KARAKTERISTIČNE FUNKCIJE

$\boxed{\text{DEF}}$ Ako je $X : \Omega \to \mathbb{R}$ sluč. varijabla, <u>karakteristična funkcija</u> od X je fja $\varphi : \mathbb{R} \to \mathbb{C}$ dana sa

$$\varphi(t) = \varphi_X(t) = Ee^{itX} = E(\cos tX + i \sin tX)$$

TEOREM 10 (elementarna svojstva karak. fje)

a) $\varphi(0) = 1$

b) $\varphi(-t) = \overline{\varphi(t)}$

c) $|\varphi(t)| = |Ee^{itX}| \leq E|e^{itX}| \leq 1$

d) $|\varphi(t+h)-\varphi(t)| \le E|e^{ihX}-1| \Rightarrow \varphi$ je uniformno neprekidna na $\mathbb{R}$

e) $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$

f) X_1, X_2 nezavisne $\Rightarrow \varphi_{X_1+X_2} = \varphi_{X_1}\cdot\varphi_{X_2}$

Primjer (normalna razdioba)

$$Z \sim N(0,1) \Rightarrow \varphi_Z(t) = e^{-t^2/2}$$

"dokaz fizičara"

$$\int e^{itx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = e^{-t^2/2}\int \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-it)^2}{2}}\,dx = e^{-t^2/2}$$

gustoća $N(it,1)$ razdiobe

Primjer (eksponencijalna razdioba)

$$T \sim Exp(1) \Rightarrow \varphi_T(t) = 1/(1-it)$$

$$\int_0^\infty e^{itx} e^{-x} dx = \frac{e^{(it-1)x}}{it-1}\Big|_0^\infty = \frac{1}{1-it}$$

LEMA 11

Ako fje.dist. $F_1, F_2, \dots, F_n$ imaju karakteristične funkcije $\varphi_1, \dots, \varphi_n$, a $\lambda_i \geq 0$ zadovoljavaju $\sum_1^n \lambda_i = 1 \Rightarrow \sum_1^n \lambda_i F_i$ ima karakt. fju

$$\sum_1^n \lambda_i \varphi_i$$

NAP) Ako je φ kar. fja $\Rightarrow$ to su i $\mathrm{Re}\,\varphi$ i $|\varphi|^2$ DZ

TEOREM 12 (teorem inverzije)

Neka je $\varphi(t) = \int e^{itx}\mu(dx)$ za neku vjeroj. mjeru na $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, za $a<b$ vrijedi

$$\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\varphi(t)\,dt = \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$$

NAP Integral ne mora konvergirati apsolutno

npr $\mu = \delta_0$, $a=-1, b=1 \Rightarrow \frac{e^{-ita}-e^{-itb}}{it} = \frac{2\sin t}{t}$

$\varphi \equiv 1$, dakle podintegralna f-ja nije aps. integrabilna.

Dokaz

$$I_t := \int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\,\varphi(t)\,dt = \int_{-T}^{T}\int \frac{e^{-ita}-e^{-itb}}{it}\, e^{itx}\mu(dx)\,dt$$

Uocimo
$$\int \frac{e^{-ita}-e^{-itb}}{it} = \int_a^b e^{-ity}\,dy \Rightarrow$$

$$\left|\frac{e^{-ita}-e^{-itb}}{it}\right| \le \int_a^b \left|e^{-ity}\right| dy = b-a$$

Fubinijev tm $\Rightarrow$
$$I_t = \int\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\, e^{itx}\,dt\,\mu(dx)$$

$$= \int \left[\int_{-T}^{T} \frac{\sin t(x-a)}{t}\,dt - \int_{-T}^{T} \frac{\sin t(x-b)}{t}\,dt\right] \mu(dx) \quad (*)$$

$$R(\vartheta, T) := \int_{-T}^{T} \frac{\sin \vartheta t}{t}\, dt \implies$$

$$I_t = \int \left[R(x-a, T) - R(x-b, T) \right] \mu(dx)$$

Stavimo li $S(t) = \int_0^t \frac{\sin x}{x}\, dx$, za $\vartheta > 0$, uz zamjenu varijabli $\vartheta t = x \implies$

$$R(\vartheta, T) = 2 \int_0^{T\vartheta} \frac{\sin x}{x}\, dx = 2 S(T\vartheta)$$

za $\vartheta < 0$

$$R(\vartheta, T) = - R(|\vartheta|, T) \implies$$

$$R(\vartheta, T) = 2 \operatorname{sgn} \vartheta \, S(T|\vartheta|)$$

Nadalje:

$$\lim_{T\to\infty} S(T) = \lim_{T\to\infty} \int_0^T \frac{\sin x}{x}\,dx = \frac{\pi}{2} \quad \Rightarrow$$

$$\lim_{T\to\infty} R(\vartheta,T) = 2\,\mathrm{sgn}\,\vartheta \cdot \frac{\pi}{2} = \mathrm{sgn}\,\vartheta \cdot \pi$$

$\Downarrow$

$$R(x-a,T)-R(x-b,T) \to \begin{cases} 2\pi & a<x<b \\ \pi & x=a \text{ ili } b \\ 0 & x<a,\ x>b \end{cases}$$

Kako

$|R(\vartheta,T)| \le 2\sup_y S(y) < \infty$, a konstanta je integrabilna

u odn. na μ, teorem o dom konv. $\Rightarrow$

$$\lim_{T\to\infty} \frac{1}{2\pi} I_t = \frac{1}{2\pi}\int \lim_{T\to\infty}\Big(R((x-a),T) - R(x-b,T)\Big)\mu(dx)$$

$$= \mu(a,b) + \tfrac{1}{2}\mu(\{a,b\}) \qquad \square$$

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NAP)

i) Formula inverzije $\Rightarrow$ distribucija je jednoznačno odredena karakt. fjom

ii) formula inverzije je jednostavnija za integrabilne φ tj. ako postoji gustoća razdiobe

iii) iz dokaza $\Rightarrow$ $\mu\{a\} = \lim\limits_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t)\, dt$

iv) φ realna $\Rightarrow$ $X \stackrel{d}{=} -X$

TEOREM 13 (formula inverzije za neprek. razdiobe)

Ako $\int |\varphi(t)| dt < \infty \Rightarrow \mu$ ima ograničenu neprekidnu fju gustoće

$$f(y) = \frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt$$

Dokaz: I_T $\left| \frac{e^{-ita} - e^{-itb}}{it} \right| = \left| \int_a^b e^{-ity} dy \right| \leq |b-a| \Rightarrow$

$$\int_{-\infty}^{\infty} \left| \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) \right| dt < \infty$$ pa tm o dom konv $\Rightarrow$

$$\lim_{T\to\infty} \frac{1}{2\pi} I_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt$$

Thm 12 $\Rightarrow$

$$\mu(a,b)+\frac{1}{2}\mu(\{a,b\}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ita}-e^{-itb}}{it}\varphi(t)\,dt \leq$$

$$\leq \frac{b-a}{2\pi}\int_{-\infty}^{\infty}|\varphi(t)|\,dt$$

Za $b \to a$ desna strana ide u 0 $\Rightarrow \mu(\{a\})=0$

$\forall a \in \mathbb{R}$ tj. μ nema diskretnu komponentu.

Nadalje: $$\mu(x,x+h) = \frac{1}{2\pi}\int\frac{e^{-itx}-e^{-it(x+h)}}{it}\varphi(t)\,dt$$

$$= \frac{1}{2\pi}\int\left(\int_x^{x+h} e^{-ity}\,dy\right)\varphi(t)\,dt$$

$$= \int_x^{x+h}\left(\frac{1}{2\pi}\int e^{-ity}\varphi(t)\,dt\right)dy \qquad (\text{Fubini})$$

Dakle μ ima funkciju gustoće

$$f(y) = \frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt$$

Jasno je $|f(y)| \leq \frac{1}{2\pi} \int |\varphi(y)|\, dy < \infty$, a

neprekidnost slijedi iz tm o dom. konv. $\square$

NAP Pokažite da je moguće da μ bude neprekidna,

a ipak $\int |\varphi(t)|\, dt = \infty$. (DZ)

Ako $\varphi(t) \to 0$ za $t \to \infty$, već znamo da

μ nema diskretnu komponentu (DZ.)