On the number of Diophantine m-tuples

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Abstract

A set of m positive integers is called a Diophantine m-tuple if the product of any two of them is one less than a perfect square. It is known that there does not exist a Diophantine sextuple and that there are only finitely many Diophantine quintuples. On the other hand, there are infinitely many Diophantine m-tuples for $m = 2$, 3 and 4.

In this paper, we derive asymptotic estimates for the number of Diophantine pairs, triples and quadruples with elements less than given positive integer N.

1 Introduction

A set of m positive integers is called a Diophantine m-tuple if the product of its any two distinct elements increased by 1 is a perfect square. Diophantus himself found four positive rationals 1/16, 33/16, 17/4, 105/16 with the above property, while the first Diophantine quadruple, the set $\{1, 3, 8, 120\}$, was found by Fermat (see [5, 7]). In 1969, Baker and Davenport [2] proved that the Fermat's set cannot be extended to a Diophantine quintuple. In 1998, Dujella and Pethő [10] proved that even the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple.

A "folklore" conjecture is that there does not exist a Diophantine quintuple. Recently, we proved in [9], improving the results from [8], that there does not exist a Diophantine sextuple and there are only finitely many, effectively computable, Diophantine quintuples.

The analogous problem for higher powers was considered by Bugeaud and Dujella in [3]. They proved that if $k \geq 3$ an a given integer and a set D of positive integers has the property that $ab + 1$ is a perfect k-th power

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for all $a, b \in D$, $a \neq b$, then $|D| \leq 7$. Moreover, $|D| \leq 3$ for $k \geq 177$. In [4, 11, 6, 14, 12], estimates for the size of a set $D \subseteq \{1, 2, \ldots, N\}$ with the property that $ab + 1$ is a perfect power for all $a, b \in D$, $a \neq b$, were given.

In this paper, we are interested in estimating the number of Diophantine m-tuples. According to the above mentioned results from [9], the only interesting cases are $m = 2$, $m = 3$, $m = 4$ and, perhaps, $m = 5$.

Let us define

$$D_m(N) = |\,\{D \subseteq \{1, 2, \ldots, N\} \,:\, D \text{ is a Diophantine } m\text{-tuple }\}\,|.$$

In Section 2, we prove that $D_2(N) \sim \frac{6}{\pi^2} N \log N$.

It was known already to Euler that every Diophantine pair $\{a, b\}$ can be extended to a Diophantine quadruple. Namely, if $ab + 1 = r^2$, then

$$\{a, b, a + b + 2r, 4r(a + r)(b + r)\} \tag{1}$$

is a Diophantine quadruple. Diophantine triple of the form $\{a, b, a+b+2r\}$ is called a regular Diophantine triple. In 1979, Arkin, Hogatt and Strauss [1] proved that every Diophantine triple can be extended to a Diophantine quadruple. More precisely, if $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$, then

$$\{a, b, c, a + b + c + 2abc + 2rst\} \tag{2}$$

is a Diophantine quadruple. Diophantine quadruple of the form (2) is called a regular Diophantine quadruple.

Regular triples and quadruples play an essential role in the estimates of the numbers $D_3(N)$ and $D_4(N)$. Namely, we will show that the main contribution to $D_3(N)$ comes from regular Diophantine triples, while the main contrubution to $D_4(N)$ comes from quadruples of the form (1). Using these facts, we are able to prove in Sections 3 and 4 that $D_3(N) \sim \frac{3}{\pi^2} N \log N$ and that the true order of magnitude of $D_4(N)$ is $\sqrt[3]{N} \log N$. Determining the constant C such that $D_4(N) \sim C \sqrt[3]{N} \log N$ remains an open problem. At present, we are able to show that if such constant exist, then $0.1608 < C < 0.5354$.

It follows from [9] that there exist a constant K such that $D_5(N) < K$ for all positive integers N. In Section 5 we prove that we may take $K = 10^{1930}$.

2 Diophantine pairs

Lemma 1 *The number of solutions of the congruence*

$$x^2 \equiv 1 \pmod{b}$$

in the range $1 \leq x \leq b$ *is* $2^{\omega(b)}$ *if* b *is odd or* $b \equiv 4 \pmod 8$*;* $2^{\omega(b)-1}$ *if* $b \equiv 2 \pmod 4$*;* $2^{\omega(b)+1}$ *if* $b \equiv 0 \pmod 8$*. Here* $\omega(b)$ *denotes the number of distinct prime factors of* b*.*

PROOF. See [17, g, §4, ch. V]. ∎

The statement of the following lemma is a consequence of Selberg's formula [16]. We give a proof for the sake of completeness.

Lemma 2

$$\sum_{x=1}^{N} 2^{\omega(x)} = \frac{6}{\pi^2} N \log N + O(N)$$

PROOF. We have

$$\sum_{x=1}^{N} 2^{\omega(x)} = \sum_{x=1}^{N} \sum_{d|x,\, \mu^2(d)=1} 1 = \sum_{d=1,\, \mu^2(d)=1}^{N} \left\lfloor \frac{N}{d} \right\rfloor = N \cdot \sum_{d=1}^{N} \frac{\mu^2(d)}{d} + O(N).$$

Let

$$A(d) = \sum_{x=1}^{d} \mu^2(x) = \frac{6}{\pi^2} d + O(\sqrt{d}).$$

It follows

$$\begin{aligned} \sum_{x=1}^{N} 2^{\omega(x)} &= N \sum_{d=1}^{N} \frac{A(d) - A(d-1)}{d} + O(N) \\ &= N \cdot \sum_{d=1}^{N-1} \frac{A(d)}{d(d+1)} + \frac{A(N)}{N} + O(N) \\ &= \frac{6}{\pi^2} N \cdot \sum_{d=1}^{N-1} \frac{1}{d+1} + O(N) \\ &= \frac{6}{\pi^2} N \log N + O(N). \end{aligned}$$

∎

Theorem 1

$$D_2(N) = \frac{6}{\pi^2} N \log N + O(N)$$

Proof. Let $b \leq N$ be a positive integer. If $\{a, b\}$ is a Diophantine pair, then there exist an integer r such that

$$ab + 1 = r^2. \tag{3}$$

On the other hand, all solutions r of the congruence $r^2 \equiv 1 \pmod{b}$, such that $1 < r \leq b$, induce (by (3)) a Diophantine pair $\{a, b\}$ such that $a < b$. Hence, by Lemmas 1 and 2, we have

$$\begin{aligned}
D_2(N) &= \sum_{b=1}^{N} 2^{\omega(b)} - \sum_{b=1}^{\lfloor N/2 \rfloor} 2^{\omega(b)} + 2 \sum_{b=1}^{\lfloor N/4 \rfloor} 2^{\omega(b)} - N \\
&= \frac{6}{\pi^2} N \log N - \frac{6}{\pi^2} \Big(\frac{N}{2} + O(1) \Big) \log \Big(\frac{N}{2} + O(1) \Big) \\
&\quad + 2 \cdot \frac{6}{\pi^2} \Big(\frac{N}{4} + O(1) \Big) \log \Big(\frac{N}{4} + O(1) \Big) + O(N) \\
&= \frac{6}{\pi^2} N \log N \cdot \Big(1 - \frac{1}{2} + \frac{1}{2} \Big) + O(N) \\
&= \frac{6}{\pi^2} N \log N + O(N)
\end{aligned}$$

■

3 Diophantine triples

We have $D_3(N) = D_3^{(1)}(N) + D_3^{(2)}(N)$ where $D_3^{(1)}(N)$ denotes the number of regular Diophantine triples in $\{1, 2, \ldots, N\}$, i.e. triples of the form $\{a, b, a+b+2r\}$ where $ab + 1 = r^2$, $r > 0$, while $D_3^{(2)}(N)$ denotes the number of all other (irregular) Diophantine triples in $\{1, 2, \ldots, N\}$.

Proposition 1

$$D_3^{(1)}(N) = \frac{3}{\pi^2} N \log N + O(N)$$

Proof. Let $c = a+b+2r$. Then $b = a+c-2s$, where $ac+1 = s^2$, $s > 0$. Every pair $\{a, c\}$, such that $ac + 1 = s^2$ and $a < c \leq N$ induces a regular triple $\{a, a+c-2s, c\} \subset \{1, 2, \ldots, N\}$. (Note that $a+c-2s = 0$ iff $a = c-2$.) Every regular triple $\{a, b, c\}$ is obtained twice by this construction: starting with $\{a, c\}$ and starting with $\{b, c\}$. Therefore,

$$D_3^{(1)}(N) = \frac{1}{2}(D_2(N) - N + 2) = \frac{3}{\pi^2} N \log N + O(N). \tag{4}$$

■

From (4), it follows directly

Corollary 1

$$D_2(N) \equiv N \pmod 2$$

Lemma 3

$$\begin{aligned}\sum_{x=1}^{N} 2^{\omega(x)} \frac{1}{x} &= O(\log^2 N),\\ \sum_{x=1}^{N} 2^{\omega(x)} \frac{1}{x^2} &= O(1),\\ \sum_{x=1}^{N} 2^{\omega(x)} \frac{1}{x^{3/4}} &= \frac{24}{\pi^2}\sqrt[4]{N}\log N + O(\sqrt[4]{N}).\end{aligned}$$

Proof. The proof is analogous to the proof of Lemma 3, using the facts that the series $\sum_{x=1}^{\infty} \frac{\log x}{x^2}$ is convergent, while $\sum_{x=1}^{N} \frac{\log x}{x} \sim \frac{1}{2}\log^2 N$ and $\sum_{x=1}^{N} \frac{\log x}{x^{3/4}} \sim 4\sqrt[4]{N}\log N$. ■

Proposition 2

$$D_3^{(2)}(N) = O(N)$$

Proof. Let $\{a, b, c\}$, $a < b < c$, be an irregular Diophantine triple. By [13, Lemma 4], there exists a positive integer $c_0 < \frac{c}{4ab}$ such that $\{a, b, c_0, c\}$ is a regular Diophantine quadruple. If the Diophantine triple $\{a, b, c_0\}$ is regular, then $c_0 = a + b \pm 2r$. Otherwise, by the same result of Jones [13], we have $c_0 \geq 4ab$ or $b \geq 4ac_0$. We will consider these four cases separately.

1) If $c_0 = a + b + 2r$, then $N \geq c > 4abc_0 > b^2$. According to Theorem 1, the number of such triples is $O(\sqrt{N}\log N)$.

2) Let $c_0 = a+b-2r$. We have $ac_0+1 = s_0^2$, $s_0 > 0$, and $b = a+c_0+2s_0$. Furthermore, $N \geq c > 4abc_0 > (\max(a, c_0))^2$, and again Theorem 1 implies that the contribution of such triples is $O(\sqrt{N}\log N)$.

3) Assume that $c_0 \geq 4ab$. Then $N \geq 16a^2b^2 > b^2$. Hence, we have $O(\sqrt{N}\log N)$ possible pairs $\{a, b\}$. For a fixed pair $\{a, b\}$, c is an element of the union of finitely many binary recursive sequences. Each such sequence corresponds to a solution of the congruence $z_0^2 \equiv 1 \pmod b$. According to [8, Lemma 1], there are at most $2b^{3/4}$ such sequences. Hence, the number of possible c's is $O(N^{3/8}\log N)$, and the contribution of the third case in $O(N^{7/8}\log N)$.

4) Assume that $b \geq 4ac_0$. Then $N \geq 16a^2c_0^2$ and $c_0 \leq \frac{\sqrt{N}}{4a}$. Furthermore, $N \geq 4r^2$ and $r \leq \frac{\sqrt{N}}{2}$. Let $g(a) = \frac{\frac{\sqrt{N}}{2}+a}{a} \cdot \frac{\sqrt{N}}{4a}$ and $x = \lfloor \frac{\sqrt{N}}{4} \rfloor$. Then the contribution of this case is, by Lemma 1, bounded by

$$\sum_{a=1}^{x} 2^{\omega(a)} g(a) - \sum_{a=1}^{x/2} 2^{\omega(a)} g(2a) + 2\sum_{a=1}^{x/4} 2^{\omega(a)} g(4a).$$

By Lemma 3, we have

$$\begin{aligned}\sum_{a=1}^{x} 2^{\omega(a)} g(a) &= \sum_{a=1}^{x} 2^{\omega(a)} \cdot \frac{N}{8a^2} + \sum_{a=1}^{x} 2^{\omega(a)} \cdot \frac{\sqrt{N}}{4a} \\ &= N \cdot O(1) + \sqrt{N} \cdot O(\log^2 N) = O(N).\end{aligned}$$

Analogously, $\sum_{a=1}^{x/2} 2^{\omega(a)} g(2a) = O(N)$ and $\sum_{a=1}^{x/4} 2^{\omega(a)} g(4a) = O(N)$.

Therefore, the contribution of the fourth case is $O(N)$.

■

Theorem 2

$$D_3(N) = \frac{3}{\pi^2} N \log N + O(N)$$

PROOF. Directly from Propositions 1 and 2. ■

4 Diophantine quadruples

Theorem 3

$$D_4(N) = \Theta(\sqrt[3]{N} \log N)$$

More precisely,

$$\begin{aligned} D_4(N) &> 0.1608 \sqrt[3]{N} \log N, \\ D_4(N) &< 0.5354 \sqrt[3]{N} \log N, \end{aligned}$$

for sufficiently large N.

Proof. There are three types of Diophantine quadruples $\{a,b,c,d\}$: 1) irregular quadruples; 2) regular quadruples in which the triple $\{a,b,c\}$ is not regular; 3) quadruples of the form (1). We will estimate the numbers of quadruples of each of these three types separately.

1) By [9, Proposition 1], if $\{a,b,c,d\}$ is an irregular Diophantine quadruple and $a<b<c<d$, then $d>c^{3.5}$. Hence, $c<N^{2/7}$ and, by Theorem 2, the number of possible triples $\{a,b,c\}$ is $O(N^{2/7}\log N)$. It remains to estimate the number of possible d's.

Let $cd+1=z^2$. According to [8, Lemma 1], z belongs to the union of finitely many binary recursive sequences, and each such sequence is generated by z_0, which satisfies $z_0^2\equiv 1 \pmod{c}$ and $z_0<c^{3/4}$. Let $\omega(c)=k$. Then the number of possible z_0 is bounded by 2^{k+1}.

Let p_i denotes the i-th prime. Then $c\ge p_1\cdots p_k$. If $k\ge 2$, we have

$$\log c \ge \sum_{p\le p_k}\log p > \frac{1}{2}p_k > \frac{1}{2}k\log k \tag{5}$$

(see [15]), but this is also true for $k=1$.

Therefore, $2^k \le 2^{\log c/\log k} < c^{0.7/\log k}$. If $2^k\ge c^{0.01}$, then $k<e^{70}$ and $c\le 2^{100k}<10^{10^{33}}$. Hence, we may assume that $2^k<c^{0.01}$. But, it implies that the number of possible z_0's is $O(N^{0.02/7})$, while the number of possible d's is $O(N^{0.02/7}\log N)$. Finally, we obtain that the contribution of quadruples of the first type is $O(N^{0.292}\log^2 N)$.

2) Since $\{a,b,c\}$ is not regular, by [13, Lemma 4], we have $c\ge 4ab+a+b+1$. It implies

$$d>4abc+c\ge(4ab+a+b+1)(4ab+1)\ge(4ab+4)(4ab+4)=16r^4.$$

As in **1)**, we can prove that for a fixed pair $\{a,b\}$ there are at most $O(b^{0.01}\log N)=O(N^{0.01})$ possible c's. The number of pairs $\{a,b\}$ is bounded by

$$2\sum_{a=1}^{\sqrt[4]{N}/2} 2^{\omega(a)}\frac{\frac{\sqrt[4]{N}}{2}+a}{a} = O(\sqrt[4]{N}\log^2 N)+O(\sqrt[4]{N}\log N)=O(\sqrt[4]{N}\log^2 N),$$

and the contribution of quadruples of the second type is $O(N^{0.26}\log^2 N)$.

3) Denote the number of quadruples of the third type by $E^{(3)}(N)$.

Since both $\{a,b,c\}$ and $\{a,b,c,d\}$ are regular, we have $c = a+b+2r$ and $4abc+c < d < 4abc+4c$. It is easy to check that $a+b \geq 2r$. Therefore we have $c \geq 4r$ and $d < 4cr^2 \leq \frac{c^3}{4}$. By Theorem 2 (or Proposition 1), for sufficiently large N, we have

$$E^{(3)}(N) \geq \frac{3}{\pi^2} \cdot \sqrt[3]{4N} \cdot \frac{1}{3} \log N = \frac{\sqrt[3]{4}}{\pi^2} \sqrt[3]{N} \log N. \tag{6}$$

In the opposite direction, we have the following inequalities $d > 16a^3$ and $d > 4r^2 \cdot \frac{r^2}{a} = \frac{4r^4}{a}$.

Let $h(a) = \frac{\sqrt[4]{Na/4}+a}{a}$ and $y = \left\lfloor \sqrt[3]{\frac{N}{16}} \right\rfloor$. Then

$$E^{(3)}(N) \leq \sum_{a=1}^{y} 2^{\omega(a)} h(a) - \sum_{a=1}^{y/2} 2^{\omega(a)} h(2a) + 2 \sum_{a=1}^{y/4} 2^{\omega(a)} h(4a).$$

Using Lemma 3, we obtain

$$\begin{aligned}
\sum_{a=1}^{y} 2^{\omega(a)} h(a) &= \sum_{a=1}^{y} 2^{\omega(a)} + \frac{\sqrt[4]{N}}{\sqrt{2}} \sum_{a=1}^{y} 2^{\omega(a)} \frac{1}{a^{3/4}} \\
&= \frac{6}{\pi^2} \cdot \frac{\sqrt[3]{N}}{2\sqrt[3]{2}} \cdot \frac{1}{3} \log N + \frac{\sqrt[4]{N}}{\sqrt{2}} \cdot \frac{24}{\pi^2} \cdot \frac{\sqrt[12]{N}}{\sqrt[3]{2}} \cdot \frac{1}{3} \log N + O(\sqrt[3]{N}) \\
&= \frac{1}{\sqrt[3]{2}\pi^2} \sqrt[3]{N} (\log N)(1 + 4\sqrt{2}) + O(\sqrt[3]{N}).
\end{aligned}$$

Analogously,

$$\sum_{a=1}^{y/2} 2^{\omega(a)} h(2a) = \frac{1}{\sqrt[3]{2}\pi^2} \sqrt[3]{N} (\log N)(\frac{1}{2} + 2\sqrt{2}) + O(\sqrt[3]{N})$$

and

$$\sum_{a=1}^{y/4} 2^{\omega(a)} h(4a) = \frac{1}{\sqrt[3]{2}\pi^2} \sqrt[3]{N} (\log N)(\frac{1}{4} + \sqrt{2}) + O(\sqrt[3]{N}).$$

Therefore, for sufficiently large N, we have

$$E^{(3)}(N) \leq \frac{1}{\sqrt[3]{2}\pi^2} (4\sqrt{2}+1) \sqrt[3]{N} \log N. \tag{7}$$

The statement of the theorem follows directly from the inequalities (6) and (7). ∎

Remark 1 From the proof of Theorem 3, it follows that the main contribution to the number $D_4(N)$ comes from the number $E^{(3)}(N)$ of quadruples of the form $\{a, b, a+b+2r, 4r(a+b)b+r)\}$. In order to get better insight in asymptotic behavior of the numbers $D_4(N)$, it is natural to consider the numbers $e^{(3)}(N) = E^{(3)}(N)/\sqrt[3]{N}\log N$. Here are some experimental results about these numbers:

$$e^{(3)}(10^6) \approx 0.1254, \quad e^{(3)}(10^9) \approx 0.1747, \quad e^{(3)}(10^{12}) \approx 0.2057,$$
$$e^{(3)}(10^{15}) \approx 0.2277, \quad e^{(3)}(10^{18}) \approx 0.2440, \quad e^{(3)}(10^{21}) \approx 0.2565,$$
$$e^{(3)}(10^{24}) \approx 0.2662.$$

These results suggest that there is a constant C, $0.2662 < C < 0.5354$, such that $D_4(N) \sim C\sqrt[3]{N}\log N$.

5 Diophantine quintuples

Theorem 4

$$D_5(N) < 10^{1930}$$

Proof. Let $\{a, b, c, d, e\}$ be a Diophantine quintuple, where $a < b < c < d < e$. Then, by [9, Corollary 4], we have $d < 10^{2171}$ and $e < 10^{10^{26}}$. By the main result of [10], we may assume that $\{a, b\} \neq \{1, 3\}$.

Assume first that the quadruple $\{a, b, c, d\}$ is regular. Then $d > 4abc > b^2$ and $b < 10^{1086}$. Let us estimate the number of possible pairs $\{a, b\}$ which satisfies these conditions. We have at most 10^{618} such pairs satisfing $b \leq 10^{309}$. Assume that $10^{309} < b < 10^{1086}$. Let $k = \omega(b)$. By (5), we have

$$\log b > \frac{1}{2} k \log k. \tag{8}$$

If $2^k \geq b^{0.25}$, then (8) implies $k < 256$ and $b < 10^{309}$, a contradiction. Hence, $2^k < b^{0.25}$, and the number of corresponding pairs $\{a, b\}$ is less than

$$\sum_{b=10^{309}+1}^{10^{1086}-1} 2^{\omega(b)+1} < 2 \sum_{b=10^{309}+1}^{10^{1086}-1} b^{0.25} < 2\int_{10^{309}}^{10^{1086}} b^{0.25}\, db < 10^{1358}.$$

For a fixed pair $\{a, b\}$, the third number c is an element of the union of finitely many binary recursive sequences. According to [8, Lemma 1], the number of these sequences is less than or equal to the number of solutions of the congruence $z_0^2 \equiv 1 \pmod{b}$ in the range $-0.71b^{0.75} < z_0 < 0.71b^{0.75}$. If $b \leq 10^{309}$ this number is obviously $< 10^{232}$, while if $10^{309} < b < 10^{1086}$ it is $\leq 2^{\omega(b)+1} < 2b^{0.25} < 10^{272}$. Elements is these sequences grow exponentially, and the corresponding base is $> 4ab \geq 32$. Therefore, in any of these binary sequences, there are at most $\log_{32} 10^{2171}$ elements less than 10^{2171}. Therefore, the number of c's is $< 10^{276}$.

Since $\{a, b, c, d\}$ is regular, for fixed $\{a, b, c\}$, d is unique, while for e we have at most $10^{272} \cdot \log_{32} 10^{10^{26}} < 10^{298}$ possibilities.

Hence, the number of Diophantine quintuples $\{a, b, c, d, e\}$ in which the subset $\{a, b, c, d\}$ is regular, is less than

$$10^{1358} \cdot 10^{276} \cdot 1 \cdot 10^{298} = 10^{1930}.$$

Assume now that the quadruple $\{a, b, c, d\}$ is irregular. Then, by [9, Lemma 6], $d > c^{2.5}b^{1.5} > b^4$ and $b < 10^{543}$. As above, the number of possible pairs $\{a, b\}$ which satisfy these conditions with $b \leq 10^{309}$ is less than 10^{618}, and for $10^{309} < b < 10^{543}$, there are less than

$$\sum_{b=10^{309}+1}^{10^{543}-1} 2^{\omega(b)+1} < 2 \sum_{b=10^{309}+1}^{10^{543}-1} b^{0.25} < 2 \int_{10^{309}}^{10^{543}} b^{0.25}\, db < 10^{679}$$

such pairs.

For fixed pair $\{a, b\}$, the number of binary sequences in which c's, d's and e's may be contained is bounded by 10^{232} if $b \leq 10^{309}$, and by $2b^{0.25} < 10^{137}$ if $10^{309} < b < 10^{543}$.

Hence, the number of Diophantine quintuples $\{a, b, c, d, e\}$ in which the subset $\{a, b, c, d\}$ is irregular, is less than

$$10^{679} \cdot \left(10^{232} \log_{32} 10^{2171}\right)^2 10^{232} \log_{32} 10^{10^{26}} < 10^{1408}.$$

■

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