# THE CB-NORM APPROXIMATION OF GENERALIZED SKEW DERIVATIONS BY ELEMENTARY OPERATORS

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*Dedicated to the memory of my mentor and a friend, Professor R. M. Timoney*

Abstract. Let $A$ be a ring and $\sigma : A \to A$ a ring endomorphism. A generalized skew (or $\sigma$-)derivation of $A$ is an additive map $d : A \to A$ for which there exists a map $\delta : A \to A$ such that $d(xy) = \delta(x)y + \sigma(x)d(y)$ for all $x, y \in A$. If $A$ is a prime $C^*$-algebra and $\sigma$ is surjective, we determine the structure of generalized $\sigma$-derivations of $A$ that belong to the cb-norm closure of elementary operators $\mathcal{E}\ell(A)$ on $A$; all such maps are of the form $d(x) = bx + axc$ for suitable elements $a, b, c$ of the multiplier algebra $M(A)$. As a consequence, if an epimorphism $\sigma : A \to A$ lies in the cb-norm closure of $\mathcal{E}\ell(A)$, then $\sigma$ must be an inner automorphism. We also show that these results cannot be extended even to relatively well-behaved non-prime $C^*$-algebras like $C(X, \mathbb{M}_2)$.

## 1. Introduction

A well-known consequence of Skolem-Noether theorem (see e.g. [8, Theorem 4.46]) is that a finite-dimensional central simple algebra $A$ over a field $\mathbb{F}$ admits only inner derivations and inner automorphisms. This fact can be also proved by observing that all $\mathbb{F}$-linear maps $\phi : A \to A$ are elementary operators, i.e. they can be written as finite sums of two-sided multiplications $x \mapsto axb$, with $a, b \in A$ (see [8, Lemma 1.25, Theorem 1.30]). Hence, one can ask the following general question:

**Problem 1.1.** Under which conditions on a semiprime ring (or an algebra) $R$ are all derivations and/or automorphisms of $R$ that are also elementary operators necessarily inner?

In order to investigate Problem 1.1 it is sometimes convenient to consider maps $d : R \to R$ that comprise both (generalized) derivations and automorphisms. One particularly interesting class of such maps $d$ is the following (see e.g. [19, 20]):

**Definition 1.2.** Let $R$ be a ring and let $\sigma : R \to R$ be a ring endomorphism. An additive map $d : R \to R$ is called a *generalized $\sigma$-derivation* (or a *generalized skew derivation*) if there exists a map $\delta : R \to R$ such that

$$d(xy) = \delta(x)y + \sigma(x)d(y) \tag{1.1}$$

for all $x, y \in R$.

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In case when $R$ is semiprime and $\sigma$ is surjective, in [10] we considered the problem of determining the structure of generalized $\sigma$-derivations $d : R \to R$ that are also elementary operators. In order to give a description of such maps, we used standard techniques of the theory of rings of quotients. As a consequence of the proof of [10, Theorem 1.2], which is the main result of that paper, we showed that if $R$ is semiprime and centrally closed, then a derivation $\delta : R \to R$ (resp. a ring epimorphism $\sigma : R \to R$) is an elementary operator if and only if $\delta$ (resp. $\sigma$) is an inner derivation (resp. inner automorphism), thus giving an affirmative answer to Problem 1.1 for this class of rings.

It is also interesting to consider Problem 1.1 in the setting of $C^*$-algebras. Moreover, when working with $C^*$-algebras, it is well-known that their derivations, automorphisms and elementary operators are completely bounded. This motivates us to consider the following analytic variation of Problem 1.1:

**Problem 1.3.** Under which conditions on a $C^*$-algebra $A$ are all derivations and/or automorphisms of $A$ that admit a cb-norm approximation by elementary operators necessarily inner?

In our previous work we considered Problem 1.3 only for derivations. More precisely, we showed that all such derivations of $A$ are inner in a case when $A$ is prime [12, Theorem 4.3] or central [12, Theorem 5.6]. This result was further extended in [14, Theorem 1.5] for unital $C^*$-algebras whose every Glimm ideal is prime. The latter result in particular applies to derivations of local multiplier algebras (see e.g. [2]), since their Glimm ideals are prime [2, Corollary 3.5.10]. Hopefully, this result might be useful in order to give an answer to Pedersen's problem from 1978, which asks whether all derivations of local multiplier algebras are inner [24].

Motivated by these results, in this paper we consider the problem of determining the structure of generalized $\sigma$-derivations of $C^*$-algebras $A$, with $\sigma$ surjective, that can be approximated by elementary operators in the cb-norm.

The main result of this paper is Theorem 3.1, where we fully describe the structure of such maps when $A$ is a prime $C^*$-algebra. In particular, if $A$ is prime, we show that if an epimorphism $\sigma : A \to A$ admits a cb-norm approximation by elementary operators, then $\sigma$ must be an inner automorphism of $A$ (Corollary 3.8). The proof of Theorem 3.1 is given through several steps in Section 3.

In Section 4 we consider the possible generalization of Theorem 3.1 and its consequences for $C^*$-algebras that are not necessarily prime. However, this generalization will not be possible even for some well-behaved $C^*$-algebras, like homogeneous $C^*$-algebras, even thought they have only inner derivations (see [31, Theorem 1] for unital case and [13, Proposition 3.2] for general case). In fact, we show that for each $n \geq 2$ there is a compact manifold $X_n$ such that the $C^*$-algebra $C(X_n, \mathbb{M}_n)$ (where $\mathbb{M}_n$ is the algebra of $n \times n$ complex matrices) admits outer automorphisms that are simultaneously elementary operators (Proposition 4.2). We also give an example of a unital separable $C^*$-algebra $A$ that admits both outer derivations and outer automorphisms that are also elementary operators (Proposition 4.6).

## 2. Preliminaries

Let $R$ be a ring. As usual, by $Z(R)$ we denote its centre. By an ideal of $R$ we always mean a two-sided ideal. An ideal $I$ of $R$ is said to be *essential* if $I$ has a non-zero intersection with every other non-zero ideal of $R$. If $R$ is unital, by $R^\times$ we denote the set of all invertible elements of $R$.

Recall that a ring $R$ is said to be *semiprime* if for $x \in R$, $xRx = \{0\}$ implies $x = 0$. If in addition for $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$, $R$ is said to be *prime*.

If $R$ is a semiprime ring by $M(R)$ we denote its multiplier ring (see [2, Section 1.1]). Note that $M(R)$ is also a semiprime ring and that $R$ is prime if and only if $M(R)$ is prime [2, Lemma 1.1.7]. For each $a \in M(R)^\times$ we denote by $\mathrm{Ad}(a)$ the automorphism $x \mapsto axa^{-1}$. We call such automorphisms inner (when $R$ is unital this coincides with the standard notion of the inner automorphism).

**Remark 2.1.** If $R$ is a semiprime ring and $d : R \to R$ a generalized $\sigma$-derivation (Definition 1.2), then a map $\delta$ is obviously uniquely determined by $d$. Moreover, $\delta$ is a *$\sigma$-derivation*, that is $\delta$ is an additive map that satisfies

$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \tag{2.1}$$

for all $x, y \in R$. Indeed, using (1.1) and additivity of $d$ and $\sigma$, for all $x, y, z \in R$ we have

$$d((x + y)z) = \delta(x + y)z + \sigma(x + y)d(z) = \delta(x + y)z + \sigma(x)d(z) + \sigma(y)d(z)$$

and

$$d(xz + yz) = d(xz) + d(yz) = \delta(x)z + \sigma(x)d(z) + \delta(y)z + \sigma(y)d(z).$$

Subtracting these two equations we get the additivity of $\delta$. Similarly, subtracting the next two equations

$$d(xyz) = \delta(xy)z + \sigma(xy)d(z) = \delta(xy)z + \sigma(x)\sigma(y)d(z),$$

$$d(xyz) = \delta(x)yz + \sigma(x)d(yz) = \delta(x)yz + \sigma(x)\delta(y)z + \sigma(x)\sigma(y)d(z)$$

shows (2.1). Further, it is now easy to verify that the map $\rho := d - \delta$ is a *left $R$-module $\sigma$-homomorphism*, that is $\rho : R \to R$ is an additive map that satisfies

$$\rho(xy) = \sigma(x)\rho(y) \tag{2.2}$$

for all $x, y \in R$. Therefore, every generalized $\sigma$-derivation can be uniquely decomposed as

$$d = \delta + \rho,$$

where $\delta$ is a $\sigma$-derivation and $\rho$ is a left $R$-module $\sigma$-homomorphism. In particular, generalized $\sigma$-derivations simultaneously generalize $\sigma$-derivations of $R$ (we get them for $d = \delta$) and left $R$-module $\sigma$-homomorphisms of $R$ (we get them for $\delta = 0$).

**Example 2.2.** The simplest examples of $\sigma$-derivations $\delta : R \to R$ are *inner $\sigma$-derivations*, i.e. those of the form

$$\delta(x) = ax - \sigma(x)a,$$

where $a$ is some element of $M(R)$. Further, any map of the form

$$\rho(x) = \sigma(x)a,$$

where $a \in M(R)$, is a left $R$-module $\sigma$-homomorphism. If $R$ is unital, then all left $R$-module $\sigma$-homomorphism of $R$ are of this form ($M(R) = R$ in this case).

Throughout this paper $A$ will be a $C^*$-algebra. Then $M(A)$ has a structure of a $C^*$-algebra and is called the *multiplier algebra* of $A$. It is well-known (and easily checked) that $A$, as a ring, is semiprime. As usual, by $\mathrm{CB}(A)$ we denote the set of all completely bounded maps $\phi : A \to A$ (see e.g. [23]). For $S \subseteq \mathrm{CB}(A)$ we denote by $\overline{\overline{S}}_{cb}$ the cb-norm closure of $S$.

The most prominent class of completely bounded maps on $A$ are *elementary operators*, i.e. those that can be expressed as finite sums of *two-sided multiplications* $M_{a,b} : x \mapsto axb$, where $a$ and $b$ are elements of $M(A)$. We denote the set of all elementary operators on $A$ by $\mathcal{E}\ell(A)$. It is well-known that elementary operators on $C^*$-algebras are completely bounded. In fact, we have the following estimate for their cb-norm:

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h , \tag{2.3}$$

where $\|\cdot\|_h$ is the Haagerup tensor norm on the algebraic tensor product $M(A) \otimes M(A)$, i.e.

$$\|t\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} \ : \ t = \sum_i a_i \otimes b_i \right\}.$$

By inequality (2.3) the mapping

$$(M(A) \otimes M(A), \|\cdot\|_h) \to (\mathcal{E}\ell(A), \|\cdot\|_{cb}) \quad \text{given by} \quad \sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}.$$

is a well-defined contraction. Its continuous extension to the completed Haagerup tensor product $M(A) \otimes_h M(A)$ is known as a *canonical contraction* from $M(A) \otimes_h M(A)$ to $\mathrm{CB}(A)$ and is denoted by $\Theta_A$. We have the following result (see [2, Proposition 5.4.11]):

**Theorem 2.3** (Mathieu)**.** *$\Theta_A$ is isometric if and only if $A$ is a prime $C^*$-algebra.*

If $A$ is not a prime note that $\Theta_A$ is not even injective. Indeed, in this case there are non-zero elements $a, b \in A$ such that $aAb = \{0\}$. Then $a \otimes b$ defines a non-zero tensor in $M(A) \otimes M(A)$ but $M_{a,b} = 0$. For a unital but not necessarily prime $C^*$-algebra one can construct a central Haagerup tensor product $A \otimes_{Z,h} A$ and consider the induced contraction $\Theta_A^Z : A \otimes_{Z,h} A \to \mathrm{CB}(A)$ (the questions when $\Theta_A^Z$ is isometric or injective were treated in [30, 5, 4]).

## 3. Results

We begin this section by stating the main result of this paper:

**Theorem 3.1.** *Let $A$ be a prime $C^*$-algebra and let $d : A \to A$ be a generalized $\sigma$-derivation, with $\sigma$ surjective. The following conditions are equivalent:*

(i) $d \in \overline{\overline{\mathcal{E}\ell(A)}}_{cb}$.

(ii) *Either $d$ is a left multiplication implemented by some element of $M(A)$ or $\sigma$ is an inner automorphism of $A$. In the latter case, if $d = \delta + \rho$ is a decomposition as in Remark 2.1, then $\delta$ is an inner $\sigma$-derivation and $\rho$ is a right multiplication of $\sigma$ by some element of $M(A)$.*

**Remark 3.2.** (i) Note that any left multiplication $d = M_{l,1}$, where $l \in M(A)$, is a generalized $\sigma$-derivation with respect to any ring endomorphism $\sigma : A \to A$. Indeed, for any such $\sigma$, let $\delta(x) = lx - \sigma(x)l$ and $\rho(x) = \sigma(x)l$. Then obviously $d = \delta + \rho$, so in this case we cannot say anything about the epimorphism $\sigma$ and the corresponding maps $\delta$ and $\rho$.

(ii) If $d$ is not a left multiplication, then following the second case of part (ii) of Theorem 3.1 we have

$$\sigma = \mathrm{Ad}(a), \quad \delta(x) = bx - \sigma(x)b \quad \text{and} \quad \rho(x) = \sigma(x)b',$$

for some $a \in M(A)^\times$ and $b, b' \in M(A)$, $b \neq b'$. In particular, $d$ is of the form

$$d(x) = bx + axc,$$

where $c := a^{-1}(b' - b)$.

**Remark 3.3.** In the sequel of this section we assume that $A$ is an infinite-dimensional prime $C^*$-algebra, since otherwise [22, Theorem 6.3.8] and the primeness of $A$ would imply that $A$ is isomorphic to the matrix algebra $\mathbb{M}_n$ for some non-negative integer $n$. Then every linear map $\phi : A \to A$ is an elementary operator (see e.g. [8, Lemma 1.25]), so Theorem 3.1 is just a simple consequence of [10, Theorem 1.2] (the maximal right ring of quotients $Q_{mr}(A)$ in this case coincides with $A$).

For the proof of Theorem 3.1 we will need some auxiliary results. We start with the following:

**Lemma 3.4.** *Let $A$ be a prime $C^*$-algebra. Suppose that $\sigma : A \to A$ is a ring epimorphism for which there exists a non-zero element $a \in M(A)$ such that*

$$ax = \sigma(x)a \quad \forall x \in A. \tag{3.1}$$

*Then $a$ is invertible in $M(A)$, so that $\sigma = \mathrm{Ad}(a)$ is an inner automorphism of $A$.*

Before proving Lemma 3.4 recall from [2] (see also [6]) that an *essentially defined double centralizer* on a semiprime ring $R$ is a triple $(\mathcal{L}, \mathcal{R}, I)$, where $I$ is an essential ideal of $R$, $\mathcal{L} : I \to R$ is a left $R$-module homomorphism, $\mathcal{R} : I \to R$ is a right $R$-module homomorphism such that $\mathcal{L}(x)y = x\mathcal{R}(y)$ for all $x \in I$. One can form the *symmetric ring of quotients* $Q_s(R)$ which is characterized (up to isomorphism) by the following properties:

(i) $R$ is a subring of $Q_s(R)$;
(ii) for any $q \in Q_s(R)$ there is an essential ideal $I$ of $R$ such that $qI + Iq \subseteq R$;
(iii) if $0 \neq q \in Q_s(R)$ and $I$ is an essential ideal of $R$, then $qI \neq 0$ and $Iq \neq 0$;
(iv) for any essentially defined double centralizer $(\mathcal{L}, \mathcal{R}, I)$ on $R$ there exists $q \in Q_s(R)$ such that $\mathcal{L}(x) = qx$ and $\mathcal{R}(x) = xq$ for all $x \in I$.

In a case when $R = A$ is a $C^*$-algebra, $Q_s(A)$ has a natural structure as a unital complex $*$-algebra, whose involution is positive definite. An element $q \in Q_s(A)$ is called *bounded* if there is $\lambda \in \mathbb{R}_+$ such that $q^*q \leq \lambda 1$, in a sense that there is a finite number of elements $q_1, \ldots, q_n \in Q_s(A)$ such that

$$q^*q + \sum_{i=1}^{n} q_i^* q_i = \lambda 1.$$

The set $Q_b(A)$ of all bounded elements of $Q_s(A)$ has a pre-$C^*$-algebra structure with respect to the norm

$$\|q\|^2 = \inf\{\lambda \in \mathbb{R}_+ : \ q^*q \leq \lambda 1\},$$

which clearly extends the norm of $A$. One can easily check that an element $q \in Q_s(A)$ is bounded if and only if it can be represented by a bounded (continuous) essentially defined double centralizer (see [2, p. 57]). We call $Q_b(A)$ the *bounded symmetric algebra of quotients* of $A$ and its completion $M_{\mathrm{loc}}(A)$ the *local multiplier algebra* of $A$. Note that $M_{\mathrm{loc}}(A)$ has a structure of a $C^*$-algebra as a completion of a pre-$C^*$-algebra.

*Proof of Lemma 3.4.* First note that a non-zero element $a \in M(A)$ that satisfies (3.1) cannot be a zero-divisor. Indeed, if there exists $x \in M(A)$ such that $ax = 0$ then for each $y \in A$ we have $ayx = \sigma(y)ax = 0$, so that $aAx = \{0\}$. Since $A$ is prime and since $A$ is an essential ideal of $M(A)$, $a \neq 0$ implies $x = 0$. Similarly, if $xa = 0$ then for all $y \in A$ we have $x\sigma(y)a = xay = 0$. Since $\sigma$ is surjective, this is equivalent to $xAa = \{0\}$, so the primeness of $A$ again implies $x = 0$.

We now show that $a$ is invertible in $M(A)$. Since $M(A)$ is a unital $C^*$-subalgebra of $M_{\mathrm{loc}}(A)$, we have $M(A)^\times = M(A) \cap M_{\mathrm{loc}}(A)^\times$, so it suffices to show that $a$ is invertible in $M_{\mathrm{loc}}(A)$. In order to do this, first note that $aA$ is a non-zero ideal of $A$, hence essential, since $A$ is prime. Indeed, since $\sigma$ is surjective, we have $A = \sigma(A)$, hence

$$AaA = \{\sigma(x)ay : \ x, y \in A\} = \{axy : \ x, y \in A\} \subseteq aA.$$

In particular, for $\alpha \in \mathbb{C}$ and $x \in A$ we have $(\sigma(\alpha x) - \alpha\sigma(x))aA = \{0\}$, which implies $\sigma(\alpha x) = \alpha\sigma(x)$. Therefore $\sigma$ is a linear map, hence an algebra epimorphism.

We define maps $\mathcal{L}, \mathcal{R} : aA \to A$ by

$$\mathcal{L}(ax) = \sigma(x) \quad \text{and} \quad \mathcal{R}(ax) = x.$$

That the maps $\mathcal{L}$ and $\mathcal{R}$ are well-defined follows from the fact that $a$ is not a (left) zero-divisor. Clearly, $\mathcal{R}$ is a right $A$-module homomorphism. Next, for $x, y \in A$ we have

$$\mathcal{L}(\sigma(x)ay) = \mathcal{L}(axy) = \sigma(xy) = \sigma(x)\sigma(y) = \sigma(x)\mathcal{L}(ay)$$

and

$$\mathcal{L}(ax)ay = \sigma(x)ay = axy = ax\mathcal{R}(ay).$$

This shows that $(\mathcal{L}, \mathcal{R}, aI)$ is an essentially defined double centralizer on $A$. Since $A$ is prime, by [2, Corollary 2.2.15] all essentially defined double centralizers are automatically continuous. In particular, there exists an element $b \in Q_b(A) \subseteq M_{\mathrm{loc}}(A)$ such that

$$\sigma(x) = \mathcal{L}(ax) = axb = \sigma(x)ab \quad \text{and} \quad x = \mathcal{R}(ax) = bax$$

for all $x \in A$. Since $\sigma(A) = A$, this is equivalent to $A(1 - ab) = \{0\}$ and $(1 - ba)A = \{0\}$. Hence, $a$ is invertible in $M_{\mathrm{loc}}(A)$ and $a^{-1} = b$. This completes the proof. $\square$

Recall from [29, Definition 3.2] that a sequence $(a_n)$ in an infinite-dimensional $C^*$-algebra $B$ such that the series $\sum_{n=1}^\infty a_n^* a_n$ is norm convergent is said to be *strongly independent* if for every sequence $(\alpha_n) \in \ell^2$, equality $\sum_{n=1}^\infty \alpha_n a_n = 0$ implies $\alpha_n = 0$ for all $n \in \mathbb{N}$.

The next fact can be deduced from [7, Proposition 1.5.6], [29, Lemma 4.1] and [1, Lemma 2.3].

**Remark 3.5.** Let $B$ be an infinite-dimensional $C^*$-algebra.

(i) Every tensor $t \in B \otimes_h B$ has a representation as a convergent series $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where $(a_n)$ and $(b_n)$ are sequences in $B$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent. Moreover, the sequence $(b_n)$ can be chosen to be strongly independent.

(ii) If $t = \sum_{n=1}^{\infty} a_n \otimes b_n$ is a representation of $t$ as above, with $(b_n)$ strongly independent, then $t = 0$ if and only if $a_n = 0$ for all $n \in \mathbb{N}$.

**Corollary 3.6.** *Let $A$ be a prime $C^*$-algebra and suppose that $(a_n)$, $(b_n)$ are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. If*

$$\sum_{n=1}^{\infty} a_n x b_n = 0 \tag{3.2}$$

*for all $x \in A$, then $a_n = 0$ for all $n \in \mathbb{N}$.*

*Proof.* If $t := \sum_{n=1}^{\infty} a_n \otimes b_n \in M(A) \otimes_h M(A)$, then (3.2) is equivalent to $\Theta_A(t) = 0$. Since $A$ is prime, by Theorem 2.3 $\Theta_A$ is isometric (hence injective), so $t = 0$. The claim now follows from part (ii) of Remark 3.5. □

**Proposition 3.7.** *Let $A$ be a prime $C^*$-algebra and let $\sigma : A \to A$ be a ring epimorphism. If $\rho : A \to A$ is a non-zero left $A$-module $\sigma$-homomorphism, then the following conditions are equivalent:*

(i) $\rho \in \overline{\overline{\mathcal{E}\ell(A)}}_{cb}$.

(ii) *There are elements $a, p \in M(A)$, with $a$ invertible and $p \neq 0$, such that $\sigma = \mathrm{Ad}(a)$ and*
$$\rho(x) = \sigma(x) p = a x a^{-1} p$$
*for all $x \in A$.*

*Proof.* Since $A$ is prime, by Theorem 2.3 the canonical contraction $\Theta_A : M(A) \otimes_h M(A) \to \mathrm{CB}(A)$ is isometric. In particular, the image of $\Theta_A$ is closed in the cb-norm so $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ coincides with the image of $\Theta_A$. Hence, there is a tensor $t \in M(A) \otimes_h M(A)$ such that $\rho = \Theta_A(t)$. By Remark 3.5, we can write $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where $(a_n)$ and $(b_n)$ are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. Then (2.2) implies

$$\sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) y b_n = 0$$

for all $x, y \in A$. By Corollary 3.6 we have

$$a_n x = \sigma(x) a_n \tag{3.3}$$

for all $n \in \mathbb{N}$. Since $\rho$ is non-zero, there is $n_0 \in \mathbb{N}$ such that $a_{n_0} \neq 0$. By Lemma 3.4 $a := a_{n_0}$ is invertible in $M(A)$. Hence $\sigma = \mathrm{Ad}(a)$ is an inner automorphism of $A$. Finally, if $p := \sum_{n=1}^{\infty} a_n b_n \in M(A)$, using (3.3) we get

$$\rho(x) = \sum_{n=1}^{\infty} a_n x b_n = \sigma(x) \left( \sum_{n=1}^{\infty} a_n b_n \right) = \sigma(x) p = a x a^{-1} p.$$

□

As a direct consequence of Proposition 3.7 we get:

**Corollary 3.8.** *If $A$ is a prime $C^*$-algebra then every ring epimorphism $\sigma : A \to A$ that lies in $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ must be an inner automorphism of $A$.*

The next fact can be deduced from the proof of [12, Theorem 4.3]. For completeness, we include a proof.

**Lemma 3.9.** *Let $B$ be a unital infinite-dimensional $C^*$-algebra and let $f, g, h : B \to B$ be any functions with $f \neq 0$. Suppose that for all $x \in B$ we have the following equality of tensors in $B \otimes_h B$*

$$f(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n g(x) + h(x) a_n) \otimes b_n, \tag{3.4}$$

*where $(a_n)$ and $(b_n)$ are sequences in $B$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. Then there is a non-zero element $b \in B$ such that*

$$f(x) = bg(x) + h(x)b \tag{3.5}$$

*for all $x \in B$.*

*Proof.* Choose $x_0 \in B$ such that $f(x_0) \neq 0$ and let $\varphi \in B^*$ be an arbitrary bounded linear functional such that $\varphi(f(x_0)) \neq 0$. If for $x = x_0$ we act on the equality (3.4) with the right slice map $R_\varphi : B \otimes_h B \to B$, $R_\varphi : a \otimes b \mapsto \varphi(a)b$ (see e.g. [29, Section 4]), we obtain

$$\varphi(f(x_0)) \cdot 1 = \sum_{n=1}^{\infty} \varphi(a_n g(x_0) + h(x_0) a_n) b_n. \tag{3.6}$$

For $n \in \mathbb{N}$ let

$$\alpha_n := \frac{\varphi(a_n g(x_0) + h(x_0) a_n)}{\varphi(f(x_0))}.$$

Note that $(\alpha_n) \in \ell^2$, since all bounded linear functionals on $C^*$-algebras are completely bounded (see e.g. [23, Proposition 3.8]) and the series $\sum_{n=1}^{\infty} (a_n g(x_0) + h(x_0) a_n)(a_n g(x_0) + h(x_0) a_n)^*$ is norm convergent. Then (3.6) can be rewritten as $\sum_{n=1}^{\infty} \alpha_n b_n = 1$, so by (3.4) we have

$$\sum_{n=1}^{\infty} (\alpha_n f(x) - a_n g(x) - h(x) a_n) \otimes b_n = 0$$

for all $x \in B$. Consequently, since $(b_n)$ is strongly independent, Remark 3.5 (ii) implies that

$$\alpha_n f(x) = a_n g(x) + h(x) a_n$$

for all $n \in \mathbb{N}$ and $x \in B$. Since $\sum_{n=1}^{\infty} \alpha_n b_n = 1$, there is some $n_0 \in \mathbb{N}$ such that $\alpha_{n_0} \neq 0$. If $b := (1/\alpha_{n_0}) a_{n_0}$, then the above equation is obviously equivalent to (3.5). Also, $b \neq 0$ since $f \neq 0$. □

*Proof of Theorem 3.1.* (ii) $\Longrightarrow$ (i). This is trivial (see also Remark 3.2).

(i) $\Longrightarrow$ (ii). Assume that $d \in \overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ and that $d \neq M_{l,1}$ for all $l \in M(A)$. In particular $d \neq 0$. Using the same arguments from the beginning of the proof of Proposition 3.7, we see that there is a tensor $t \in M(A) \otimes_h M(A)$ such that $d = \Theta_A(t)$. By Remark 3.5, we can write $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where $(a_n)$ and $(b_n)$

are sequences of $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. Using (1.1) for all $x, y \in A$ we get

$$\delta(x)y = \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) y b_n,$$

or equivalently

$$\Theta_A(\delta(x) \otimes 1)) = \Theta_A \left( \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) \otimes b_n \right). \tag{3.7}$$

Again, since $\Theta_A$ is isometric (hence injective), (3.7) is equivalent to the equality

$$\delta(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) \otimes b_n$$

of tensors in $M(A) \otimes_h M(A)$ for all $x \in A$. If $\delta = 0$, then $d$ must be a non-zero left $A$-module $\sigma$-homomorphism of $A$ (see Remark 2.1) so the claim follows directly from Proposition 3.7. If $\delta \neq 0$, Lemma 3.9 implies that there is a non-zero element $b \in M(A)$ such that

$$\delta(x) = bx - \sigma(x)b$$

for all $x \in A$. If we decompose $d = \delta + \rho$ as in Remark 2.1, the map $\rho' : A \to A$ defined by

$$\rho'(x) := \rho(x) - \sigma(x)b = d(x) - bx$$

is obviously a left $A$-module $\sigma$-homomorphism of $A$ that lies in $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$ (since $d$ does). Since, by assumption, $d$ is not a left multiplication, $\rho'$ is non-zero. Hence, by Proposition 3.7 there are elements $a, p \in M(A)$ with $a$ invertible and $p \neq 0$ such that $\sigma = \mathrm{Ad}(a)$ and $\rho'(x) = \sigma(x)p$ for all $x \in A$. In particular, if we put $b' := b + p$, we get $\rho(x) = \sigma(x)b'$, which completes the proof. $\square$

## 4. Counterexamples and further remarks

In [12, 14] we considered derivations of unital $C^*$-algebras $A$ that lie in $\overline{\overline{\mathcal{E}\ell(A)}}_{cb}$. We showed that all such derivations are inner in a case when $A$ is prime [12, Theorem 4.3] or central [12, Theorem 5.6], or more generally, when $A$ is a unital $C^*$-algebra whose every Glimm ideal is prime [14, Theorem 1.5].

In light of this, it is natural to ask if one can extend Corollary 3.8 in its original form (and consequently Theorem 3.1) for similar classes of $C^*$-algebras. However, this will not be possible, even for relatively well-behaved $C^*$-algebras like homogeneous $C^*$-algebras. In fact, we will now show that for all $n \geq 2$, a $C^*$-algebra $A_n = C(PU(n), \mathbb{M}_n)$, where $PU(n) = U(n)/\mathbb{S}^1$ is the projective unitary group, admits outer automorphisms which are simultaneously elementary operators on $A_n$ (Proposition 4.2).

In order to show this, first suppose that $A$ is a general separable $n$-homogeneous $C^*$-algebra (i.e. all irreducible representations of $A$ have the same finite dimension $n$). Then by [17, Theorem 4.2] the primitive spectrum $X := \mathrm{Prim}(A)$ is a (locally compact) Hausdorff space and by a well-known theorem of Fell [11, Theorem 3.2] and Tomiyama-Takesaki [32, Theorem 5] there is a locally trivial bundle $\mathcal{E}$ over $X$ with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}^*(\mathbb{M}_n) \cong PU(n)$ such that $A$ is isomorphic to the $C^*$-algebra $\Gamma_0(\mathcal{E})$ of continuous sections of $\mathcal{E}$ that vanish at infinity. Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with primitive spectra $X_i$ $(i = 1, 2)$

are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ (the pullback bundle) as bundles over $X_1$ (see [32, Theorem 6]). Thus, we may identify $A$ with $\Gamma_0(\mathcal{E})$. Further, by Dauns-Hofmann theorem [27, Theorem A.34] we can identify $Z(A)$ with $C_0(X)$ and $Z(M(A))$ with $C_b(X)$.

Let us denote by $\mathrm{Aut}_Z^*(A)$ the set of all $Z(M(A))$-linear $*$-automorphisms of $A$ (i.e. $\sigma(x^*) = \sigma(x)^*$ and $\sigma(zx) = z\sigma(x)$ for all $z \in Z(M(A))$ and $x \in A$) and by $\mathrm{InnAut}^*(A)$ the set of automorphisms $\sigma$ of $A$ that are of the form $\sigma = \mathrm{Ad}(u)$, for some unitary element $u \in M(A)$. Obviously $\mathrm{InnAut}^*(A) \subseteq \mathrm{Aut}_Z^*(A)$. It is a very interesting (and non-trivial) problem to describe when do we have $\mathrm{Aut}_Z^*(A) = \mathrm{InnAut}^*(A)$ (see e.g. [28, 18, 25, 26] for some results regarding this question). In particular, we have the following consequence of [25, Theorem 2.1] (see also [25, 2.19]):

**Theorem 4.1** (Phillips-Raeburn)**.** *If $A = \Gamma_0(\mathcal{E})$ is a separable $n$-homogeneous $C^*$-algebra with primitive spectrum $X$, we have the exact sequence*

$$0 \longrightarrow \mathrm{InnAut}^*(A) \longrightarrow \mathrm{Aut}_Z^*(A) \xrightarrow{\eta} \check{H}^2(X;\mathbb{Z})$$

*of abelian groups, where $\check{H}^2(X;\mathbb{Z})$ is the second integral Čech cohomology group of $X$. Further, the image of $\eta$ is contained in the torsion subgroup of $\check{H}^2(X;\mathbb{Z})$.*

**Proposition 4.2.** *For $n \geq 2$ let $A_n = C(PU(n), \mathbb{M}_n)$. Then every derivation of $A_n$ is inner, but $A_n$ admits an outer automorphism that is also an elementary operator.*

**Remark 4.3.** A $C^*$-algebra $A$ is said to be *quasicentral* if any element $a \in A$ can be decomposed as $a = zb$ for some $b \in A$ and $z \in Z(A)$ (see [9, 3, 12] for other characterizations of such algebras). If every closed ideal of $A$ is quasicentral, note that any $Z(A)$-linear map $\phi : A \to A$ preserves all closed ideals of $A$ (i.e. $\phi(I) \subseteq I$ for any such ideal $I$). Indeed, since any $a \in I$ can be decomposed as $a = zb$, with $z \in Z(I)$ and $b \in I$ and since $Z(I) \subseteq Z(A)$, we have $\phi(a) = \phi(zb) = z\phi(b) \in I$.

This observation in particular applies to $n$-homogeneous $C^*$-algebras $A \cong \Gamma_0(\mathcal{E})$. Indeed, using the fact that the bundle $\mathcal{E}$ is locally trivial one can easily check that $n$-homogeneous $C^*$-algebras are quasicentral. Also, every closed ideal of an $n$-homogeneous $C^*$-algebra is also an $n$-homogeneous $C^*$-algebra, hence quasicentral. Further, if $A$ is unital (and $n$-homogeneous), every bounded $Z(A)$-linear map $\phi : A \to A$ is an elementary operator on $A$. This follows directly from the above observation and Magajna's theorem [21, Theorem 1.1]. Therefore, for every unital $n$-homogeneous $C^*$-algebra $A$ we have $\mathrm{Aut}_Z^*(A) \subseteq \mathcal{E}\ell(A)$.

*Proof of Proposition 4.2.* That all derivations of $A_n$ are inner follows from [31, Theorem 1]. On the other hand, by [16, Section IV] $A_n$ admits an automorphism $\sigma \in \mathrm{Aut}_Z^*(A_n) \setminus \mathrm{InnAut}^*(A_n)$ (note that $\check{H}^2(PU(n);\mathbb{Z}) = \mathbb{Z}_n$, so all elements of $\check{H}^2(PU(n);\mathbb{Z})$ are torsion elements). By Remark 4.3, $\sigma \in \mathcal{E}\ell(A_n)$. Suppose that $\sigma = \mathrm{Ad}(a)$ for some $a \in A_n^\times$. Then, since $\sigma$ is $*$-preserving, for all $x \in A_n$ we have

$$ax^*a^{-1} = \sigma(x^*) = \sigma(x)^* = (a^*)^{-1}x^*a^*.$$

Hence $a^*a \in Z(A_n)$, so $|a| = \sqrt{a^*a} \in Z(A_n)$. Therefore, if $u := |a|^{-1}a$, then $u$ is a unitary element of $A_n$ and $\sigma = \mathrm{Ad}(u) \in \mathrm{InnAut}^*(A_n)$; a contradiction. □

On the other hand, if $A_n = C(PU(n), \mathbb{M}_n)$ as before, every $\sigma \in \mathrm{Aut}_Z^*(A_n)$ is implemented by some unitary element of $Q_b(A)$. This follows from the following fact:

**Proposition 4.4.** *Let $A = \Gamma_0(\mathcal{E})$ be a separable $n$-homogeneous $C^*$-algebra whose primitive spectrum $X$ is locally contractable. Then for every $\sigma \in \mathrm{Aut}_Z^*(A)$ there is a unitary element $u \in Q_b(A)$ such that $\sigma = \mathrm{Ad}(u)$.*

*Proof.* We first show that there is a dense open subset $U$ of $X$ such that $\check{H}^2(U;\mathbb{Z}) = 0$. This can be shown by using the similar arguments as in the proof of [15, Lemma 3.1]. Indeed, let $\mathfrak{F}$ be a collection of all families $\mathcal{V}$ consisting of mutually disjoint open contractable subsets of $X$. We use the standard set-theoretic inclusion for partial ordering. If $\mathfrak{C}$ is a chain in $\mathfrak{F}$, then obviously $\bigcup \mathfrak{C}$ is an upper bound of $\mathfrak{C}$ in $\mathfrak{F}$. Therefore, applying Zorn's lemma, we obtain a maximal family $\mathcal{M}$ in $\mathfrak{F}$. Let $U$ be the union of all members of $\mathcal{M}$. Since $U$ is a disjoint union of contractable spaces, we have $\check{H}^2(U;\mathbb{Z}) = 0$. Since $X$ is locally contractable (and regular), the maximality of $\mathcal{M}$ implies that $U$ is a dense (evidently open) subset of $X$.

Now let $I := \Gamma_0(\mathcal{E}|_U)$, where $\mathcal{E}|_U$ is a restriction bundle of $\mathcal{E}$ to $U$. Since $U$ is dense in $X$, $I$ is an essential closed ideal of $A$. If $\sigma \in \mathrm{Aut}_Z^*(A)$, by Remark 4.3 we have $\sigma(I) \subseteq I$, so $\sigma|_I \in \mathrm{Aut}_Z^*(I)$. Since $\mathrm{Prim}(I) = U$ and $\check{H}^2(U;\mathbb{Z}) = 0$, by Theorem 4.1 there is a unitary element $u \in M(I)$ such that $\sigma(x) = uxu^*$ for all $x \in I$. Since $I$ is an essential ideal of $A$, we also have $\sigma(x) = uxu^*$ for all $x \in A$. If we define $\mathcal{L}_u, \mathcal{R}_u : I \to A$ by $\mathcal{L}_u(x) = xu$ and $\mathcal{R}_u(x) = ux$, then obviously $(\mathcal{L}_u, \mathcal{R}_u, I)$ is a bounded essentially defined double centralizer of $A$, so $u \in Q_b(A)$ and $\sigma = \mathrm{Ad}(u)$. □

**Problem 4.5.** Can we omit the assumption of local contractibility of the space $X$ in Proposition 4.4, that is if $A$ is a general separable $n$-homogeneous $C^*$-algebra, are all automorphisms $\sigma \in \mathrm{Aut}_Z^*(A)$ of the form $\sigma = \mathrm{Ad}(u)$ for some unitary $u \in Q_b(A)$?

In [12, Example 6.1] we gave an example of a unital separable $C^*$-algebra $A$ which admits outer derivations that are also elementary operators. We now show that the same $C^*$-algebra admits outer automorphisms that are also elementary operators:

**Proposition 4.6.** *Let $B := C([1,\infty], \mathbb{M}_2)$ be a $C^*$-algebra that consists of all continuous functions from the extended interval $[1,\infty]$ to the $C^*$-algebra $\mathbb{M}_2$. If $A$ is a $C^*$-subalgebra of $B$ that consists of all $a \in B$ such that*

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \qquad (n \in \mathbb{N})$$

*for some convergent sequence $(\lambda_n(a))$ of complex numbers, then $A$ admits an outer automorphism which is also an elementary operator on $A$.*

*Proof.* Let $f : [1,\infty] \to \mathbb{C}$ be a continuous function such the series $\sum_{n=1}^{\infty} f(n)$ does not converge and such that the range of $f$ is a subset of the imaginary axis. We define an element $b \in B$ by

$$b := \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}.$$

It was observed in [12, Section 6] that $\delta := \mathrm{ad}(b)$ defines an outer derivation of $A$ which lies in the operator norm closure of the space of all inner derivations of $A$. Since $b^* = -b$, $\delta$ is a $*$-derivation (i.e. $\delta(x^*) = \delta(x)^*$ for all $x \in A$). Hence

$\sigma := \exp(\delta)$ defines a $*$-automorphism of $A$ (see e.g. [2, Section 4.3]). Note that $\sigma = \mathrm{Ad}(u)$, where

$$u := \exp b = \begin{bmatrix} \exp f & 0 \\ 0 & 1 \end{bmatrix}$$

is a unitary element of $B$. Since the exponential map is continuous, and since $\delta$ in fact lies in the operator norm closure of the space of all inner $*$-derivations of $A$, we conclude that $\sigma$ lies in the operator norm closure of the set of all inner $*$-automorphisms of $A$.

*Claim 1.* $\sigma$ is an outer automorphism of $A$.

On the contrary, suppose that there exists an invertible element $a \in A$ such that $\sigma = \mathrm{Ad}(a)$. Then $u^*ax = xu^*a$ for all $x \in A$. Since

$$J := \{a \in A \ : \ a(n) = 0 \text{ for all } n \in \mathbb{N}\}$$

defines an closed essential ideal of both $A$ and $B$, we conclude that $u^*a$ is an invertible central element of $B$. Hence, there exists an invertible continuous function $\varphi \in C([1, \infty])$ such that

$$a = (\varphi \oplus \varphi)u = \begin{bmatrix} \exp f \cdot \varphi & 0 \\ 0 & \varphi \end{bmatrix}.$$

Since $a \in A$, we conclude that $\varphi(n+1) = (\exp f(n))\varphi(n)$, and consequently

$$\varphi(n+1) = \exp\left(\sum_{k=1}^{n} f(k)\right)\varphi(1) \tag{4.1}$$

for all $n \in \mathbb{N}$. Finally, since $\lim_{x\to\infty} \varphi(x)$ exists and since $\varphi(1) \neq 0$, (4.1) implies that the series $\sum_{n=1}^{\infty} f(n)$ converges, a contradiction.

*Claim 2.* $\sigma$ is an elementary operator on $A$.

As noted, $\sigma$ lies in the operator norm closure of the set of all inner $*$-automorphisms of $A$. In particular, $\sigma$ lies in the operator norm closure of the set of all elementary operators on $A$. But the latter set is closed in the operator norm by [12, Lemma 6.6]. Hence, $\sigma$ is an elementary operator. $\square$

We end this paper with the following question:

**Problem 4.7.** Is Corollary 3.8 true for all von Neumann algebras? In particular, if an automorphism $\sigma$ of a von Neumann algebra $A$ is also an elementary operator, is $\sigma$ necessarily an inner automorphism?

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