left(R^{\sharp(s,1)} D (R^{\sharp(s,1)})^{-1}\right)_{ss}$$

is continuous and hence constant and equal to its value in the identity I. ☐

Claim 1.4.18. Let $1 \le s \le n$ and let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a matrix with s-th row and s-th column equal to zero. Then $\phi(A)$ has the same property.

Proof. We already know that $\phi(A) \leftrightarrow E_{ss}$ (Claim 1.4.16), so it remains to show that $\phi(A)_{ss} = 0$. By density, it suffices to prove the claim for matrices A with n distinct eigenvalues. Hence, Claim 2.4 implies that there exists $R \in \mathcal{A}_{k_1,\dots,k_r}^{\times}$ such that $R \leftrightarrow E_{ss}$, $R_{ss} = 1$ and RAR^{-1} is a diagonal matrix with $(RAR^{-1})_{ss} = 0$. Therefore

$$\phi(A)_{ss} \overset{\text{Claim 1.4.17}}{=} \phi(RAR^{-1})_{ss} \overset{\text{Claim 1.4.4}}{=} (RAR^{-1})_{ss} = 0.$$

☐

By way of induction, suppose that $n \ge 4$ and that Theorem 1.4 holds for $n - 1$.

Claim 1.4.19. There exists an invertible diagonal matrix $\Lambda \in \mathcal{D}_n^{\times}$ and a map $\circ \in \{\mathrm{id}, \cdot^t\}$ such that

$$\phi(E_{ij}) = \Lambda E_{ij}^{\circ} \Lambda^{-1}, \qquad \text{for all } E_{ij} \in \mathcal{A}_{k_1,\dots,k_r}.$$

Proof. For $1 \le s \le n$ consider the map

$$\phi_s : \mathcal{A}_{k_1,\dots,k_r}^{\setminus s} \to M_{n-1}, \qquad \phi_s(A) = \phi(A^{\sharp(s,0)})^{\flat s}.$$

For each such s, the map ϕ_s is continuous, injective, and preserves spectrum and commutativity so by the inductive hypothesis we conclude:

- There exists $T_s \in M_{n-1}^{\times}$ and $\circ_s \in \{\mathrm{id}, \cdot^t\}$ such that $\phi_s(X) = T_s X^{\circ_s} T_s^{-1}$ for all $X \in \mathcal{A}_{k_1,\dots,k_r}^{\setminus s}$.

By Claim 1.4.4, we have

$$E_{jj} = \phi_s(E_{jj}) = T_s E_{jj}^{\circ_s} T_s^{-1} = T_s E_{jj} T_s^{-1}$$

for all $1 \le j \le n-1$ which implies that the matrix T_s is diagonal. Therefore, for all $1 \le s \le n$ we can denote

$$T_s = \operatorname{diag}(t_1^s, \dots, t_{s-1}^s, t_{s+1}^s, \dots, t_n^s)$$

for some $t_1^s, \dots, t_{s-1}^s, t_{s+1}^s, \dots, t_n^s \in \mathbb{C}^{\times}$.
By passing to the map $\mathcal{A}_{k_1,\dots,k_r} \to M_n$ given by

$$\left(\begin{bmatrix} T_n & 0 \\ 0 & 1 \end{bmatrix}^{-1} \phi(\cdot) \begin{bmatrix} T_n & 0 \\ 0 & 1 \end{bmatrix} \right)^{\circ_n}$$

which satisfies all the same assumptions as ϕ including $\Delta \mapsto \Delta$, we can assume that $T_n = I$. In other words

$$\phi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{for all } X \in \mathcal{A}_{k_1,\dots,k_r}^{\setminus n} = \mathcal{A}_{k_1,\dots,k_{r-1},k_r-1}. \tag{1.3}$$

Let us fix $1 \le s \le n$. By considering cases whether i, j are less or greater than s, it is not hard to show that for all indices $i, j \in \{1, \dots, n\} \setminus \{s\}$ such that $E_{ij} \in \mathcal{A}_{k_1,\dots,k_r}^{\setminus s}$ we have

$$\phi(E_{ij}) = \left(\frac{t_i^s}{t_j^s}\right)^{\pm 1} E_{ij}^{\circ_s} \tag{1.4}$$

Suppose $1 \le s \le n-1$. Then for all distinct indices $i, j \in \{1, \dots, n-1\} \setminus \{s\}$ such that $E_{ij} \in \mathcal{A}_{k_1,\dots,k_r}$ we have

$$E_{ij} \overset{(1.3)}{=} \phi(E_{ij}) \overset{(1.4)}{=} \left(\frac{t_i^s}{t_j^s}\right)^{\pm 1} E_{ij}^{\circ_s}.$$

which implies that $\circ_s$ is the identity map and that $t_i^s = t_j^s$ for all such (i, j). We conclude that $t_i^s = t_j^s$ for all $i, j \in \{1, \dots, n-1\} \setminus \{s\}$. Without loss of generality we can assume that $t_i^s = 1$ for all $i \in \{1, \dots, n-1\} \setminus \{s\}$.

Now fix $1 \le p \ne s \le n-1$ and choose $q \in \{1, \dots, n-1\} \setminus \{p, s\}$. Then we have

$$\frac{1}{t_n^p} E_{qn} = \frac{t_q^p}{t_n^p} E_{qn} \overset{(1.4)}{=} \phi(E_{qn}) \overset{(1.4)}{=} \frac{t_q^s}{t_n^s} E_{qn} = \frac{1}{t_n^s} E_{qn}$$

so $t_n^p = t_n^s$ which we can denote simply by t_n. In other words, we assumed that for all $1 \le s \le n-1$ we have

$$T_s = \operatorname{diag}(1, \dots, 1, t_n)$$

and $T_n = I$.

Now it is easy to see that if we set $\Lambda = \operatorname{diag}(1, \dots, 1, t_n) \in \mathcal{D}_n^\times$, we have

$$\phi(E_{ij}) = \Lambda E_{ij} \Lambda^{-1}, \qquad \text{for all } E_{ij} \in \mathcal{A}_{k_1,\dots,k_r}.$$

$\square$

By passing to the map $(\Lambda^{-1}\phi(\cdot)\Lambda)^\circ : \mathcal{A}_{k_1,\dots,k_r} \to M_n$ which satisfies $\Delta \mapsto \Delta$, we can assume that $\phi(E_{ij}) = E_{ij}$ for all matrix units $E_{ij} \in \mathcal{A}_{k_1,\dots,k_r}$. After doing the same proof as for the Claim 1.4.19 on this new ϕ, we obtain the following result:

Claim 1.4.20. Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a matrix with s-th row and s-th column equal to zero for some $1 \le s \le n$. Then $\phi(A) = A$.

Proof. Returning to the proof of Claim 1.4.19, the relation (1.4) now implies that $T_s = I$ and hence ϕ_s is the identity map. Now we have

$$\phi(A) = \phi((A^{\flat s})^{\sharp(s,0)}) = \phi_s(A^{\flat s})^{\sharp(s,0)} = (A^{bs})^{\sharp(s,0)} = A.$$

$\square$

Claim 1.4.21. Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a matrix which is zero outside the first row. Then $\phi(A) = A$.

Proof. Follows as in [27, p. 45]. $\square$

Claim 1.4.22. Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a matrix which is zero outside the n-th column. Then $\phi(A) = A$.

Proof. For a fixed $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ denote

$$V(x) = \begin{bmatrix} 0 & \cdots & 0 & x_1 \\ 0 & \cdots & 0 & x_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & x_n \end{bmatrix}.$$

Consider the map

$$\psi : \mathcal{T}_n \to M_n, \qquad \psi(X) = \phi(X^{\odot})^{\odot}$$

which, by Claim 2.1, satisfies all assumptions as ϕ including $\Delta \mapsto \Delta$. Notice that $V(x_1, \dots, x_n) = R(x_n, \dots, x_1)^{\odot}$ so we have

$$\phi(V(x_1, \dots, x_n))^{\odot} = \psi(R(x_n, \dots, x_1)) \overset{\text{Claim 1.4.9}}{=} R(x_n, \dots, x_1)$$

implying $\phi(V(x_1, \dots, x_n)) = V(x_1, \dots, x_n)$. □

1.4. **Rank-one matrices and conclusion.** In this section we can assume without loss of generality that $k_1 < n$ as $k_1 = n$ implies $\mathcal{A}_{k_1,\dots,k_r} = M_n$.

Claim 1.4.23. Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a rank-one matrix. Then there exists $1 \le s \le r$ such that $\operatorname{supp} A \subseteq [1, k_1 + \cdots + k_s] \times [k_1 + \cdots + k_{s-1} + 1, n]$.

Proof. Let $1 \le s \le r$ be the smallest number such that A exists within rows $1 \le i \le k_1 + \cdots + k_s$. Then there exists a nonzero element $A_{uv} \neq 0$ at the position $k_1 + \cdots + k_{s-1} + 1 \le u \le k_1 + \cdots + k_s$ and $k_1 + \cdots + k_{s-1} + 1 \le v \le n$.
Denote by $S_1, \dots, S_n$ the columns of A. Consider $1 \le j \le k_1 + \cdots + k_{s-1}$. Since A has rank one, there exists $\lambda \in \mathbb{C}$ such that the $S_j = \lambda S_v$. In particular, since $A \in \mathcal{A}_{k_1,\dots,k_r}$, we have

$$0 = A_{uj} = \lambda \underbrace{A_{uv}}_{\neq 0} \implies \lambda = 0$$

so $S_j = 0$. This proves the claim. □

Claim 1.4.24. Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a rank-one matrix and assume that $1 \le s \le r - 1$ has the property that $\operatorname{supp} A \subseteq [1, k_1 + \cdots + k_s] \times [k_1 + \cdots + k_{s-1} + 1, n]$. Then $\phi(A) = A$.

Proof. Every matrix of the above form can be expressed as

$$\Lambda x^t = \begin{bmatrix} \lambda_1 x \\ \vdots \\ \lambda_{k_1 + \cdots + k_s} x \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

for some

$$x = \Big(\underbrace{0, \dots, 0}_{k_1 + \cdots + k_{s-1}}, x_{k_1 + \cdots + k_{s-1} + 1}, \dots, x_n \Big) \in \mathbb{C}^n \setminus \{0\} \tag{1.5}$$

and $\Lambda = (\lambda_1, \dots, \lambda_{k_1 + \cdots + k_s}, 0, \dots, 0) \in \mathbb{C}^n$. Note that the last row of Λx^t is certainly zero since $s \le r - 1$.

For $v=(v_1,\dots,v_n)\in\mathbb{C}^n$ let $V(v)=ve_n^t$ be the matrix from Claim 1.4.22. We have

$$(ve_n^t)(\Lambda x^t)=0, \qquad (\Lambda x^t)(ve_n^t)=(x^tv)\Lambda e_n^t=V(\lambda_1 v^tx,\dots,\lambda_{k_1+\dots+k_s}v^tx,0,\dots,0)$$

so if $v^tx=0$, it follows

$$\Lambda x^t\leftrightarrow ve_n^t \implies \phi(\Lambda x^t)\leftrightarrow\phi(ve_n^t)\overset{\text{Claim 1.4.22}}{=} ve_n^t.$$

If we denote

$$\phi(\Lambda x^t)=\begin{bmatrix}R_1\\ \vdots\\ R_n\end{bmatrix}=\begin{bmatrix}b_{11}&\cdots&b_{1n}\\ \vdots&\ddots&\vdots\\ b_{n1}&\cdots&b_{nn}\end{bmatrix},$$

it follows that

$$V(v^tR_1,\dots,v^tR_n)=\phi(\Lambda x^t)ve_n^t=ve_n^t\phi(\Lambda x^t)=\begin{bmatrix}b_{n1}v & b_{n2}v & \cdots & b_{nn}v\end{bmatrix}. \tag{1.6}$$

We can assume $v\neq 0$ to conclude $b_{n1}=\dots=b_{n,n-1}=0$. Therefore $R_n=b_{nn}e_n$ and we know that b_{nn} is either zero or equal to $\operatorname{Tr}(\Lambda x^t)=\sum_{k_1+\dots+k_{s-1}+1\le j\le k_1+\dots+k_s}\lambda_jx_j$.
The map

$$\mathbb{C}\to\left\{0,\sum_{k_1+\dots+k_{s-1}+1\le j\le k_1+\dots+k_s}\lambda_jx_j\right\},\qquad u\mapsto\phi(\Lambda(x_1,\dots,x_{n-1},u)^t)_{nn}$$

is well-defined and continuous so it is constant. In particular we have

$$b_{nn}=\phi(\Lambda x^t)_{nn}=\phi(\Lambda(x_1,\dots,x_{n-1},0)^t)_{nn}\overset{\text{Claim 1.4.20}}{=}0.$$

Therefore $R_n=0$ and from 1.6 we obtain

$$v^tR_1=\dots=v^tR_{n-1}=0.$$

Since $v\in\mathbb{C}^n$ was an arbitrary nonzero vector such that $v^tx=0$, we conclude $R_1,\dots,R_{n-1}\in\operatorname{span}\{x\}$. Therefore we can write

$$\phi(\Lambda x^t)=\begin{bmatrix}\mu_1x\\ \vdots\\ \mu_{n-1}x\\ 0\end{bmatrix}=Mx^t$$

for some vector $M=(\mu_1,\dots,\mu_{n-1},0)\in\mathbb{C}^n$.

For $y=(y_1,\dots,y_n),z=(z_1,\dots,z_n)\in\mathbb{C}^n$ with $y_n=z_n=0$ and $y^t\Lambda=z^tx=0$ we have

$$(\Lambda x^t)(zy^t)=(zy^t)(\Lambda x^t)=0\implies\Lambda x^t\leftrightarrow zy^t$$

and therefore $Mx^t=\phi(\Lambda x^t)\leftrightarrow\phi(zy^t)$. Furthermore, notice that the matrix zy^t has n-th row and n-th column equal to zero so by Claim 1.4.20 we obtain $\phi(zy^t)=zy^t$.
Now if we assume $z\neq 0$, from $x\neq 0$ we obtain $zx^t\neq 0$ and therefore

$$0=(Mx^t)(zy^t)=(zy^t)(Mx^t)=(y^tM)\underbrace{(zx^t)}_{\neq 0}$$

which implies $y^tM=0$. Since $y\in\mathbb{C}^n$ was an arbitrary vector with the property $y^t\Lambda=0$ and $y_n=0$ (which is not a problem since $M_n=\Lambda_n=0$, so we might as well remove this restriction), it follows that $M \in \operatorname{span}\{\Lambda\}$. The equality

$$\sum_{k_1+\cdots+k_{s-1}+1\le j\le k_1+\cdots+k_s} \lambda_j x_j = \operatorname{Tr}(\Lambda x^t) = \operatorname{Tr}\phi(\Lambda x^t) = \sum_{k_1+\cdots+k_{s-1}+1\le j\le k_1+\cdots+k_s} \mu_j x_j$$

and the fact that $\sum_{k_1+\cdots+k_{s-1}+1\le j\le k_1+\cdots+k_s} \lambda_j x_j \neq 0$ imply that $M = \Lambda$. Therefore $\phi(\Lambda x^t) = \Lambda x^t$.

We have proved the desired claim for all matrices Λx^t where $x \in \mathbb{C}^n \setminus \{0\}$ is of the form (1.5) and $\lambda_1, \dots, \lambda_{k_1+\cdots+k_s} \in \mathbb{C}$ satisfy $\sum_{k_1+\cdots+k_{s-1}+1\le j\le k_1+\cdots+k_s} \lambda_j x_j \neq 0$. Since these form a dense set, the claim follows in general.

□

Claim 1.4.25. Let $A \in \mathcal{A}_{k_1,\dots,k_r}$ be a rank-one matrix and assume that $\operatorname{supp} A \subseteq [1,n] \times [k_1+\cdots+k_{r-1}+1, n]$. Then $\phi(A) = A$.

Proof. Consider the map

$$\psi : \mathcal{A}_{k_r,\dots,k_1} \to M_n, \qquad \psi(X) = \phi(X^\odot)^\odot.$$

By Claim 2.1 we have that ψ satisfies all assumptions as ϕ including $\Delta \mapsto \Delta$. Furthermore, the rank-one matrix $A^\odot$ is now supported only in the first k_r rows so by Claim 1.4.24 it follows $\psi(A^\odot) = A^\odot$ and thus we obtain $\phi(A) = A$. □

Claim 1.4.26. ϕ acts as identity on rank-one matrices.

Proof. Follows from Claims 1.4.23, 1.4.24 (the cases $1 \le s \le r-1$) and 1.4.25 (the case $s = r$). □

Claim 1.4.27. ϕ is the identity map.

Proof. By density, it suffices to prove that ϕ is the identity map on the set of matrices in $\mathcal{A}_{k_1,\dots,k_r}$ with n distinct eigenvalues. Let $S \in \mathcal{A}^\times_{k_1,\dots,k_r}$ be arbitrary. By Claim 1.4.5, there exists $T \in M_n^\times$ such that

$$\phi(SDS^{-1}) = TDT^{-1}, \quad \text{for all diagonal matrices } D \in \mathcal{D}_n.$$

By Claim 1.4.26 we also have

$$SE_{jj}S^{-1} = \phi(SE_{jj}S^{-1}) = TE_{jj}T^{-1}, \qquad 1 \le j \le n$$

so by linearity it must be $TDT^{-1} = SDS^{-1}$ and consequently

$$\phi(SDS^{-1}) = TDT^{-1} = SDS^{-1}$$

for all diagonal matrices $D \in \mathcal{D}_n$. This proves the claim. □

This completes the inductive step. We conclude that Theorem 1.4 holds for all $n \ge 3$.

□

Theorem 4.1. *Let $\mathcal{A}, \mathcal{B} \subseteq M_n$ be two parabolic algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a continuous injective spectrum and commutativity preserving map. Then one of the following is true:*

- *$\mathcal{A} \subseteq \mathcal{B}$ and there exists $T \in \mathcal{B}^\times$ such that $\phi(X) = TXT^{-1}$ for all $X \in \mathcal{A}$.*
- *$\mathcal{A}^\odot \subseteq \mathcal{B}$ and there exists $T \in \mathcal{B}^\times J$ such that $\phi(X) = TX^tT^{-1}$ for all $X \in \mathcal{A}$.*

Proof. Theorem 1.4 implies that there exists $T \in M_n^\times$ and $\circ \in \{\mathrm{id}, \cdot^t\}$ such that $\phi(X) = TX^\circ T^{-1}$ for all $X \in \mathcal{A}$. The rest of the result follows from Proposition 3.2. □

When we further assume that $\mathcal{B} = \mathcal{A}$, so that $\phi : \mathcal{A} \to \mathcal{A}$, we can relax the spectrum preserving assumption to spectrum shrinking ($\sigma(\phi(X)) \subseteq \sigma(X)$ for all $X \in \mathcal{A}$). More precisely, we obtain the following result (similarly as in [31], the proof relies on the invariance of domain theorem).

Corollary 4.2. *Let $\mathcal{A} \subseteq M_n$ be a parabolic algebra and let $\phi : \mathcal{A} \to \mathcal{A}$ be a continuous injective commutativity preserving spectrum shrinking map. Then one of the following is true:*

- *There exists $T \in \mathcal{A}^\times$ such that $\phi(X) = TXT^{-1}$ for all $X \in \mathcal{A}$.*
- *$\mathcal{A}^\odot = \mathcal{A}$ and there exists $T \in \mathcal{A}^\times J$ such that $\phi(X) = TX^tT^{-1}$ for all $X \in \mathcal{A}$.*

Proof. We shall prove that ϕ actually preserves characteristic polynomial so that Theorem 4.1 applies.
By the invariance of domain theorem, the image $\mathcal{R} = \phi(\mathcal{A})$ is an open set in M_n and $\phi|^{\mathcal{R}} : \mathcal{A} \to \mathcal{R}$ is a homeomorphism. Let $\mathcal{E}$ denote the set of all matrices in M_n with n distinct eigenvalues. As $\mathcal{E}$ is dense in M_n, $\mathcal{E} \cap \mathcal{R}$ is dense in $\mathcal{R}$.
Now, since ϕ shrinks spectrum, its inverse $(\phi|^{\mathcal{R}})^{-1}$ expands spectrum. In particular, the restriction $(\phi|^{\mathcal{R}})^{-1}|_{\mathcal{E}\cap\mathcal{R}}$ preserves characteristic polynomial. Since the characteristic polynomial $k_\cdot : M_n \to \mathbb{C}_{\leq n}[x]$ is a continuous map (Remark 4.0.1), we conclude that the continuous maps

$$\mathcal{R} \to \mathbb{C}_{\leq n}[x] : \qquad X \mapsto k_{(\phi|^{\mathcal{R}})^{-1}(X)}, \qquad \text{and} \qquad X \mapsto k_X$$

are equal on the dense set $\mathcal{E} \cap \mathcal{R}$. Hence, they are equal everywhere so $(\phi|^{\mathcal{R}})^{-1}$ preserves characteristic polynomial. The same follows for ϕ, of course. $\square$

5. Counterexamples

We show the optimality of Theorem 1.4 via counterexamples. In short, all assumptions except injectivity are indispensable for all parabolic algebras except M_n, while injectivity is superfluous in the M_n case and necessary in all other cases. We assume $n \geq 3$ unless stated otherwise and let $\mathcal{A} \subseteq M_n$ be an arbitrary parabolic algebra not equal to M_n.

Example 5.1 (Spectrum shrinking is necessary)**.** Let D be the open unit disk in $\mathbb{C}$ and let

$$g : D \to D, \qquad g(z) = \frac{1-3z}{3-z}.$$

It is not difficult to check that g is a holomorphic bijection (actually, it is an involution). Consider the map

$$\phi : \mathcal{A} \to M_n, \qquad \phi(X) = g\left(\frac{X}{1+\|X\|}\right).$$

This is well-defined, as for each $X \in \mathcal{A}$ the matrix $\frac{X}{1+\|X\|}$ has norm < 1 and hence its spectrum is contained in D at which point we can apply g using the holomorphic functional calculus. Using the properties of the holomorphic functional calculus we conclude that ϕ is continuous and preserves commutativity. Moreover, since the map $X \mapsto \frac{X}{1+\|X\|}$ is injective, via the application of $g^{-1} = g$ we conclude that ϕ is injective.
However, ϕ is clearly not linear as

$$\phi(0) = g(0) = \frac{1}{3}I.$$

Example 5.2 (Commutativity preserving is necessary)**.** Consider the map

$$f : M_n \to M_n^\times, \qquad X \mapsto (e^{\det X}, 1, \ldots, 1).$$

The map

$$\phi : \mathcal{A} \to M_n, \qquad \phi(X) = f(X)Xf(X)^{-1}$$

is continuous and preserves spectrum. Moreover, since f is a similarity invariant and $\phi(X)$ is similar to X, we can see that ϕ is bijective with the inverse $Y \mapsto f(Y)^{-1}Yf(Y)$. However, ϕ is not linear: clearly ϕ acts as the identity on diagonal matrices and singular matrices, so we have

$$\phi(I + E_{12}) = I + eE_{12} \neq I + E_{12} = \phi(I) + \phi(E_{12}).$$

Example 5.3 (Continuity is necessary)**.** Consider the map $\phi : \mathcal{A} \to M_n$ given by

$$\phi(X) = \begin{cases} \mathrm{diag}(\lambda_2, \lambda_1, \ldots, \lambda_n), & \text{if } X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \text{ and all } \lambda_i \text{ are distinct,} \\ X, & \text{otherwise} \end{cases}$$

which is bijective, spectrum and commutativity preserving but clearly not continuous. This example is taken from [28].

Example 5.4 (Injectivity is necessary)**.** Consider the map

$$\phi : \mathcal{A} \to M_n, \qquad \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ 0 & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{rr} \end{bmatrix} \mapsto \begin{bmatrix} X_{11} & 0 & \cdots & 0 \\ 0 & X_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{rr} \end{bmatrix}.$$

Then ϕ is clearly a unital Jordan homomorphism (and hence satisfies all assumptions of Theorem 1.4), but is not injective.

Example 5.5 ($n \geq 3$ is necessary)**.** This is also an example from [28]. Let $f : [0, +\infty\rangle \to [1, 2]$ be a nonconstant continuous map such that $\lim_{x\to+\infty} f(x) = 1$. For specificity we can consider

$$f(x) = 1 + \frac{1}{1+x}.$$

Consider the map $\phi : M_2 \to M_2$ given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \phi(A) = S_A A S_A^{-1}, \quad \text{where} \quad S_A = \begin{cases} I, & \text{if } b = 0, \\ \begin{bmatrix} f\left(\left|\frac{c}{b}\right|\right) & 0 \\ 0 & 1 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

Then ϕ is a continuous injective spectrum and commutativity preserving map, but is clearly not linear.

A similar example of a map $\phi : \mathcal{T}_2 \to M_2$ can be found in [26].

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