# VII ERGODSKI TEOREMI

J.Z.V.B $\Rightarrow$ $X_i$ njd, $E|f(X_i)| < \infty$ $\Rightarrow$

$$\frac{1}{n}\sum_{m=0}^{n-1} f(X_i) \xrightarrow{g.s.} \mu = Ef(X_0)$$

Postavlja se pitanje možemo li i za ne nužno nezavisne nizove dobiti istu/sličnu tvrdnju?

Ako da, za koje?

---> odgovor za ergodske

$\boxed{\text{DEF}}$ Niz sl. varijabli $(X_k)_{k\geq 0}$ je <u>stacionaran</u> ako

$$(X_0, \ldots, X_n) \stackrel{d}{=} (X_k, \ldots, X_{k+n}) \quad \forall n, \forall k \geq 0$$

<u>Primjer</u>

a) njd niz je jasno stacionaran

b) Ako je $(X_n)$ M. lanac s prijelaznom vjerojatnosti $p(\cdot,\cdot)$ i stacionarnom razdiobom $\pi$ tj. $\pi(A) = \int \pi(dx)\, p(x, A)$, te $X_0 \sim \pi$

$\Rightarrow$ $(X_n)_{n\geq 0}$ je <u>stacionaran</u>

c) rotacija na kružnici

$\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}([0,1))$, $\mathbb{P} = $ Leb. mjera

$\vartheta \in (0,1)$

$$X_n(\omega) := (\omega + n\vartheta) \bmod 1$$

gdje $x \bmod 1 = x - \lfloor x \rfloor$.

Preslikavanje

$$x \longmapsto e^{2\pi i x}$$

će zaista $X_n$ smjestiti na kružnicu u $\mathbb{C}$.

I to je M. lanac očito

$$X_0 \sim \mathbb{P}, \quad p(x, \{y\}) = \begin{cases} 1 & y = x + \vartheta \bmod 1 \\ 0 & \text{inače} \end{cases}$$

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# TEOREM 1

Ako je $(X_n)_{n\geq 0}$ stacionaran niz i $g : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}$ izmjeriva $\Rightarrow$ $Y_k = g(X_k, X_{k+1}, \ldots)$ je stacionaran niz

Dokaz: $x \in \mathbb{R}^{\mathbb{N}_0}$, $g_k(x) = g(x_k, x_{k+1}, \ldots)$

$$A = \{ x : (g_0(x), g_1(x), \ldots) \in B \} \quad B \in \mathcal{R}^{\mathbb{N}_0}$$

$$P(\omega : (Y_0, Y_1, \ldots) \in B) = P(\omega : (X_0, X_1, \ldots) \in A)$$
$$= P(\omega : (X_k, X_{k+1}, \ldots) \in A) = P(\omega : (Y_k, Y_{k+1}, \ldots) \in B)$$

$\square$

Primjer (Bernoulli shift)

$\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}([0,1))$, $\mathbb{P}$ - Leb. mjera

$Y_0(\omega) = \omega$, $Y_n(\omega) = 2Y_{n-1}(\omega) \bmod 1 \qquad n \geq 1$

Ovo je stac. niz po tm-u 1. Zaista

$$X_i \overset{n.j.d.}{\sim} \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$g(x) = \sum_{i=0}^{\infty} x_i \frac{1}{2^{i+1}} \Rightarrow (Y_n)_{n\geq 0} \overset{d}{=} (g(X_n, X_{n+1}, \dots))_{n\geq 0}$$

Ovo je stac. niz i kao Mark. lanac

$$Y_0 \sim \mathbb{P}, \qquad p(x,\{y\}) = \begin{cases} 1 & y = 2x \bmod 1 \\ 0 & \text{inače} \end{cases}$$

DEF Na vj. prostoru $(\Omega, \mathcal{F}, \mathbb{P})$, za preslikavanje $\varphi$ kažemo da čuva mjeru ako $\varphi : \Omega \to \Omega$ i

$$\mathbb{P}(\varphi^{-1} A) = \mathbb{P}(A) \qquad \forall A \in \mathcal{F}.$$

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Neka $X_n(\omega) = X(\varphi^n(\omega))$ $\qquad \varphi^0 \equiv \mathrm{id}.$

$\Rightarrow$ $(X_n)$ je stacionaran niz

$B \in \mathcal{B}^{n+1}$, $A = \{\omega : (X_0, .., X_n) \in B\}$

$$\mathbb{P}((X_k, .., X_{k+n}) \in B) = \mathbb{P}(\varphi^k \omega \in A) = \mathbb{P}(\omega \in A)$$
$$= \mathbb{P}((X_0, .., X_n) \in B)$$

Zadnji primjer je ujedno i jedini.

Ako je $(Y_n)_{n\geq 0}$ stac. niz u zgodnom prostoru $S$, Kolmogorov tm. o proširenju pokazuje postoji $\mathbb{P}$ na $(S^{\mathbb{N}_0}, \mathcal{S}^{\mathbb{N}_0})$ t.d

$$X_n(\omega) = \omega_n \quad \text{i} \quad (X_n)_{n\geq 0} \overset{d}{=} (Y_n)_{n\geq 0}$$

Ako je $\varphi$ pomak tj. $\varphi(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$

$$X_n(\omega) = \omega_n = X(\varphi^n \omega) \quad \text{gdje} \quad X(\omega) = x_0$$

te $\varphi$ čuva mjeru $\mathbb{P}$.

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TEOREM 2

Svaki stac. niz $(X_n)_{n\geq 0}$ može se proširiti na dvostrani stacionaran niz $(Y_n)_{n\in\mathbb{Z}}$.

Dem/ ponovo Kolmog. teorem o proširenju (trivijalno poopćen).

U nastavku

$(\Omega, \mathcal{F}, \mathbb{P})$ vj. prostor

$\varphi$ preslik. koje čuva $\mathbb{P}$

$X_n(\omega) = X(\varphi^n \omega)$ gdje je $X$ sluč. varijabla

$\boxed{\text{DEF}}$ Skup $A \in \mathcal{F}$ je invarijantan (gotovo) ako

$$\mathbb{P}(A \triangle \varphi^{-1} A) = 0$$

striktno invarijatan ako

$$A = \varphi^{-1}(A)$$

$\mathcal{J}$ = klasa svih invarijantnih događaja

$\rightarrow$ $\mathcal{J}$ je $\sigma$-algebra, a

$X \in \mathcal{J}$ akko $X \circ \varphi = X$ g.s.

$\boxed{\text{DEF}}$ Preslikavanje $\varphi$ koje čuva mjeru na $(\Omega, \mathcal{F}, \mathbb{P})$ je <u>ergodsko</u> ako je

$$A \in \mathcal{J} \Rightarrow \mathbb{P}(A) \in \{0,1\}$$

tj. $\mathcal{J}$ je trivijalna.

U suprotnom $\exists A$, $\mathbb{P}(A) \in (0,1)$ t.d.

$\varphi(A) = A$ & $\varphi(A^c) = A^c$ $\Leftarrow$ $\varphi$ nije ireducibilno

Primjer

a) njd niz

$\Omega = \mathbb{R}^{\mathbb{N}_0}$, $\varphi$ = pomak

$A \in \mathcal{J} \Rightarrow A = \varphi^{-1}A = \{\omega : \varphi(\omega) \in A\} \in \sigma(X_1, X_2, \ldots)$

$\ldots \rightarrow A \in \sigma(X_n, X_{n+1}, \ldots) \; \forall n \Rightarrow A \in \mathcal{T}$

$\Rightarrow \mathcal{J} \subseteq \mathcal{T}$ repna $\sigma$-algebra

Kolmog. zakon 0-1 $\Rightarrow$ $\mathcal{J}$ trivijalna

$\Rightarrow \varphi$ ergodsko

$\rightarrow$ niz $(X_n)$ je ergodski

b) M. lanac na prebr. skupu $S$

Neka $(X_n)$ ima stac. razdiobu $\pi$ i

$\pi(x) > 0 \quad \forall x \in S \Rightarrow$ sva su stanja povratna

(tm 18 pog VI)

Tm 13 pog VI $\Rightarrow$

$$S = \bigcup R_i \quad , R_i \text{ zatvoreni i ireducibilni}$$

Ako $X_0 \in R_i \Rightarrow X_n \in R_i$ za sve $n$

$\Rightarrow$ Ako M. lanac nije ireducibilan nije niti ergodski

c) rotacija na kružnici

Ako je $\vartheta = m/n$ $m, n \in \mathbb{N}$

$(X_n)$ nije ergodski (dz.), a ako je

iracionalan je (dz.).

d) Bernoulli shift

$$y_n = \sum_{m=0}^{\infty} 2^{-(m+1)} X_{n+m} \qquad X_i \overset{n.j.d.}{\sim} \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

je ergodski po

<u>TEOREM 3</u> Ako je $g : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}$ izmjeriva, $(X_n)_{n \geq 0}$ ergodski stac. niz $\Rightarrow$ $Y_k = g(X_k, X_{k+1}, \dots)$ je to također.

# BIRKHOFFov ERGODSKI TEOREM

Neka $\varphi$ čuva mjeru na $(\Omega, \mathcal{F}, \mathbb{P})$

<u>TEOREM 4</u> (ergodski teorem)

Za sve $X \in L_1$

$$\frac{1}{n} \sum_{m=0}^{n-1} X(\varphi^m \omega) \to E(X \mid \mathcal{J}) \quad \underline{g.s.} \text{ i } \underline{u \ L_1}.$$

<u>Primjer</u>

a) njd niz, $\mathcal{J}$ je trivij. $\Rightarrow$

$$\frac{1}{n} \sum_{m=0}^{n-1} X_m \to EX_0 \quad g.s. \ \& \ u \ L^1$$

b) Mark. lanac, ireducibilan i stacionaran,

mera

$$\sum_x |\varphi(x)|\,\pi(x) < \infty$$

pretp. $S$ je prebrojiv $\Rightarrow$ $\exists$ je invij. $\Rightarrow$

$$\frac{1}{n}\sum_{m=0}^{n-1} f(X_m) \longrightarrow \sum f(x)\pi(x) = E_\pi f(X_0)$$

c) rotacija na kružnici

$\Omega = [0,1)$, $\varphi(\omega) = (\omega + \vartheta) \bmod 1$, $\vartheta$ iracion.

$X(\omega) = \mathbb{1}_A(\omega) \Rightarrow$

$$\frac{1}{n}\sum_{m=0}^{n-1} \mathbb{1}(\varphi^n(\omega) \in A) \longrightarrow |A| \quad \text{g.s.}$$

(cf. Weylov teorem)

d) Bernoulli shift

$\Omega = [0,1) \quad \varphi(\omega) = 2\omega \bmod 1$

$i_1, \dots, i_k \in \{0,1\}$ proizvoljni

$$X(\omega) = \begin{cases} 1 & \omega_1 = i_1, \dots, \omega_k = i_k \\ 0 & \text{inače} \end{cases}$$

$$\frac{1}{n} \sum_{m=0}^{n-1} X(\varphi^m \omega) \longrightarrow 2^{-k} \quad \text{g.s.}$$

(c.f. Borelov teorem o normalnim brojevima)

# SUBADITIVNI ERGODSKI TEOREM

**TEOREM 5** Neka $X_{m,n}$, $n \in \mathbb{N}$, $m = 0, \dots, n-1$ zadovoljavaju

i) $X_{0,m} + X_{m,n} \geq X_{0,n}$

ii) $(X_{nk,(n+1)k})_{n \geq 1}$ je stac. niz za sve $k$

iii) razdioba od $(X_{m,m+k})_{k \geq 1}$ ne ovisi o $m$

iv) $EX_{0,1}^{+} < \infty$ & $\forall n$ $EX_{0,n} \geq \mu_0 n$ gdje $\mu_0 > -\infty$

Tada a) $\lim_n \frac{EX_{0,n}}{n} = \inf \frac{EX_{0,n}}{n} =: \mu$

b) $\exists$ $\lim \frac{X_{0,n}}{n} = X$ g.s. i u $L_1$

& $EX = \mu$

c) ako su stacionarni nizovi u ii) i ergodski:

$$X = \mu \quad g.s.$$

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Primjer

a) stac. nizovi : $(\xi_k)$ stac. niz, $E|\xi_k| < \infty$

$$X_{m,n} = \xi_{m+1} + \cdots + \xi_n$$

$$X_{0,n} = X_{0,m} + X_{m,n} \quad \Rightarrow \text{iv)}$$

b) raspon sluč. šetnje : $\xi_i$ kao gore, $S_n = \sum_1^n \xi_i$

$$X_{m,n} = \left| \{ S_{m+1}, \ldots, S_n \} \right|$$

tm 5 u oba slučaja $\Rightarrow$ $X_{0,n}/n \xrightarrow{g.s./L_1} X$

(no ne znamo $X$!)

c) najdulji zajednički niz : ako su $(X_i)_{i\geq 1}, (Y_i)_{i\geq 1}$ ergodski stac. nizovi u $S$

$$L_{m,n} = \max \left\{ K : X_{i_k} = Y_{j_k} \text{ za neke } \begin{matrix} m < i_1 < \cdots < i_k \leq n \\ m < j_1 < \cdots < j_k \leq n \end{matrix} \right\}$$

$$\begin{matrix} 1\,1\,0\,0\,1\,1\,1 \\ 1\,0\,1\,0\,0\,1\,1 \end{matrix} \qquad \qquad \begin{matrix} 1\,1 \mid 0\,0\,1 \\ 1\,0 \mid 1\,1\,0 \end{matrix}$$

Jasno i) $L_{0,m} + L_{m,n} \leq L_{0,n}$ , iv) $0 \leq L_{0,n} \leq n$

$\Rightarrow$ $-L_{m,n}$ je subaditivan

$\Rightarrow$

$$\frac{L_{0,n}}{n} \longrightarrow \mu = \sup_{m\geq 1} E \frac{L_m}{m}$$

NAP) Chvatal & Sankoff ('75) pokazali za n.j.d. Bernoulli $X_n, Y_n$ $\mu \in (0.72..., 0.86...)$

Dokaz tm 5 u 4 koraka

1 korak |

- i) $\Rightarrow X_{0,m}^{+} + X_{m,n}^{+} \geq X_{0,n}^{+}$

$$\Rightarrow X_{0,1}^{+} + X_{1,2}^{+} + \cdots + X_{n-1}^{+} \geq X_{0,n}^{+} \overset{iii)}{\Rightarrow} EX_{0,n}^{+} \leq n \cdot EX_{0,1}^{+}$$

$< \infty$ po iv)

- $|x| = 2x^{+} - x \Rightarrow$ iz iv)

$$E|X_{0,n}| \leq 2EX_{0,n}^{+} - EX_{0,n} \leq C \cdot n < \infty$$

- $a_n = EX_{0,n}$ $\quad$ i, iii) $\Rightarrow$ $\quad a_m + a_{n-m} \geq a_n$

$$\Rightarrow \frac{a_n}{n} \to \inf \frac{a_m}{m} \equiv \mu \qquad (\liminf \checkmark)$$

$$\Rightarrow a)$$

2 korak |

• i) $\Rightarrow$ $$X_{0,n} \leq X_{0,km} + X_{km,n}$$

$$\leq X_{0,m} + X_{m,2m} + \dots + X_{(k-1)m,km} + X_{km,n}$$

podijelimo s $n = km + l$ $\Rightarrow$

$$\frac{X_{0,n}}{n} \leq \frac{k}{km+l} \underbrace{\frac{X_{0,m} + \dots + X_{(k-1)m,km}}{k}} + \frac{X_{km,n}}{n} \quad (*)$$

$\downarrow$ g.s./$L_1$ po tmu 4

$$A_m = E(X_{0,m} \mid J_m)$$

↖ shift inv. σ-algebra za taj niz

$$EA_m = EX_{0,m}$$

$$A_m = E(X_{0,m})$$ ako su nizovi u ii) ergodski

Za fiksni $\ell$, $\varepsilon > 0$

$$\text{iii)} \Rightarrow \sum_{k=1}^{\infty} P(X_{km,km+\ell} > (km+\ell)\varepsilon)$$

$$= \sum_{k=1}^{\infty} P(X_{0,\ell} > (km+\ell)\varepsilon) < \infty \qquad (**)$$

jer $EX_{0,\ell}^{+} < \infty$
po koraku 1

$(*)$ & $(**)$ $\Rightarrow$

$$\overline{X} := \limsup \frac{X_{0,n}}{n} \leq \frac{A_m}{m} \qquad (***)$$

$$\Rightarrow E\overline{X} \leq \frac{EX_{0,m}}{m} \leq \mu$$

(ako su nizovi u ii) ergodski) $\Rightarrow$ $\overline{X} \leq \mu$

3 korak |

$$\underline{X} := \liminf X_{0,n}/n$$

Pokazuje se $E\underline{X} \geq \mu$. Naime, ako je tako

Kako
$$-\infty < \mu_0 \leq \mu \leq EX_{0,1} < \infty \quad , a$$

$$E\overline{X} \leq \mu \Rightarrow \underline{X} = \overline{X} \quad \text{tj. } \exists \text{ g.s. } \lim \frac{X_{0,n}}{n}$$

naime
$$\underline{X} \leq \overline{X}$$
$$\mu = E\underline{X} = E\overline{X} \in (-\infty, \infty) \Rightarrow \overline{X} = \underline{X} \text{ g.s.}$$

(za dokaz $E\underline{X} \geq \mu$ v. knjigu)

4. korak |

Ostalo je pokazati $L_1$ konvergenciju

prisjetimo se (***) $\Rightarrow$

$$\overline{X} \leq \frac{A_m}{m} =: \Gamma_m \ , \ E\Gamma_m = \frac{EX_{0,m}}{m}$$

Neka $\Gamma = \inf \Gamma_m$ , $|z| = 2z^+ - z \Rightarrow$

$$E|X_{0,n}/n - \Gamma| = 2E(X_{0,n}/n - \Gamma)^+ - E(X_{0,n}/n - \Gamma) \leq$$

$$\leq 2E(X_{0,n}/n - \Gamma)^+$$

Kako $EX_{0,n}/n \geq \mu = \inf \Gamma_m \geq E\Gamma \Rightarrow$

po def. od $\mu$

$$E\left(\frac{X_{0,n}}{n}-\Gamma\right)^+ \leq E\left(\frac{X_{0,n}}{n}-\Gamma_m\right)^+ + \underset{\substack{\shortparallel \\ E(\Gamma_m-\Gamma) \text{ jer } \Gamma_m > \Gamma}}{E(\Gamma_m-\Gamma)^+}$$

No $E\Gamma_m \to \mu \quad , m \to \infty \quad$ po a)

koraci 3,3 $\Rightarrow E\Gamma \geq E\bar{X} \geq E\underline{X} \geq \mu \quad \Rightarrow \quad E\Gamma = \mu$

$\Rightarrow \quad E(\Gamma_m - \Gamma) \to 0$

Ograničimo i 2. pribrojnik

$$E(X_{0,n}/n - \Gamma_m)^+ \leq E\left(\frac{X_{0m} + \dots + X_{(k-1)m,km}}{km+l} - \Gamma_m\right)^+$$

$$+ E\left(\frac{X_{km,n}}{n}\right)^+ = \frac{E_{0,l}^+}{n} \to 0_{n\to\infty}$$

Uočite još $y^+ \le |y|$

$$E\left|\frac{X_{0,m}+\cdots+X_{k-1\,m,km}}{m\cdot k}-\Gamma_m\right| \longrightarrow 0$$

po ergodskom teoremu

greška u D.

$\square$