We show that the only solutions of this equation are $u = 0, 1$. For on dividing out by

$$2\binom{u}{2}\left(\frac{y_1}{x_1}\right)^2,$$

we have

$$\frac{1}{2} + \binom{u-2}{3}\frac{d}{4.5}\left(\frac{y_1}{x_1}\right)^3 + \binom{u-2}{6}\frac{d^2}{7.8}\left(\frac{y_1}{x_1}\right)^6 + \cdots = 0.$$

This gives an impossible congruence mod 3, since 4, 5, 7, 8, . . . are 3-adic units.

Suppose next that $y_1 \not\equiv 0 \pmod 3$. Since

$$\begin{aligned}\eta_1^3 &= x_1^3 + 3x_1^2y_1\theta + 3x_1y_1^2\theta^2 + y_1^3\theta^3\\ &= 1 + 3x_1^2y_1\theta + 3x_1y_1^2\theta^2 = 1 + 3\delta,\end{aligned}$$

say, η_1^u is defined for $u \equiv 0 \pmod 3$. Write $u = 3v + u_0$, $u_0 = 0, 1, 2$.

Then $$\eta_1^{3v+u_0} \equiv \eta_2 \pmod 3, \qquad \eta_1^{u_0} \equiv \eta_2 \pmod 3.$$

Comparing coefficients of θ^2, we see that $u_0 = 0, 1$.
Take first $u_0 = 0$. Then

$$(1 + 3\delta)^v = x_2 + y_2\theta.$$

Denote by b_t the coefficient of θ^2 in δ^t. Then

$$\sum_{t=0}^{\infty} 3^t b_t \binom{v}{t} = 0,$$

or $$3x_1y_1^2v + 3^2x_1^4y_1^2\binom{v}{2} + \cdots = 0.$$

Divide out by $3x_1y_1^2$, and we have, say,

$$v + 3B_2\binom{v}{2} + 3^2B_3\binom{v}{3} + \cdots = 0,$$

where the B are polynomials in x_1, y_1 with integer coefficients. This is impossible, for if 3^λ is the highest power of 3 dividing v, all the other terms are divisible by $3^{\lambda+1}$. For the general term is

$$3^{t-1}B_t\binom{v}{t} = 3^{t-1}\frac{vB_t}{t}\binom{v-1}{t-1}$$

and $3^{t-2}/t$ is a 3-adic integer. This is obvious on putting $t = t_0 3^\mu$ where 3^μ is the highest power of 3 dividing t.

Suppose next $u_0 = 1$. Then

$$(x_1 + y_1\theta)\left(\sum_{t=0}^{\infty} 3^t\delta^t\binom{v}{t}\right) = x_2 + y_2\theta.$$

Denote by c_t the coefficient of θ in δ^t. Then

$$x_1\sum_{t=0}^{\infty} 3^t b_t\binom{v}{t} + y_1\sum_{t=0}^{\infty} 3^t c_t\binom{v}{t} = 0.$$

Dividing out by $3x_1^2y_1^2$, we have, say,

$$2v + 3c_1\binom{v}{2} + 3^2c_2\binom{v}{3} + \cdots = 0,$$

and this is impossible as before.

3. When it is required to find all the integer solutions of an equation, the process may not be very complicated if fundamental units are not involved as in the following theorem. This deals with a conjecture enunciated by Ramanujan and first proved by Nagell. Many other proofs have been given and these have been analysed and discussed by Hasse. He has given a simpler version of Nagell's[4] proof as well as a generalization. We give Hasse's[5] proof of

Theorem 6

The equation $x^2 + 7 = 2^n$ has only the positive integer solutions given by $x = 1, 3, 5, 11, 181$ corresponding to $n = 3, 4, 5, 7, 15$.

When n is even, $n = 4$ is the only solution since

$$(2^{n/2})^2 - x^2 = 7, \qquad 2^{n/2} \pm x = 7, \qquad 2^{n/2} \mp x = 1.$$

We may now suppose that n is odd, and we write the equation as

$$\frac{x^2+7}{4} = 2^y, \tag{10}$$

where y is odd and $y \geqslant 3$.

We factorize the equation in the field $Q(\sqrt{-7})$, in which the integers have the form $(m + n\sqrt{-7})/2$ where $m \equiv n \pmod 2$, and in which unique factorization holds. Since

$$2 = \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right),$$

we have $$\frac{x+\sqrt{-7}}{2} = \pm\left(\frac{1\pm\sqrt{-7}}{2}\right)^y,$$

and so $$\left(\frac{1+\sqrt{-7}}{2}\right)^y - \left(\frac{1-\sqrt{-7}}{2}\right)^y = \pm\sqrt{-7}. \tag{11}$$

We show that the positive sign is impossible in equation (11). Write this as

$$a^y - b^y = a - b.$$

Then $$a^2 \equiv (1-b)^2 \equiv 1 \pmod{b^2},$$

since $ab = 2$, and so

$$a^y \equiv a(a^2)^{(y-1)/2} \equiv a \pmod{b^2},$$

$$a \equiv a - b \pmod{b^2},$$

which is false.
Hence we have

$$-2^{y-1} = \binom{y}{1} - \binom{y}{3}7 + \binom{y}{5}7^2 \cdots \pm \binom{y}{y}7^{(y-1)/2}, \tag{12}$$

and so $$-2^{y-1} \equiv y \pmod 7.$$

This has the odd solutions $y \equiv 3, 5, 13 \pmod{42}$. We prove that 3, 5, 13 are the only remaining solutions for y in equation (10).

It suffices to show that there cannot be two solutions y, y_1 with $y_1 - y \equiv 0 \pmod{42}$. Suppose that 7^l is the greatest power of 7 dividing $y_1 - y$. Then

$$a^{y_1} = a^y . a^{y_1-y} = a^y(\tfrac{1}{2})^{y_1-y}(1 + \sqrt{-7})^{y_1-y}. \tag{13}$$

Now $$(\tfrac{1}{2})^{y_1-y} = ((\tfrac{1}{2})^6)^{(y_1-y)/6} \equiv 1 \pmod{7^{l+1}}.$$

Also $$(1 + \sqrt{-7})^{y_1-y} \equiv 1 + (y_1 - y)\sqrt{-7} \pmod{7^{l+1}}$$

as follows on raising $1 + \sqrt{-7}$ successively to powers $7, 7^2, \ldots, 7^l$ and then to the power $(y_1 - y)/7^l$.

Since $$a^y \equiv \frac{1 + y\sqrt{-7}}{2^y} \pmod 7,$$

on substituting in equation (13), we have

$$a^{y_1} \equiv a^y + \frac{(y_1 - y)}{2^y}\sqrt{-7} \pmod{7^{l+1}}. \tag{14}$$

Similarly $$b^{y_1} \equiv b^y + \frac{(y_1 - y)}{2^y}\sqrt{-7} \pmod{7^{l+1}}.$$

Since from equation (11),

$$a^y - b^y = a^{y_1} - b^{y_1},$$

then $$(y_1 - y)\sqrt{-7} \equiv 0 \pmod{7^{l+1}},$$

and so since y_1, y are rational

$$y_1 - y \equiv 0 \pmod{7^{l+1}}.$$

This contradiction establishes the theorem.

4. When the numerical values of the fundamental units must be considered, in general, a great deal of detailed numerical work, and several cases may arise.

It is only exceptionally that little detail is required and then the result may perhaps be found more simply by classical methods. Thus we have the proof by Skolem[3] of

Theorem 7

The only integer solutions of the equation

$$x^4 - 2y^4 = 1 \tag{15}$$

are given by $y = 0, x = \pm 1$.

Clearly $y \equiv 0 \pmod 2$ and so we may replace the equation by the equation

$$x^4 - 32y^4 = 1. \tag{16}$$

It can be shown that the fundamental units in the field $Q(\theta)$, where $\theta = \sqrt[4]{2}$, are given by $1 + \theta$ and $1 + \theta^2$. Since $x + 2y\theta$ is a unit in the field, we have

$$(1 + \theta)^u(1 + \theta^2)^v = x + 2y\theta, \tag{17}$$

$$\left(1 + u\theta + \frac{u.u-1}{2!} + \cdots\right)(1 + v\theta^2 + \cdots) \equiv x \pmod 2,$$

and so $u \equiv 0 \pmod 2$.

Since the coefficients of θ^2, θ^3 vanish in the left-hand side of equation (17), we deduce the p-adic equations:

$$\binom{u}{2} + v + 2\left(\binom{u}{6} + \binom{u}{4}v + \binom{u}{2}\binom{v}{2} + \binom{v}{3}\right) + 2^2(\ldots) = 0.$$

$$\binom{u}{3} + uv + 2\left(\binom{u}{7} + \binom{u}{5}v + \binom{u}{3}\binom{v}{2} + u\binom{v}{3}\right) + 2^2(\ldots) = 0.$$

Multiply the first equation by u and subtract from the second. Then

$$-\tfrac{1}{3}(u + 1)u(u - 1) + 2\left\{\binom{u}{7} - u\binom{u}{6} + \left(\binom{u}{5} - u\binom{u}{4}\right)v \right.$$
$$\left. + \left(\binom{u}{3} - u\binom{u}{2}\right)\binom{v}{2}\right\} + 2^2(\ldots) = 0. \tag{18}$$

If $u \neq 0$, suppose that $2^\lambda \parallel u$ and so $\lambda \geqslant 1$. Then

$$\binom{u}{2u+1} = \frac{u}{2u+1}\binom{u-1}{2u} \equiv 0 \pmod{2^\lambda}.$$

Hence all the terms of equation (14) are $\equiv 0 \pmod{2^{\lambda+1}}$ except the first and so we must have $u = 0$.

Then $$v + 2\binom{v}{3} + 2^2\binom{v}{5} + \cdots = 0,$$

and a similar argument shows that $v = 0$.