Complete solution of a family of simultaneous Pellian equations

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Abstract

Let $c_k = P_{2k}^2 + 1$, where P_k denotes the k^{th} Pell number. It is proved that for all positive integers k all solutions of the system of simultaneous Pellian equations

$$z^2 - c_k x^2 = c_k - 1, \qquad 2z^2 - c_k y^2 = c_k - 2$$

are given by $(x, y, z) = (0, \pm 1, \pm P_{2k})$.

This result implies that there does not exist positive integers $d > c > 2$ such that the product of any two distinct elements of the set

$$\{1, 2, c, d\}$$

diminished by 1 is a perfect square.

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1 Introduction

Diophantus studied the following problem: Find four (positive rational) numbers such that the product of any two of them increased by 1 is a perfect square. He obtained the following solution: $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ (see [7]). The first set of four positive integers with the above property was found by Fermat, and it was $\{1, 3, 8, 120\}$.

In [4] and [8] the more general problem was considered.

Definition 1 Let n be an integer. A set of positive integers $\{a_1, a_2, \ldots, a_m\}$ is said to have *the property* $D(n)$ if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$. Such a set is called *a Diophantine* m*-tuple (with the property* $D(n)$*)* or *a* P_n*-set of size* m.

In 1985, Brown [4], Gupta and Singh [13] and Mohanty and Ramasamy [16] proved independently that if $n \equiv 2 \pmod 4$, then there does not exist a Diophantine quadruple with the property $D(n)$. In 1993, Dujella [8] proved that if $n \not\equiv 2 \pmod 4$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one Diophantine quadruple with the property $D(n)$. The conjecture is that for $n \in S$ there does not exist a Diophantine quadruple with the property $D(n)$.

A famous open question is whether there exists a Diophantine quintuple with the property $D(1)$. The first result in that direction was proved in 1969 by Baker and Davenport [2]. They proved that the Diophantine triple $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple with the property $D(1)$. Recently, we generalized this result to the parametric families of Diophantine triples $\{k, k+2, 4k+4\}$ and $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$, $k \in \mathbf{N}$ (see [9, 10]), and in the joint paper with A. Pethő [12] we proved that the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple.

In the present paper we will apply the similar methods to the special cases of the following conjecture.

Conjecture 1 *There does not exist a Diophantine quadruple with the property $D(-1)$.*

It follows from the theory of integer points on elliptic curves (see [1]) that for fixed Diophantine triple $\{a, b, c\}$ with the property $D(-1)$ there are only finitely many effectively computable Diophantine quadruples D with $\{a, b, c\} \subset D$.

Assume that the Diophantine triple $\{a, b, c\}$ with the property $D(-1)$ can be extended to a Diophantine quadruple. Then there exist d, x, y, z such that

$$ad - 1 = x^2, \quad bd - 1 = y^2, \quad cd - 1 = z^2.$$

Eliminating d, we obtain the following system of Pellian equations

$$\begin{aligned} ay^2 - bx^2 &= b - a, \\ az^2 - cx^2 &= c - a, \\ bz^2 - cy^2 &= c - b. \end{aligned}$$

Thus Conjecture 1 can be rephrased in the terms of Pellian equations.

Conjecture 2 *Let a, b, c be distinct positive integers with the property that there exist integers r, s, t such that*

$$ab - 1 = r^2, \quad ac - 1 = s^2, \quad bc - 1 = t^2.$$

If $1 \notin \{a,b,c\}$, then the system of Pellian equations

$$ay^2 - bx^2 = b - a, \qquad az^2 - cx^2 = c - a \tag{1}$$

has no solution. If $a = 1$, then all solutions system (1) are given by $(x,y,z) = (0, \pm r, \pm s)$.

For certain triples $\{a,b,c\}$ with $1 \notin \{a,b,c\}$, the validity of Conjecture 2 can be verified by simple use of congruences (see [4]). It seems that the case $a = 1$ is more involved and until now Conjecture 2 was verified for triples $\{1,2,5\}$ (by Brown [4]), $\{1,5,10\}$ (by Mohanty and Ramasamy [15]), $\{1,2,145\}$, $\{1,2,4901\}$, $\{1,5,65\}$, $\{1,5,20737\}$, $\{1,10,17\}$ and $\{1,26,37\}$ (by Kedlaya [14]).

In the present paper we will verify Conjecture 2 for all triples of the form $\{1,2,c\}$.

First of all, observe that the conditions $c - 1 = s^2$ and $2c - 1 = t^2$ imply

$$t^2 - 2s^2 = 1. \tag{2}$$

All solutions in positive integers of Pell equation (2) are given by $s = s_k = P_{2k}$, $t = t_k = Q_{2k}$, where (P_k) and (Q_k) are sequences of Pell and Pell-Lucas numbers defined by

$$P_1 = 1, \quad P_2 = 2, \quad P_{k+2} = 2P_{k+1} + P_k,$$

$$Q_1 = 1, \quad Q_2 = 3, \quad Q_{k+2} = 2Q_{k+1} + Q_k.$$

Hence, if $\{1,2,c\}$ is a Diophantine triple with the property $D(-1)$, then there exists $k \geq 1$ such that

$$c = c_k = P_{2k}^2 + 1 = \frac{1}{8}[(1+\sqrt{2})^{4k} + (1-\sqrt{2})^{4k} + 6]. \tag{3}$$

Now we formulate our main results.

Theorem 1 *Let k be a positive integer and $c_k = P_{2k}^2 + 1$. All solutions of the system of simultaneous Pellian equations*

$$z^2 - c_k x^2 = c_k - 1 \tag{4}$$

$$2z^2 - c_k y^2 = c_k - 2 \tag{5}$$

are given by $(x,y,z) = (0, \pm 1, \pm P_{2k})$.

Remark 1 Since $c_1 = 5$, $c_2 = 145$ and $c_3 = 4901$ we may observe that the case $k = 1$ of Theorem 1 was proved by Brown [4] and the cases $k = 2$ and $k = 3$ by Kedlaya [14].

From Theorem 1 we obtain the following corollaries immediately.

Corollary 1 *The pair $\{1, 2\}$ cannot be extended to a Diophantine quadruple with the property $D(-1)$.*

Corollary 2 *Let k be a positive integer. Then the system of simultaneous Pell equations*

$$\begin{aligned} y^2 - 2P_{2k}^2 x^2 &= 1 \\ z^2 - (P_{2k}^2 + 1)x^2 &= 1 \end{aligned}$$

has only the trivial solutions $(x, y, z) = (0, \pm 1, \pm 1)$.

Let us mention that Bennett [3] proved recently that systems of simultaneous Pell equations of the form

$$y^2 - mx^2 = 1, \quad z^2 - nx^2 = 1, \quad (0 \neq m \neq n \neq 0)$$

have at most three nontrivial solutions, and suggested that such systems have at most one nontrivial solution, provided that they are not of a very specific form which is described in [3].

2 Preliminaries

Let k be the minimal positive integer, if such exists, for which the statement of Theorem 1 is not valid. Then results of Brown and Kedlaya imply that $k \geq 4$.

Since neither c_k nor $2c_k$ is a square we see that $\mathbf{Q}(\sqrt{c_k})$ and $\mathbf{Q}(\sqrt{2c_k})$ are real quadratic number fields. Moreover $2c_k - 1 + 2s_k\sqrt{c_k} = (s_k + \sqrt{c_k})^2$ and $4c_k - 1 + 2t_k\sqrt{2c_k} = (t_k + \sqrt{2c_k})^2$ are non-trivial units of norm 1 in the number rings $\mathbf{Z}[\sqrt{c_k}]$ and $\mathbf{Z}[\sqrt{2c_k}]$ respectively.

The theory of Pellian equations guarantees that there are finite sets $\{z_0^{(i)} + x_0^{(i)}\sqrt{c_k} \, : \, i = 1, \dots, i_0\}$ and $\{z_1^{(j)} + y_1^{(j)}\sqrt{2c_k} \, : \, j = 1, \dots, j_0\}$ of elements of $\mathbf{Z}[\sqrt{c_k}]$ and $\mathbf{Z}[\sqrt{2c_k}]$ respectively, such that all solutions of (4) and (5) are given by

$$z + x\sqrt{c} = (z_0^{(i)} + x_0^{(i)}\sqrt{c})(2c - 1 + 2s\sqrt{c})^m, \quad i = 1, \dots, i_0, \, m \geq 0, \qquad (6)$$

$$z\sqrt{2}+y\sqrt{c}=(z_1^{(j)}\sqrt{2}+y_1^{(j)}\sqrt{c})(4c-1+2t\sqrt{2c})^n,\ j=1,\dots,j_0,\ n\geq 0, \quad (7)$$

respectively. For simplicity, we have omitted here the index k and will continue to do so.

From (6) we conclude that $z=v_m^{(i)}$ for some index i and integer m, where

$$v_0^{(i)}=z_0^{(i)},\ \ v_1^{(i)}=(2c-1)z_0^{(i)}+2scx_0^{(i)},\ \ v_{m+2}^{(i)}=(4c-2)v_{m+1}^{(i)}-v_m^{(i)}, \quad (8)$$

and from (7) we conclude that $z=w_n^{(i)}$ for some index j and integer n, where

$$w_0^{(j)}=z_1^{(j)},\ w_1^{(i)}=(4c-1)z_1^{(j)}+2tcy_1^{(j)},\ w_{n+2}^{(j)}=(8c-2)w_{n+1}^{(j)}-w_n^{(j)}. \quad (9)$$

Thus we reformulated the system of equations (4) and (5) to finitely many Diophantine equations of the form

$$v_m^{(i)}=w_n^{(j)}.$$

If we choose representatives $z_0^{(i)}+x_0^{(i)}\sqrt{c}$ and $z_1^{(j)}\sqrt{2}+y_1^{(j)}\sqrt{c}$ such that $|z_0^{(i)}|$ and $|z_1^{(j)}|$ are minimal, then, by [17, Theorem 108], we have the following estimates:

$$0<|z_0^{(i)}|\leq\sqrt{\frac{1}{2}\cdot 2c\cdot(c-1)}<c\,,$$

$$0<|z_1^{(j)}|\leq\frac{1}{2}\sqrt{\frac{1}{2}\cdot 4c\cdot 2(c-2)}<c\,.$$

3 Application of congruence relations

From (8) and (9) it follows easily by induction that

$$v_{2m}^{(i)}\equiv z_0^{(i)}\pmod{2c},\quad v_{2m+1}^{(i)}\equiv -z_0^{(i)}\pmod{2c},$$

$$w_{2n}^{(j)}\equiv z_1^{(j)}\pmod{2c},\quad w_{2n+1}^{(j)}\equiv -z_1^{(j)}\pmod{2c}.$$

Therefore, if the equation $v_m^{(i)}=w_n^{(j)}$ has a solution in integers m and n, then we must have $|z_0^{(i)}|=|z_1^{(j)}|$.

Let $d_0=[(z_0^{(i)})^2+1]/c$. Then we have:

$$d_0-1=(x_0^{(i)})^2,\quad 2d_0-1=(y_1^{(j)})^2,\quad cd_0-1=(z_0^{(i)})^2 \quad (10)$$

and

$$d_0 \le \frac{c^2-c+1}{c} < c\,. \tag{11}$$

Assume that $d_0 > 1$. It follows from (10) and (11) that there exist a positive integer $l < k$ such that $d_0 = c_l$. But now the system

$$z^2 - c_l x^2 = c_l - 1, \qquad 2z^2 - c_l y^2 = c_l - 2$$

has a non-trivial solution $(x,y,z) = (s_k, t_k, z_0^{(i)})$, contradicting the minimality of k. Accordingly, $d_0 = 1$ and $|(z_0^{(i)})| = |(z_1^{(j)})| = s$. Thus we proved the following lemma.

Lemma 1 *If the equation $v_2^{(i)} = w_n^{(j)}$ has a solution, then $|z_0^{(i)}| = |z_1^{(j)}| = s$.*

The following lemma can be proved easily by induction. (We will omit the superscripts (i) and (j).)

Lemma 2

$$v_m \equiv (-1)^m(z_0 - 2cm^2 z_0 - 2csmx_0) \pmod{8c^2}$$

$$w_n \equiv (-1)^n(z_1 - 4cn^2 z_1 - 2ctny_1) \pmod{8c^2}$$

Observe that $|z_0| = |z_1| = s$ implies $x_0 = 0$ and $y_1 = \pm 1$. Furthermore, since we may restrict ourself to positive solutions of the system (4) and (5), we may assume that $z_0 = z_1 = s$. If $y = 1$, then $v_l < w_l$ for $l > 0$, and $v_m = w_n$, $n \ne 0$ implies $m > n$. If $y = -1$, then from $v_0 < w_1$ it follows $v_l < w_{l+1}$ for $l \ge 0$, and thus $v_m = w_n$ implies $m \ge n$.

Lemma 3 *If $v_m = w_n$, then m and n are even.*

Proof: Lemma 2 and the relation $z_0 = z_1 = s$ imply $m \equiv n \pmod{2}$. If $v_{2m+1} = w_{2n+1}$, then Lemma 2 implies

$$(2m+1)^2 s \equiv (2n+1)[(4n+2)s \pm t] \pmod{4c}\,,$$

and we have a contradiction with the fact that s is even and t is odd. □

Lemma 4 *If $v_{2m} = w_{2n}$, then $n \le m \le n\sqrt{2}$.*

PROOF: We have already proved that $m \geq n$. From (8) and (9) we have

$$v_m = \frac{s}{2}[(2c-1+2s\sqrt{c})^m + (2c-1-2s\sqrt{c})^m] > \frac{1}{2}(2c-1+2s\sqrt{c})^m,$$

$$\begin{aligned} w_n &= \frac{1}{2\sqrt{2}}[(s\sqrt{2} \pm \sqrt{c})(4c-1+2t\sqrt{2c})^n + (s\sqrt{2} \mp \sqrt{c})(4c-1-2t\sqrt{2c})^n] \\ &< \frac{s\sqrt{2}+\sqrt{c}+1}{2\sqrt{2}}(4c-1+2t\sqrt{2c})^n < \frac{1}{2}(4c-1+2t\sqrt{2c})^{n+\frac{1}{2}}. \end{aligned}$$

Since $k \geq 4$, we have $c \geq c_4 = 166465$. Thus $v_{2m} = w_{2n}$ implies

$$\frac{2m}{2n+\frac{1}{2}} < \frac{\ln(4c-1+2t\sqrt{2c})}{\ln(2c-1+2s\sqrt{c})} < 1.0517\,. \tag{12}$$

If $n = 0$ then $m = 0$, and if $n \geq 1$ then (12) implies

$$m < 1.0517n + 0.2630 < 1.3147n < n\sqrt{2}\,.$$

□

Lemma 5 *If $v_{2m} = w_{2n}$ and $n \neq 0$, then $m \geq n > \dfrac{1}{\sqrt{2}}\sqrt[4]{c}$.*

PROOF: If $v_{2m} = w_{2n}$, then Lemma 2 implies

$$2s(m^2 - 2n^2) \equiv \pm tn \pmod{2c}$$

and

$$4(m^2 - 2n^2)^2 \equiv n^2 \pmod{2c}.$$

Assume that $n \neq 0$ and $n \leq \frac{1}{\sqrt{2}}\sqrt[4]{c}$. Since $n \leq m \leq n\sqrt{2}$ by Lemma 4, we have

$$|2s(m^2 - 2n^2)| \leq 2\sqrt{c}n^2 \leq c,$$

$$4(m^2 - 2n^2)^2 \leq 4n^4 \leq c.$$

Thus, from $n^2 < c$ and $tn < \sqrt{2c}n < c$ we conclude that

$$4(m^2 - 2n^2)^2 = n^2, \quad \text{and} \quad 2s(m^2 - 2n^2) = -tn.$$

These two relations imply $s^2 = t^2$, a contradiction. □

4 Application of a result of Rickert

In this section we will use a result of Rickert [18] on simultaneous rational approximations to the numbers $\sqrt{(N-1)/N}$ and $\sqrt{(N+1)/N}$ and we will finish the proof of Theorem 1. For the convenience of the reader, we recall Rickert's result.

Theorem 2 *For an integer $N \geq 2$ the numbers*

$$\theta_1 = \sqrt{(N-1)/N}, \quad \theta_2 = \sqrt{(N+1)/N}$$

satisfy

$$\max\left(|\theta_1 - p_1/q|, |\theta_2 - p_2/q|\right) > (271N)^{-1} q^{-1-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = \lambda(N) = \frac{\log(12N\sqrt{3} + 24)}{\log[27(N^2-1)/32]}\,.$$

Lemma 6 *Let $N = t^2$ and $\theta_1 = \sqrt{(N-1)/N}$, $\theta_2 = \sqrt{(N+1)/N}$. Then all positive integer solutions x, y, z of the simultaneous Pellian equations (4) and (5) satisfy*

$$\max\left(|\theta_1 - \frac{2sx}{ty}|, |\theta_2 - \frac{2z}{ty}|\right) < y^{-2}\,.$$

PROOF: We have $\theta_1 = \frac{s}{t}\sqrt{2}$ and $\theta_2 = \frac{1}{t}\sqrt{2c}$. Hence,

$$|\theta_1 - \frac{2sx}{ty}| = \frac{s}{t}|\sqrt{2} - \frac{2x}{y}| = \frac{s}{t}|2 - \frac{4x^2}{y^2}| \cdot |\sqrt{2} + \frac{2x}{y}|^{-1}$$
$$\leq \frac{s}{t} \cdot \frac{2|y^2 - 2x^2|}{y^2} \cdot \frac{1}{\sqrt{2}} < y^{-2}$$

and

$$|\theta_2 - \frac{2z}{ty}| = \frac{1}{t}|\sqrt{2c} - \frac{2z}{y}| = \frac{2}{t}|c - \frac{2z^2}{y^2}| \cdot |\sqrt{2c} + \frac{2z}{y}|^{-1}$$
$$< \frac{2}{t} \cdot \frac{|cy^2 - 2z^2|}{y^2} \cdot \frac{1}{2\sqrt{2c}} = \frac{c-2}{t\sqrt{2c}} \cdot \frac{1}{y^2} < \frac{1}{2}y^{-2}.$$

□

Lemma 7 *Let x, y, z be positive integers satisfying the system of Pellian equations (4) and (5). Then*

$$\log y > 0.6575\sqrt[4]{c}\log(4c-3)\,. \tag{13}$$

Proof: Let $z = v_m$. Since $x > 0$, we have $m \neq 0$. From $y^2 - 2x^2 = 1$ we obtain

$$\begin{aligned} y > x\sqrt{2} &= \frac{s}{\sqrt{2c}}[(2c-1+2s\sqrt{c})^m - (2c-1-2s\sqrt{c})^m] \\ &> (2c-1+2s\sqrt{c})^{m-1} > (4c-3)^{m-1}. \end{aligned}$$

Now from Lemma 5 and $k \geq 4$ we conclude that

$$\log y > (m-1)\log(4c-3) > 0.6575\sqrt[4]{c}\log(4c-3).$$

□

Proof of Theorem 1. We will apply Theorem 2 for $N = t^2 = 2c-1$. Lemma 6 and Theorem 2 imply

$$(271)^{-1}(ty)^{-1-\lambda} < y^{-2}.$$

It follows that

$$y^{1-\lambda} < 271t^{3+\lambda} < 271(2c-1)^2 < 1084c^2.$$

Since $c \geq 166465$, we have

$$\frac{1}{1-\lambda} = \frac{\log[27(N^2-1)/32]}{\log[\frac{27(N^2-1)}{32(12N\sqrt{3}+24)}]} < \frac{2\log(1.8372c)}{\log(0.08118c)}$$

and

$$\log y < \frac{2\log(1.8372c)\log(1084c^2)}{\log(0.08118c)}\,. \tag{14}$$

Combining (13) and (14) we obtain

$$\sqrt[4]{c} < \frac{2\log(1.8372c)\log(1084c^2)}{0.6575\log(4c-3)\log(0.08118c)}\,. \tag{15}$$

Since the function $f(c)$ on the right side of (15) is decreasing, it follows that

$$\sqrt[4]{c} < f(c_4) = f(166465) < 9.349$$

and $c < 7639$, which contradicts the fact that $k \geq 4$. □

5 Concluding remarks

In [14], Kedlaya proved the statement of Theorem 1 for $k = 1$, 2 and 3 using the quadratic reciprocity method introduced by Cohn in [5].

However, the application of elliptic curves gives us a stronger result. Namely, consider the family of elliptic curves E_k, $k \geq 1$, given by

$$y^2 = (x-1)(2x-1)(c_k x - 1).$$

The computational numbertheoretical program package SIMATH ([19]) can be used to check that for $k = 1, 2, 3$ the rank of E_k is zero, and the torsion points on E_k are $\mathcal{O}, 1, \frac{1}{2}, \frac{1}{c_k}$. It implies that for $k = 1, 2, 3$ the set $\{1, 2, c_k\}$ cannot be extended to a *rational* Diophantine quadruple with the property $D(-1)$.

Let us mention that Euler found a rational Diophantine quadruple with the property $D(-1)$ and it was $\{\frac{7}{2}, \frac{65}{56}, \frac{233}{224}, \frac{289}{224}\}$ (see [6]), and as a special case of a two-parametric formula for Diophantine quintuples in [11] the rational Diophantine quintuple $\{\frac{130}{40}, \frac{25}{8}, \frac{37}{10}, 10, \frac{533}{40}\}$ with the property $D(-1)$ was obtained.

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