THE PROBLEM OF DIOPHANTUS AND DAVENPORT FOR GAUSSIAN INTEGERS

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Abstract: A set of Gaussian integers is said to have the property $D(z)$ if the product of its any two distinct elements increased by z is a square of a Gaussian integer. In this paper it is proved that if a Gaussian integer z is not representable as a difference of the squares of two Gaussian integers, then there does not exist a quadruple with the property $D(z)$, but if z is representable as a difference of two squares and if $z \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then there exists at least one quadruple with the property $D(z)$.

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1 Introduction

The Greek mathematician Diophantus of Alexandria noted that the rational numbers $\frac{1}{16}$, $\frac{33}{16}$, $\frac{17}{4}$ and $\frac{105}{16}$ have the following property: the product of any two of them increased by 1 is a square of a rational number (see [3]). Fermat first found a set of four positive integers with the above property, and it was $\{1, 3, 8, 120\}$. Later, Davenport and Baker [2] showed that if d is a positive integer such that the set $\{1, 3, 8, d\}$ has the property of Diophantus, then d has to be 120.

There are two well known generalizations of the set $\{1, 3, 8, 120\}$: for all positive integers n the sets

$$\{n,\, n+2,\, 4n+4,\, 4(n+1)(2n+1)(2n+3)\}, \tag{1}$$

$$\{F_{2n},\, F_{2n+2},\, F_{2n+4},\, 4F_{2n+1}F_{2n+2}F_{2n+3}\} \tag{2}$$

have the property of Diophantus (see [12], [11]). For $n = 1$ we get the Fermat's solution. In [6], it was proved that these sets are two special cases of a more general fact. Let the sequence (g_n) be defined as:

$$g_0 = 0,\ \ g_1 = 1,\ \ g_n = pg_{n-1} - g_{n-2},\ \ n \geq 2,$$

where $p \geq 2$ is an integer. Then the sets

$$\{g_n, g_{n+2}, (p \pm 2)g_{n+1}, 4g_{n+1}[(p \pm 2)g_{n+1}^2 \mp 1]\}$$

have the property of Diophantus. For $p = 2$ we get the set (1), and for $p = 3$ we get the set (2).

In [1] and [4], the more general problem was considered. Let n be an integer. A set of positive integers is said to have *the property of Diophantus of order* n, symbolically $D(n)$, if the product of any two distinct elements increased by n is a perfect square. If n is an integer of the form $4k+2$, $k \in \mathbf{Z}$, then there does not exist a Diophantine quadruple with the property $D(n)$ (see [1, Theorem 1], [4, Theorem 4] or [9, p. 802]). If an integer n is not of the form $4k+2$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one Diophantine quadruple with the property $D(n)$, and if $n \notin S \cup T$, where $T = \{-15, -12, -7, 7, 13, 15, 21, 24, 28, 32, 48, 60, 84\}$, then there exist at least two distinct Diophantine quadruples with the property $D(n)$ (see [4, Theorems 5 and 6] and [5, p. 315]). The proof of the former result is based on the fact that the sets

$$\{m, m(3k+1)^2 + 2k, m(3k+2)^2 + 2k + 2, 9m(2k+1)^2 + 8k + 4\} \tag{3}$$

$$\{m, mk^2 - 2k - 2, m(k+1)^2 - 2k, m(2k+1)^2 - 8k - 4\}, \tag{4}$$

have the property $D(2(2k+1)m+1)$. The formulas of this type were systematically derived in [7].

In this paper we consider the analogous problem for Gaussian integers. Let z be a Gaussian integer and let $m \geq 2$ be an integer. A set $\{a_1, a_2, \ldots, a_m\} \subset \mathbf{Z}[i] \setminus \{0\}$ is said to have *the property* $D(z)$ if the product of any two distinct elements increased by z is a square of a Gaussian integer. Such a set is called *a complex Diophantine* m*-tuple*.

The requirement that $a_i \neq 0$ is not essential, except that if we omit this requirement, the assertion of Theorem 3 becomes trivial. If the set $\{a_1, a_2, \ldots, a_m\}$ is a complex Diophantine quadruple then the same is true for the set $\{-a_1, -a_2, \ldots, -a_m\}$. These two m-tuples are said to be *equivalent*.

It is proved (Theorem 1) that if b is an odd integer or $a \equiv b \equiv 2 \pmod 4$, then there does not exist a complex Diophantine quadruple with the property $D(a+bi)$. It is interesting that this condition is equivalent to the condition that $a+bi$ is not representable as a difference of the squares of two Gaussian integers. Therefore this result is an analogue of the corresponding result for ordinary integers, because an integer n is of the form $4k+2$ iff n is not representable as a difference of the squares of two integers.

It is also proved (Theorem 2) that if a Gaussian integer z is representable as a difference of the squares of two Gaussian integers and $z \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then there exist at least two nonequivalent complex Diophantine quadruples with the property $D(z)$.

At the end of this paper, complex Diophantine quadruples with the property $D(l^2)$, $l \in \mathbf{Z}[i]$, are considered. It is proved (Theorem 3) that every complex Diophantine pair $\{a, b\}$ with the property $D(l^2)$, where ab is not a perfect square, can be extended to the complex Diophantine quadruple with the same property in an infinite number of ways. This is an analogue of the results from [10] and [4] for Diophantine quadruples with the properties $D(1)$ and $D(l^2)$, $l \in \mathbf{Z}$.

2 The problem of the existence of the complex Diophantine quadruples

THEOREM 1 *If b is an odd integer or if $a \equiv b \equiv 2 \pmod 4$, then there does not exist a complex Diophantine quadruple with the property*

$D(a+bi)$.

Proof. **1)** Suppose that b is an odd integer and the set $\{a_j + b_j i : j = 1,2,3,4\}$ has the property $D(a+bi)$. It follows that for $j \neq k$ there exist integers c_{jk} and d_{jk} such that

$$a_j a_k - b_j b_k + a + i(a_j b_k + a_k b_j + b) = c_{jk}^2 - d_{jk}^2 + 2c_{jk}d_{jk}i\,. \tag{5}$$

This gives $a_j b_k + a_k b_j \equiv 1 \pmod 2$. Therefore, at most one a_j and at most one b_j are even. So, without loss of generality we can assume that a_1, a_2, b_1, b_2 are odd. Hence, $a_1 b_2 + a_2 b_1 \equiv 0 \pmod 2$, which is a contradiction.

2) Suppose that $a \equiv b \equiv 2 \pmod 4$ and the set $\{a_j + b_j i : j = 1,2,3,4\}$ has the property $D(a+bi)$. Then

$$a_j b_k + a_k b_j \equiv 0 \pmod 2\,. \tag{6}$$

Furthermore, from $c_{jk}^2 - d_{jk}^2 \not\equiv 2 \pmod 4$, we conclude that

$$a_j a_k - b_j b_k \not\equiv 0 \pmod 4\,. \tag{7}$$

Suppose that there does not exist $j \in \{1,2,3,4\}$ satisfying $a_j \not\equiv b_j \pmod 2$. It follows that the set

$$\{\frac{a_j + b_j i}{1+i} = \frac{a_j + b_j}{2} + \frac{b_j - a_j}{2} i \;:\; j = 1,2,3,4\}$$

has the property $D(\frac{a+bi}{2i})$, contrary to **1)**.

Accordingly, we can assume that a_1 is even and b_1 is odd. Now we conclude from (6) that a_2, a_3, a_4 are even. Hence at most one b_j is even, by (7). Thus, we can assume that b_2 and b_3 are odd. It follows that in (5), for $j,k \in \{1,2,3\}$, the product $c_{jk}d_{jk}$ is even, because $a_j a_k - b_j b_k + a = c_{jk}^2 - d_{jk}^2$ is odd. Therefore

$$a_j b_k + a_k b_j \equiv 2 \pmod 4\,.$$

Let $a_j = 2r_j$, $j = 1,2,3$. Then

$$r_j b_k + r_k b_j \equiv 1 \pmod 2\,,$$

which contradicts the fact that there exist at least two numbers with the same parity, between the numbers r_1, r_2, r_3. ■

The condition of Theorem 1 is equivalent to the condition that the Gaussian integer $a+bi$ is not representable as a difference of the square of two Gaussian integers (see [14, p. 449]). Thus we can now rephrase Theorem 1 as follows.

Corollary 1 *If a Gaussian integer z is not representable as a difference of the squares of two Gaussian integers, then there does not exist a complex Diophantine quadruple with the property $D(z)$.*

Remark 1 If the set $\{a_1, a_2, \dots a_m\} \subset \mathbf{Z}[i]$ has the property $D(z)$, then the set $\{ia_1, ia_2, \dots, ia_m\}$ has the property $D(-z)$, and the set $\{\overline{a_1}, \overline{a_2}, \dots, \overline{a_m}\}$ has the property $D(\overline{z})$. According to this, we obtain that for every $z \in \mathbf{Z}[i]$ the families of all nonequivalent Diophantine quadruples with the properties $D(z)$, $D(-z)$, $D(\overline{z})$ and $D(-\overline{z})$ have the same cardinality.

Theorem 2 *Let $z = x + yi$ be a Gaussian integer. Suppose that y is even and that $x \equiv 2 \pmod 4$ implies $y \equiv 0 \pmod 4$. If $z \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then there exist at least two nonequivalent complex Diophantine quadruples with the property $D(z)$.*

Proof. Let us introduce the following temporary definition. A Gaussian integer is said to have the property (P) if there exist at least two nonequivalent complex Diophantine quadruples with the property $D(z)$.

If a Gaussian integer z satisfies the condition of Theorem 2, then z can be represented in one of the following forms

$$z = (2a+1)+2bi, \quad z = 4a+(4b+2)i, \quad z = 4a+4bi, \quad z = (4a+2)+4bi.$$

Therefore the proof falls naturally into four parts.

1) The sets

$$\begin{gathered}\{1,\ (9a^2+8a-9b^2+1)+(18ab+8b)i, \\ (9a^2+14a-9b^2+6)+(18ab+14b)i, \\ (36a^2+44a-36b^2+13)+(72ab+44b)i\},\end{gathered} \tag{8}$$

$$\begin{gathered}\{1,\ (a^2-2a-b^2-2)+(2ab-2b)i, (a^2-b^2+1)+2abi, \\ (4a^2-4a-4b^2-3)+(8ab-4b)i\}\end{gathered} \tag{9}$$

have the property $D((4a+3)+4bi)$. These sets are obtained from (3) and (4) for $m = 1$ and $k = a + bi$. It remains to determine the pairs (a,b) for which the above sets have at least two equal elements or some of the elements equal to zero, and the pairs (a,b) for which the quadruples (8) and (9) are equivalent. It is easy to check that the above cases appear iff $(a,b) \in \{(-1,0),(0,0),(1,0),(2,0),(3,0)\}$. Consequently, all Gaussian integers of the form $z=(4a+3)+4bi$, $z \notin \{-1,3,7,11,15\}$, have the property (P).

Since there exist an infinite number of (positive integer) Diophantine quadruples with the property $D(1)$, Remark 1 implies that there exist an infinite number of complex Diophantine quadruples with the property $D(-1)$. From [4, Theorems 5 and 6] we see that there exist at least two (positive integer) Diophantine quadruples with the property $D(11)$, and at least one (positive integer) Diophantine quadruple with the properties $D(-7)$, $D(7)$, $D(-15)$ and $D(15)$. A trivial verification shows that the sets $\{1,-4,-3+2i,-3-2i\}$ and $\{2,-14,-3+2i,-3-2i\}$ have the property $D(3)$. Thus we have proved that all Gaussian integers of the form $z=(4a+3)+4bi$ have the property (P).

The sets

$$\begin{gathered}\{i,\, (18ab+6a+2b)-(9a^2+2a-9b^2-6b-1)i,\\ (18ab+12a+2b+2)-(9a^2+2a-9b^2-12b-4)i,\\ (72ab+36a+8b+4)-(36a^2+8a-36b^2-36b-9)i\},\end{gathered} \tag{10}$$

$$\begin{gathered}\{i,\, (2ab-2b-2)-(a^2-2a-b^2)i,\\ (2ab+2a-2b)-(a^2-2a-b^2-2b-1)i,\\ (8ab+4a-8b-4)-(4a^2-8a-4b^2-4b-1)i\}\end{gathered} \tag{11}$$

have the property $D((4a+1)+(4b+2)i)$. These sets are obtained from (3) and (4) for $m=i$ and $k=b-ai$. Analysis similar to the above shows that the sets (10) and (11) are two nonequivalent complex Diophantine quadruples iff $(a,b) \notin \{(0,-1),(0,0),(2,-2),(2,-1),(2,0),(2,1)\}$. Consequently, for $z=(4a+1)+(4b+2)i$, $z \notin \{1 \pm 2i, 9 \pm 2i, 9 \pm 6i\}$, there exist at least two nonequivalent complex Diophantine quadruples with the property $D(z)$.

For $z \in \{9 \pm 2i, 9 \pm 6i\}$, the relation (10) yields one complex Diophantine quadruple with the property $D(z)$. According to the fact that the sets $\{-i, 2-6i, 6-5i, 6+7i\}$ and $\{i,-2+5i,2+6i,6+9i\}$ have the properties $D(9+6i)$ and $D(9+2i)$ respectively, we deduce that all Gaussian integers of the form $z=(4a+1)+(4b+2)i$, $z \neq 1 \pm 2i$, have the property (P).

If z is of the form $(4a+3)+4bi$, then $-z$ is of the form $(4c+1)+4di$, and if z is of the form $(4a+1)+(4b+2)i$, then $-z$ is of the form

$(4c+3)+(4d+2)i$. From this and Remark 1 it follows that all Gaussian integers of the form $z = (4a+1)+4bi$ or $z = (4a+3)+(4b+2)i$, $z \neq -1 \pm 2i$, have the property (P). Thus we have proved that all $z \in \mathbf{Z}[i]$ of the form $z = (2a+1)+2bi$, $z \neq \pm 1 \pm 2i$, have the property (P).

2) Since $(1+i)^2 = 2i$, multiplying all elements of the set with the property $D((2a+1)+2bi)$ by $1+i$ we get the set with the property $D(-4b+(4a+2)i)$. Applying **1**) we conclude that all $z \in \mathbf{Z}[i]$ of the form $z = 4a+(4b+2)i$, $z \neq \pm 4 \pm 2i$, have the property (P).

Finally, the fact that the sets $\{2, -i, -2-i, -6-4i\}$ and $\{2, -2+3i, -4-i, -14+4i\}$ have the property $D(4+2i)$ implies that all Gaussian integers of the form $z = 4a+(4b+2)i$ have the property (P).

3) Multiplying all elements of the set with the property $D(4a+(4b+2)i)$ by $1+i$ we get the set with the property $D((-8b-4)+8ai)$. Applying **2**) we conclude that all $z \in \mathbf{Z}[i]$ of the form $z = (8a+4)+8bi$ have the property (P).

The sets

$$\begin{gathered}\{1,\, (9a^2-8a-9b^2)+(18ab-8b)i,\\ (9a^2-2a-9b^2+1)+(18ab-2b)i,\\ (36a^2-20a-36b^2+1)+(72ab-20b)i\},\end{gathered} \tag{12}$$

$$\begin{gathered}\{1,\, (a^2-6a-b^2+1)+(2ab-6b)i,\\ (a^2-4a-b^2+4)+(2ab-4b)i,\\ (4a^2-20a-4b^2+9)+(8ab-20b)i\}\end{gathered} \tag{13}$$

have the property $D(8a+8bi)$. These sets are obtained from [4, (20) and (10)] for $k = a+bi$. These sets are two nonequivalent complex Diophantine quadruples iff $(a,b) \notin \{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0)\}$. Hence, all $z \in \mathbf{Z}[i]$ of the form $z = 8a+8bi$, $z \notin Z_1 = \{0, 8, 16, 24, 32, 40, 48\}$, have the property (P).

Since $-8, -16, -24, -32, -40, -48 \notin Z_1$, Remark 1 implies that all Gaussian integers of the form $z = 8a+8bi$ have the property (P).

Multiplying all elements of the sets (3) and (4) by 2, for $m = \frac{i}{2}$ and $k = b-ai$, we get the sets

$$\begin{gathered}\{i,\, (18ab+6a+4b)-(9a^2+4a-9b^2-6b-1)i,\\ (18ab+12a+4b+4)-(9a^2+4a-9b^2-12b-4)i,\\ (72ab+36a+16b+8)-(36a^2+16a-36b^2-36b-9)i\},\end{gathered} \tag{14}$$

$$\begin{gathered}\{i,\ (2ab-4b-4)-(a^2-4a-b^2)i, \\ (2ab+2a-4b)-(a^2-4a-b^2-2b-1)i, \\ (8ab+4a-16b-8)-(4a^2-16a-4b^2-4b-1)i\}\end{gathered} \tag{15}$$

with the property $D((8a+4)+(8b+4)i)$. These sets yield two nonequivalent complex Diophantine quadruples for all Gaussian integers of the form $z=(8a+4)+(8b+4)i$, $z\notin Z_2=\{4\pm 4i, 36\pm 4i, 36\pm 12i\}$.

Since $-4+4i, -36+4i, -36+12i\notin Z_2$, Remark 1 implies that all Gaussian integers of the form $z=(8a+4)+(8b+4)i$ have the property (P).

Multiplying all elements of the sets (3) and (4) by 2, for $m=\frac{1+i}{2}$ and $k=(a+b)+(b-a)i$, we get the sets

$$\begin{gathered}\{1+i,\ (18a^2+36ab+16a-18b^2+4b+1) \\ -(18a^2-36ab+4a-18b^2-16b-1)i, \\ (18a^2+36ab+28a-18b^2+4b+8) \\ -(18a^2-36ab+4a-18b^2-28b-4)i, \\ (72a^2+144a+88a-72b^2+16b+17) \\ -(72a^2-144ab+16a-72b^2-88b-9)i\},\end{gathered} \tag{16}$$

$$\begin{gathered}\{1+i,\ (2a^2+4ab-4a-2b^2-4b-4) \\ -(2a^2-4ab-4a-2b^2+4b)i, \\ (2a^2+4ab-2b^2-4b+1)-(2a^2-4ab-4a-2b^2-1)i, \\ (8a^2+16ab-8a-8b^2-16b-7) \\ -(8a^2-16ab-16a-8b^2+8b-1)i\}\end{gathered} \tag{17}$$

with the property $D((16a+8)+(16b+4)i)$, and for $m=\frac{1+i}{2}$ and $k=(a+b)+(b-a+1)i$, we get the sets

$$\begin{gathered}\{1+i,\ (18a^2+36ab+16a-18b^2-32b-14) \\ -(18a^2-36ab-32a-18b^2-16b-2)i, \\ (18a^2+36ab+28a-18b^2-32b-13) \\ -(18a^2-36ab-32a-18b^2-28b-11)i, \\ (72a^2+144ab+88a-72b^2-128b-55) \\ -(72a^2-144ab-128a-72b^2-88b-25)i\},\end{gathered} \tag{18}$$

$$\begin{gathered}\{1+i,\ (2a^2+4ab-4a-2b^2-8b-5) \\ -(2a^2-4ab-8a-2b^2+4b+5)i, \\ (2a^2+4ab-2b^2-8b-2)-(2a^2-4ab-8a-2b^2+2)i, \\ (8a^2+16ab-8a-8b^2-32b-15) \\ -(8a^2-16ab-32a-8b^2+8b+15)i\}\end{gathered} \tag{19}$$

with the property $D(16a+(16b+12)i)$. From these formulas it follows that if $z=(16a+8)+(16b+4)i$, $z\notin Z_3=\{8+4i, 24-12i, 40+4i\}$, or if $z=16a+(16b+12)i$, $z\notin Z_4=\{-4i, 32-4i, 48+12i\}$, then there exist at least two nonequivalent complex Diophantine quadruples with the property $D(z)$.

Since $-8+4i, 24-12i, 40+4i\notin Z_3$ and $-32-4i, -48+12i\notin Z_4$, Remark 1 implies that all Gaussian integers of the form $z=(16a+8)+(16b+4)i$ and $z=16a+(16b+12)i$, $z\neq -4i$, have the property (P).

Repeated application of Remark 1 enables us to conclude that all Gaussian integers of the form $z=(16a+8)+(16b+12)i$ or $z=16a+(16b+4)i$, $z\neq 4i$, have the property (P). This finishes the proof of the fact that all Gaussian integers of the form $z=4a+4bi$, $z\neq \pm 4i$, have the property (P).

4) Multiplying all elements of the sets (3) and (4) by $1+i$, for $m=\frac{1-i}{2}$ and $k=(a+b)+(b-a)i$, we get the sets

$$\begin{gathered}\{1,\ (36ab+10a+6b+1)-(18a^2+6a-18b^2-10b)i,\\ (36ab+16a+12b+6)-(18a^2+12a-18b^2-16b-2)i,\\ (144ab+52a+36b+13)-(72a^2+36a-72b^2-52b-4)i\},\end{gathered} \tag{20}$$

$$\begin{gathered}\{1,\ (4ab-4a-2)-(2a^2-2b^2+4b+2)i,\\ (4ab-2a+2b+1)-(2a^2+2a-2b^2+2b)i,\\ (16ab-12a+4b-3)-(8a^2+4a-8b^2+12b+4)i\}\end{gathered} \tag{21}$$

with the property $D((8a+2)+(8b+4)i)$, and for $m=\frac{1-i}{2}$ and $k=(a+b)+(b-a-1)i$, we get the sets

$$\begin{gathered}\{1,\ (36ab-8a+24b-6)-(18a^2+24a-18b^2+8b+8)i,\\ (36ab-2a+30b-1)-(18a^2+30a-18b^2+2b+12)i,\\ (144ab-20a+108b-15)-(72a^2+108a-72b^2+20b+40)i\},\end{gathered} \tag{22}$$

$$\begin{gathered}\{1,\ (4ab-6a+2b-5)-(2a^2+2a-2b^2+6b)i,\\ (4ab-4a+4b-2)-(2a^2+4a-2b^2+4b)i,\\ (16ab-20a+12b-15)-(8a^2+12a-8b^2+20b)i\}\end{gathered} \tag{23}$$

with the property $D((8a+6)+8bi)$. These formulas yield two nonequivalent complex Diophantine quadruples with the property $D(z)$ for all $z=(8a+2)+(8b+4)i$, $z\notin \{2+4i, 2+20i, -6+12i\}$, and for all $z=(8a+6)+8bi$, $z\notin\{-2, -2+16i, 6+24i\}$.

From Remark 1 it may be concluded that for all Gaussian integers of the forms $z=(8a+2)+(8b+4)i$, $z=(8a+6)+(8b+4)i$,

$z = (8a+6)+8bi$, $z \neq -2$, or $z = (8a+2)+8bi$, $z \neq 2$, have the property (P). Hence, all Gaussian integers of the form $z = (4a+2)+4bi$, $z \neq \pm 2$, have the property (P), which completes the proof. ∎

COROLLARY 2 *Let $n \neq \pm 2$ be an integer. Then there exist at least two nonequivalent complex Diophantine quadruples with the property $D(n)$.*

3 Quadruples with the property $D(l^2)$

Consider now complex Diophantine quadruples with the property $D(z)$, where z is a square of a Gaussian integer. We will prove an analogue of the result for ordinary integers from [4].

Let $\{a, b\} \subset \mathbf{Z}[i] \setminus \{0\}$ be the set with the property $D(l^2)$, where $l \in \mathbf{Z}[i]$. This gives

$$ab + l^2 = k^2 .$$

We can certainly assume that $l \neq 0$. Suppose that ab is not a square of a Gaussian integer. Under the above assumption there exists at least one pair of Gaussian integers S and $T \neq 0$, which satisfy the Pell equation

$$S^2 - abT^2 = 1 \tag{24}$$

(see [8, Theorem 1]). Let $s + t\sqrt{ab}$ be the fundamental solution of the equation (24). Now we can apply the construction from [4]. We define three double sequences $y_{n,m}$, $z_{n,m}$ and $x_{n,m}$, $n, m \in \mathbf{Z}$ as follows:

$$\begin{aligned} y_{n,m} &= \frac{l}{2\sqrt{b}}\{(\sqrt{a}+\sqrt{b})[\frac{1}{l}(k+\sqrt{ab})]^n(s+t\sqrt{ab})^m \\ &+ (\sqrt{b}-\sqrt{a})[\frac{1}{l}(k-\sqrt{ab})]^n(s-t\sqrt{ab})^m\}, \\ z_{n,m} &= \frac{l}{2\sqrt{a}}\{(\sqrt{a}+\sqrt{b})[\frac{1}{l}(k+\sqrt{ab})]^n(s+t\sqrt{ab})^m \\ &+ (\sqrt{a}-\sqrt{b})[\frac{1}{l}(k-\sqrt{ab})]^n(s-t\sqrt{ab})^m\}, \\ x_{n,m} &= (y_{n,m}^2 - l^2)/a = (z_{n,m}^2 - l^2)/b . \end{aligned}$$

Analysis similar to those in [4, Theorems 1 and 2] shows that if $x_{n,m}$ and $x_{n+1,m}$ are Gaussian integers, then the set $\{a, b, x_{n,m}, x_{n+1,m}\}$ has the property $D(l^2)$. From [4, (6)] we deduce that $x_{-1,m}$, $x_{0,m}$ and $x_{1,m}$ are Gaussian integers for all integers m. We are now in the position to show that by means of these numbers we can construct an infinite number of Diophantine quadruples with the property $D(l^2)$. We need the following lemma which is easily proved.

Lemma 1 *Let* $n \in \{-1, 0, 1\}$. *If* j *and* m *are integers such that* $y_{n,j} = y_{n,m}$ *and* $z_{n,j} = z_{n,m}$, *then* $j = m$.

Theorem 3 *Let* l *be a Gaussian integer and suppose that the set* $\{a, b\} \subset \mathbf{Z}[i]$ *has the property* $D(l^2)$. *If the number* ab *is not a square of a Gaussian integer, then there exist an infinite number of complex Diophantine quadruples of the form* $\{a, b, c, d\}$ *with the property* $D(l^2)$.

Proof. Consider the sets $\{a, b, x_{-1,m}, x_{0,m}\}$ and $\{a, b, x_{0,m}, x_{1,m}\}$ for $m \in \mathbf{Z}$. If $x_{n,m} = 0$, then $y_{n,m}^2 = l^2$ and $z_{n,m}^2 = l^2$, i.e. $y_{n,m}, z_{n,m} \in \{-l, l\}$. From Lemma 1 we conclude that for $n \in \{-1, 0, 1\}$ there are at most four numbers which are equal to zero, between the numbers $x_{n,m}$, $m \in \mathbf{Z}$. In the same manner we can see that for $n \in \{-1, 0, 1\}$ there are at most four numbers which are equal to a, and at most four numbers which are equal to b, between the numbers $x_{n,m}$, $m \in \mathbf{Z}$. It is easily seen that the relation $x_{0,m} = x_{\pm 1,m}$ implies $y_{0,m}^2 = \frac{al(b-a)}{2(k-l)}$, and the former equality holds for at most four integers m, by Lemma 1.

Thus the sets $\{a, b, x_{0,m}, x_{-1,m}\}$ and $\{a, b, x_{0,m}, x_{1,m}\}$ are complex Diophantine quadruples with the property $D(l^2)$ for all but a finite number of integers m, and we conclude from Lemma 1 that there is an infinite number of distinct sets between them. ■

Example 1 In [1] and [13], it was proved that the Diophantine triples $\{1, 2, 5\}$ and $\{1, 5, 10\}$, with the property $D(-1)$, cannot be extended to the Diophantine quadruples with the same property. We will show how these triples can be extended to the complex Diophantine quadruples with the property $D(-1)$. Since the number $-1 = i^2$ is a perfect square in $\mathbf{Z}[i]$, and the number $1 \cdot 5 = 5$ is not, we can apply our construction to the relation

$$1 \cdot 5 + i^2 = 2^2 .$$

We have $a = 1$, $b = 5$, $l = i$, $k = 2$, and thus $x_{0,0} = 0$, $x_{1,0} = 10$, $x_{-1,0} = 2$, $x_{2,0} = -168$, $x_{-2,0} = -24$, $x_{3,0} = 3026$, $x_{-3,0} = 442$, $\ldots$.

Therefore the sets $\{1,2,5,-24\}$, $\{1,5,10,-168\}$, $\{1,5,-24,442\}$ and $\{1,5,-168,3026\}$ are complex Diophantine quadruples with the property $D(-1)$. In this case, the use of the double sequence was unnecessary because from $s=2i$ and $t=-i$ it follows that $x_{n,m}=x_{n-m,0}$.

EXAMPLE 2

$$3\cdot(-4i)+(1+3i)^2=(1-3i)^2\,.$$

In this case, $x_{-1,m}, x_{0,m}, x_{1,m}, x_{2,m}, x_{3,m} \in \mathbf{Z}[i]$, and from $s=5$ and $t=1+i$ we obtain the following sets with the property $D(-8+6i)$:

$$\{3,-4i,5-10i,40+20i\},\ \ \{3,-4i,40+20i,-111+138i\},$$
$$\{3,-4i,9+58i,-240-20i\},\ \ \{3,-4i,-240i-20i,285-970i\},$$
$$\{3,-4i,285-970i,3720+1960i\},$$
$$\{3,-4i,3720+1960i,-11079+13522i\},$$
$$\{3,-4i,681+5682i,-23720-1960i\},\ \ldots\,.$$

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