LEMA 46 $X_1+\cdots+X_n \sim F^n$, $X_1^\lambda+\cdots+X_n^\lambda \sim F_\lambda^n$

$$\frac{dF^n}{dF_\lambda^n}(x) = e^{-\lambda x} \varphi(\lambda)^n$$

TEOREM 47 (Cramér)

Pretpostavite $E e^{\vartheta X_i} < \infty$, za neki $\vartheta > 0$, a razdioba od X_i nije koncentrirana u μ, ako postoji $\vartheta_a \in (0, \vartheta_+)$ t.d. $a = \varphi'(\vartheta_a) / \varphi(\vartheta_a)$, tada

$$\frac{1}{n} \log P(S_n \geq na) \longrightarrow -a\vartheta_a + \log \varphi(\vartheta_a)$$

III CENTRALNI GRANIČNI TEOREMI

DE MOIVRE - LAPLACE.

$$X_i \overset{njd}{\sim} \begin{pmatrix} -1 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

Stirlingova formula $\boxed{n! \sim n^n e^{-n} \sqrt{2\pi n}}$

LEMA 1 a) $a_j \to \infty$, $a_j c_j \to \lambda \Rightarrow (1+c_j)^{a_j} \to e^{\lambda}$

b) $\max_j |c_{j,n}| \to 0$, $\sum_1^n c_{j,n} \to \lambda$, $\sup_n \sum_1^n |c_{j,n}| < \infty \Rightarrow$

$$\prod_1^n (1+c_{j,n}) \to e^{\lambda}$$

Iz Stirlingove formule i leme 1 $\Rightarrow$

$$2k/\sqrt{2n} \to x \quad \Rightarrow \quad P(S_{2n} = 2k) \sim \frac{1}{\sqrt{\pi n}} e^{-x^2/2}$$

i konačno uz malo truda $\Rightarrow$

TEOREM 2 (de Moivre - Laplace)

Za $a < b$, $n \to \infty$

$$P\left(a \le \frac{S_n}{\sqrt{n}} \le b\right) \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

SLABA KONVERGENCIJA

DEF Niz funkcija distribucije konvergira slabo, pišemo $F_n \xrightarrow{w} F$ (ili $F_n \Rightarrow F$) prema nekoj funkciji distribucije F ako

$$F_n(x) \to F(x) \qquad \forall x \in C(F),$$

$C(F)$ = skup točaka neprekidnosti od F

NAP Uočite F je potp. određena vrijednostima na $C(F)$ jer je $C(F)^c$ najviše prebrojiv, a F je zdesna neprekidna.

$\boxed{\text{DEF}}$ Niz slučajnih varijabli $(X_n)_n$ konvergira po distribuciji prema sluč. varijabli X ako $F_{X_n} \xrightarrow{w} F_X$, pišemo

$$X_n \xrightarrow{d} X \quad \text{ili} \quad X_n \Rightarrow X$$

Primjer $X_i \overset{nzjd}{\sim} \begin{pmatrix} -1 & 1 \\ 1/2 & 1/2 \end{pmatrix}$, po tm. 2 $\Rightarrow$

$$P\left(S_n/\sqrt{n} \le y \right) \to \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

tj. $S_n/\sqrt{n} \xrightarrow{d} Z \qquad Z \sim N(0,1)$

Primjer Neka je X_p geometr. sl. var. s parametrom p t.j.

$$P(X_p \geq n) = (1-p)^{n-1} \quad n \in \mathbb{N}$$

Promotrimo pX_p, za $p \to 0$

$$P(pX_p > x) = P\left(X_p > \frac{x}{p}\right) = P\left(X_p \geq \left\lfloor \frac{x}{p} \right\rfloor + 1\right)$$

$$= (1-p)^{\lfloor x/p \rfloor} \longrightarrow e^{-x} \quad (\text{Lem } 1)$$

$$\Rightarrow \quad pX_p \xrightarrow{d} X \quad \text{gdje} \quad X \sim \mathrm{Exp}(1)$$

SCHEFFÉova LEMA

Ako su $(f_n)_n, f$ gustoće sl. var t.d. $f_n \to f$ po točkama tada za $B \in \mathcal{B}(\mathbb{R})$

$$\left| \int_B f_n(x)\,dx - \int_B f(x)\,dx \right| \leq \int_B |f_n(x) - f(x)|\,dx \leq$$

$$\int |f_n(x) - f(x)|\,dx = 2 \int (f(x) - f_n(x))_+\,dx \to 0$$

po tm o dom. konv.

$\Downarrow$

Odgovarajuće f-je distribucije odn sl. varijable konvergiraju slabo odn. po distribuciji.

Za niz (vjeroj.) mjera $(\mu_n)_n$ kažemo da konvergira prema mjeri μ u normi potpune varijacije ako

$$\|\mu_n - \mu\|_{TV} := \sup_B |\mu_n(B) - \mu(B)| \to 0$$

Dakle za $\mu_n(B) = \int_B f_n(x)\,dx$ i $\mu(B) = \int_B f(x)\,dx$

vrijedi:

$$\|\mu_n - \mu\|_{TV} \to 0$$

$\Rightarrow$ pripadne distribucije konvergiraju tj.

$$F_{\mu_n} \xrightarrow{w} F_\mu$$

$$F_{\mu_n}(y) = \int_{-\infty}^{y} f_n(x)\,dx, \qquad F_\mu(y) = \int_{-\infty}^{y} f(x)\,dx$$

$$= \mu_n((-\infty, y]) \qquad\qquad = \mu((-\infty, y])$$

Uočite ne vrijedi obrat

$$F_{\mu_n} \xrightarrow{w} F_\mu \qquad \not\Rightarrow \qquad \|\mu_n - \mu\|_{Tv} \to 0$$

tj. $\mu_n \xrightarrow{w} \mu$

npr.

$$\mu_n = \delta_{1/n}, \quad \mu = \delta_0$$

Primjer (središnja uređajna statistika)

Neka je Y_{n+1} $(n+1)$-va po veličini od $(2n+1)$ točaka nez. i unif. distribuiranih na intervalu $(0,1)$.

Pokazuje se Y_{n+1} ima gustoću

$$f_{Y_{n+1}}(x) = (2n+1)\binom{2n}{n} x^n (1-x)^n$$

Za $y_n = 2\left(Y_{n+1} - \tfrac{1}{2}\right)/\sqrt{2n} \Rightarrow$

$$f_{y_n}(y) = \underbrace{\binom{2n}{n} 2^{-2n}}_{= P(S_{2n}=0)} \left(1 - \frac{y^2}{2n}\right)^n \frac{2n+1}{2n} \sqrt{\frac{n}{2}}$$

za sim. jedn. sluč. šetnju.

Prema tvrdnji ispred Tm 2 $\Rightarrow$

$$P(S_{2n}=0) \sim \frac{1}{\sqrt{\pi n}}$$

$\Downarrow$

$$f_{Y_n}(y) \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

tj. po Scheffeovoj lemi

$$Y_n \xrightarrow{d} Z \sim N(0,1)$$

__Primjer__ (razdiobe ekstremnih vrijednosti)

Ako su $X_1, X_2, \ldots$ njd s fjom distrib. F

a $M_n = \max(X_1, \ldots, X_n)$ postavlja se

pitanje postojanja granične razdiobe u

$$\frac{M_n - b_n}{a_n} \xrightarrow{d} ?$$

za dobro odabrane nizove $(a_n), (b_n)$.

Tj. fje distribucije G t.d.

$$P(M_n \leq a_n x + b_n) \rightarrow G(x) \quad , x \in C(G)$$

LEMA 3 Neka je $\tau \in [0,+\infty]$ i (u_n) niz tada vrijedi

$$P(M_n \le u_n) \to e^{-\tau} \Longleftrightarrow n\overline{F}(u_n) \to \tau$$

Dokaz $\tau \in [0,\infty)$

$$P(M_n \le u_n) = F^n(u_n) = (1-\overline{F}(u_n))^n$$

$\Leftarrow$ slijedi po lemi 1, a

$\Rightarrow$ slijedi jer tada $\overline{F}(u_n) \to 0$, pa logaritmiranjem dobijemo

$$-n\ln(1-\overline{F}(u_n)) \to \tau \qquad \Big| \; -\ln(1-x) \sim x, \; x \to 0$$

$$\Rightarrow n\overline{F}(u_n) \to \tau$$

($\tau = +\infty$ za D.Z.)

Dakle

$$\mathbb{P}(M_n \le a_n x + b_n) \longrightarrow G(x)$$

$$\Leftrightarrow \quad n\overline{F}(a_n x + b_n) \to -\log G(x) \qquad (*)$$

Pokazuje se da parcij. maksimumi mogu imati graničnu razdiobu koja je jednog od sljedeća tri tipa

$$\Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad \begin{matrix} \alpha > 0 \\ x > 0 \end{matrix} \qquad \text{Fréchet}$$

$$\Psi_\alpha(x) = e^{-(-x)^\alpha}, \quad \begin{matrix} \alpha > 0 \\ x \le 0 \end{matrix} \qquad \text{Weibull}$$

$$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R} \qquad \text{Gumbel}$$

To je tzv. Fisher-Tippettov teorem
(dokazao ga je tek Gnedenko, a de Haan je također dao doprinos)

$(*) \Rightarrow$

i) $\overline{F}(x) = x^{-\alpha}, \; x \geq 1, \; \alpha > 0 \Rightarrow P\left(\frac{M_n}{n^{1/\alpha}} \leq y \right) \to \Phi_\alpha(y), \; y > 0$

ii) $\overline{F}(x) = |x|^\beta \; x \in [-1, 0], \; \beta > 0 \to P(n^{1/\beta} M_n \leq y) \to \Psi_\beta(y), \; y < 0$

iii) $F \sim Exp(1) \Rightarrow P(M_n - \log n \leq y) \to \Lambda(y), \; y \in \mathbb{R}$