DOOBova NEJEDNAKOST. L_1 KONVERGENCIJA

<u>TEOREM 25</u> Ako je (X_n) submartingal i N vrij. zaust. t.d. $P(N \le k) = 1$ za neki k $\Rightarrow$

$$EX_0 \le EX_N \le EX_k$$

Dokaz: Tm 14 $\Rightarrow$ $X_{N \wedge n}$ je submartingal $\Rightarrow$

$$EX_0 \le EX_{N\wedge 0} \le EX_{N \wedge k} = EX_N \Rightarrow \text{1. nejedn.}$$

Niz $K_n = 1_{\{N < n\}} = 1_{\{N \le n-1\}}$ je predvidiv $\Rightarrow$

$$(K \cdot X)_n = X_n - X_{N \wedge n} \text{ je submartingal (tm 13)} \Rightarrow$$

$$EX_k - EX_N = E(K \cdot X)_k \ge E(K \cdot X)_0 = 0 \Rightarrow \text{2. nejedn.}$$

TEOREM 26 (DOOBOVA NEJEDNAKOST)

Ako je (X_m) submartingal i

$$\overline{X}_n = \max_{0,\dots,n} X_m^+$$

za $\lambda > 0$ & $A = \{\overline{X}_n \geq \lambda\}$ vrijedi

$$\lambda P(A) \leq E X_n \cdot 1_A \leq E X_n^+$$

Dokaz $N = \inf\{m : X_m \geq \lambda \text{ ili } m = n\}$, jasno $X_N \geq \lambda$ na A

$$\underset{\int \lambda 1_A \, dP}{\lambda P(A)} \leq E X_N \cdot 1_A \leq \begin{pmatrix} \text{tm } 25, \ N \leq n \text{ g.s.} \\ \& \ X_N = X_n \text{ na } A^c \end{pmatrix} \leq E X_n \cdot 1_A$$

$$\leq E X_n^+ \quad (\text{trivijalno})$$

$\square$

Submartingalima možemo dakle ograničiti repove, ali i momente kao što ćemo vidjeti

TEOREM 27 (L^p - maksimalna nejednakost)

Ako je (X_n) submartingal ≥ 0 i $1 < p < \infty \Rightarrow$

$$E\bar{X}_n^{\,p} \leq \left(\frac{p}{p-1}\right)^p E(X_n^+)^p$$

Posebno ako je (Y_n) martingal & $Y_n^* = \max_{0,\dots,n} |Y_m|$

$$E|Y_n^*|^p \leq \left(\frac{p}{p-1}\right)^p E(|Y_n|^p)$$

Dokaz $|Y_n|$ je submartingal pa 1. nejedn. $\Rightarrow$ 2. nejedn.

Uzmimo $M>0$, uočimo

$$\{\bar{X}_n \wedge M \geq \lambda\} = \begin{matrix} \emptyset \text{ ili } \\ \{\bar{X}_n \geq \lambda\} \end{matrix} \Rightarrow$$

za A u tm 26 možemo uzeti $\{\bar{X}_n \wedge M \geq \lambda\}$

$$E(\bar{X}_n \wedge M)^p = \int_0^\infty p\lambda^{p-1} \cdot P(\bar{X}_n \wedge M \geq \lambda)\, d\lambda$$

$$\overset{\text{Doob}}{\leq} \int_0^\infty p\lambda^{p-1}\left(\frac{1}{\lambda}\int X_n^+ \mathbb{1}_{(\bar{X}_n \wedge M \geq \lambda)}\, dP\right) d\lambda$$

$$\overset{\text{Fubini}}{=} \int X_n^+ \int_0^{\bar{X}_n \wedge M} p\lambda^{p-2}\, d\lambda \quad dP$$

$$= \underbrace{\frac{p}{p-1}}_{q} \int \underbrace{X_n^+}_{x} \underbrace{(\bar{X}_n \wedge M)^{p-1}}_{y}\, dP \leq \ldots$$

$\frac{1}{q} + \frac{1}{p} = 1 \Leftarrow q$

Hölder $\Rightarrow E|XY| \leq \|X\|_p \cdot \|Y\|_q$

$$\cdots \le q \cdot \left(E|X_n^+|^p\right)^{1/p} \cdot \left(E|\bar{X}_n \wedge M|^{\cancel{(p-1)}\frac{p}{\cancel{p-1}}}\right)^{1/q}$$

podijelmo sa $\Rightarrow$

$$\left(E|\bar{X}_n \wedge M|^p\right)^{\underbrace{1-1/q}_{=1/p}} \le \left(\frac{p}{p-1}\right) \cdot \left(E(X_n^+)^p\right)^{1/p} \quad \Big/ \; {}^\wedge p$$

$\Rightarrow$

$$E|\bar{X}_n \wedge M|^p \le \left(\frac{p}{p-1}\right)^p E(X_n^+)^p$$

za $M \to \infty$ tm o mon. konv. daje rezultat. $\square$

NAP Za $p=1$ tvrdnja ne vrijedi, ali

$$E\bar{X}_n \le \frac{1}{1-1/e}\left[1 + E\left(X_n^+ (\log(X_n^+) \vee 0)\right)\right]$$

-

TEOREM 28 (L^p konvergencija martingala)

(X_n) martingal, $p>1$ & $\sup E|X_n|^p < \infty$

$$\Rightarrow \quad X_n \to X \quad \text{g.s. in } L^p$$

Dokaz: $(EX_n^+)^p \le (E|X_n|)^p \le E|X_n|^p$, $\underset{\text{(o konv. marting.)}}{\text{tm } 16} \Rightarrow X_n \xrightarrow{\text{g.s.}} X$

No tm ~~27~~ 28 $\Rightarrow$

$$E\Big(\sup_{0,\dots,n} |X_m|\Big)^p \le \left(\tfrac{p}{p-1}\right)^p E|X_n|^p$$

za $n \to \infty$, ~~tm~~ tm o mon konv. $\Rightarrow$ $\sup_{m \ge 0} |X_m| \in L^p$

Kako $|X_n - X|^p \le \Big(2 \sup_{m\ge 0} |X_m|\Big)^p$ tm. o dom konv $\Rightarrow E|X_n - X|^p \to 0$

$\square$

NAP. Slučaj $p=2$, tj. kvadratno integrabilni martingali je posebno interesantan jer tada možemo još preciznije opisati asimptotsko ponašanje martingala (X_n)

Uočite, npr da za martingal (X_n) t.d. $EX_n^2<\infty$ i $m\le n$, $\Rightarrow$

$$E\big((X_n-X_m)\cdot Y\big)=0 \qquad \forall\, Y\in\mathcal{F}_m,\ EY^2<\infty$$

(ortogonalnost mart. razlika)

—

UNIFORMNA INTEGRABILNOST. KONVERGENCIJA U L^1

$\boxed{\text{DEF}}$ Familija sl. varijabli $(X_i)_{i\in I}$ je <u>uniformno integrabilna</u> ako

$$\sup_{i\in I} E(|X_i| \mathbb{1}_{|X_i|>M}) \longrightarrow 0 \qquad \text{za } M\to\infty$$

Posebno $\exists M$ dovoljno velik t.d. $\sup(\cdots) \le 1$ tj.

$$\sup_{i\in I} E|X_i| \le M+1$$

Jasno ako $\exists\, Y$ td. $|X_i| \leq Y \quad \forall\, i \in I$

& $EY < \infty \Rightarrow (X_i)$ su unif. integrabilne

TEOREM 29 Za $X \in L^1(\Omega, \mathcal{F}_0, \mathbb{P})$ je

$$\{ E(X|\mathcal{F}) : \mathcal{F}\ \sigma\text{-alg.} \subseteq \mathcal{F}_0 \} \text{ unif. integrabilna.}$$

Dokaz Uočite:

i) ako $A_n \in \mathcal{F}_0$, $P(A_n) \to 0 \Rightarrow E(|X|; A_n) \to 0$

(tm. o dom. konv.) $\Rightarrow$

ii) $\forall \varepsilon > 0\ \exists \delta > 0$ td. $P(A) < \delta \Rightarrow E(|X|; A) < \varepsilon$

(u suprotnom $\exists\, A_n$, td. $P(A_n) < 1/n$ ali $E(|X|; A_n) \geq \varepsilon$ $\lightning$ sa i) dokaz tm 29, uzmimo $M>0$ t.d.

$$E|X|/M \le \delta \implies$$

$$E(|E(X|\mathcal{F})|; |E(X|\mathcal{F})|>M) \overset{\text{Jensen}}{\le} E(E(|X| \mid \mathcal{F}); \underbrace{E(|X| \mid \mathcal{F})>M}_{\in \mathcal{F}})$$

$$\overset{\text{def. uvj. oč.}}{=} E(|X|; E(|X| \mid \mathcal{F})>M)$$

Čebišev. nejedn $\implies$

$$P(E(|X| \mid \mathcal{F})>M) \le E(E(|X| \mid \mathcal{F}))/M \le \frac{E|X|}{M} \le \delta$$

Iz izbora $\delta \implies$

$$E(|E(X|\mathcal{F})|; |E(X|\mathcal{F})|>M) \le \varepsilon \quad \forall \mathcal{F}$$

ε proizvoljan $\implies$ tvrdnja $\square$

NAP (recept za provjeru unif. integrabilnosti)

Ako $\varphi \geq 0$ i $\varphi(x)/x \to \infty$ za $x \to \infty$

npr $\varphi(x) = x^p$, $p > 1$ ili $\varphi(x) = x \cdot (\log x \vee 0)$

Iz $E|\varphi(X_i)| \leq C \quad \forall i \in I \Rightarrow$

$\{X_i : i \in I\}$ unif. integrabilna familija.

TEOREM 30

Ako $X_n \xrightarrow{P} X$ sljedeće su tvrdnje ekvivalentne

i) $(X_n)_{n \geq 0}$ je unif. integrabilna

ii) $X_n \xrightarrow{L_1} X$

iii) $E|X_n| \to E|X| < \infty$

TEOREM 31 Za (X_n) submartingal, ekvivalentno je

i) (X_n) je uniformno integrabilan

ii) (X_n) konvergira g.s. & u L^1

iii) (X_n) konvergira u L^1

Dokaz ii) $\rightarrow$ iii) trivij., iii) $\Rightarrow$ i) po tm 30

i) $\Rightarrow$ iii) Zbog unif. integrab. $\sup E|X_n| < \infty$

tm o konv. martingala $\Rightarrow$ $X_n \xrightarrow{g.s.} X$

tm 30 $\Rightarrow$ $X_n \xrightarrow{L_1} X$

$\square$

LEMA 32 Za $X_n \xrightarrow{L_1} X$ vrijedi

$$E(X_n;A) \to E(X;A)$$

Dokaz: $|EX_n 1_A - EX 1_A| \leq E|X_n 1_A - X 1_A| \leq E|X_n - X| \to 0$

$\square$

LEMA 33 Za martingal (X_n) t.d. $X_n \xrightarrow{L_1} X$

$$\Rightarrow \quad X_n = E(X \mid \mathcal{F}_n)$$

Dokaz: Martingalno svojstvo $\Rightarrow$ $E(X_m \mid \mathcal{F}_n) = X_n \quad \forall m > n$

$\Rightarrow$ za $A \in \mathcal{F}_n$ $\quad E(X_n;A) = \underset{E(X_m 1_A) = E(1_A E(X_m \mid \mathcal{F}_n))}{\underset{\shortparallel}{E(X_m;A)}} \quad \forall m > n$

Lema 32

$\Rightarrow \quad E(X_m;A) \to E(X;A)$

$\Rightarrow \quad E(X_n;A) = E(X;A) \quad \forall A \in \mathcal{F}_n \Rightarrow X_n = E(X \mid \mathcal{F}_n)$ $\square$

TEOREM 34

Za martingal (X_n) ekvivalentno je:

i) (X_n) je unif. integrabilan

ii) (X_n) konv. g.s. & u L^1

iii) (X_n) konv. u L^1

iv) postoji integrabilna X t.d. $X_n = E(X \mid \mathcal{F}_n)$

⇒ tzv. regularni martingali

Dem.

i) ⇒ ii) tm 31 ✓

ii) ⇒ iii) ✓

iii) ⇒ iv) Lema 33 ✓

iv) ⇒ i) tm 29 ✓

$\square$

TEOREM 35

Ako je $\mathcal{F}_n \nearrow \mathcal{F}_\infty$ rastući niz σ-algebri &

$\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n) \Rightarrow$

$$E(X \mid \mathcal{F}_n) \xrightarrow{g.s/L_1} E(X / \mathcal{F}_\infty)$$

$\Downarrow$

TEOREM 36 (Lévyev zakon 0-1)

Ako $\mathcal{F}_n \nearrow \mathcal{F}_\infty$ i $A \in \mathcal{F}_\infty \Rightarrow$

$$E(1_A \mid \mathcal{F}_n) \longrightarrow 1_A$$

" Očito ili nevjerojatno ?" Chung

MARTINGALI UNATRAG [backwards/reversed martingale]

Sluč. proces $(X_n)_{n\le 0}$ adaptiran na rastući niz σ-algebri $(\mathcal{F}_n)_{n\le 0}$ t.d. $E|X_n|<\infty$ $\forall n$ i

$$E(X_{n+1} \mid \mathcal{F}_n) = X_n \quad \text{za } n \le -1$$

Uočite $\mathcal{F}_n \searrow$ za $n \to -\infty$, pa je tm o konvergenciji jednostavniji

TEOREM 37

$$X_n \to X_{-\infty} \quad u \ L^1 \text{ i g.s. za } n \to -\infty$$

Dokaz U_n = # prelazaka pruge $[a,b]$ niza $X_{-n}, \dots, X_0$

tm 15 $\Rightarrow \quad (b-a) E U_n \leq E(X_0 - a)^+$

no $U_n \nearrow U_\infty \Rightarrow \quad E U_\infty < \infty$ po tm. o mon. konv.

$\Rightarrow \quad \lim\limits_{n \to -\infty} X_n$ postoji g.s. po istom argumentu

martingalno svojstvo $\Rightarrow \quad X_n = E(X_0 \mid \mathcal{F}_n) \Rightarrow^{\text{tm 34 iv)}} \Rightarrow$

(X_n) je unif. integr. $\Rightarrow \quad X_n \xrightarrow{L^1} X_{-\infty}$

TEOREM 38

Ako $X_{-\infty} = \lim_{n \to -\infty} X_n$, $\mathcal{F}_{-\infty} = \bigcap_n \mathcal{F}_n$

$\Rightarrow$ $X_{-\infty} = E(X_0 | \mathcal{F}_{-\infty})$

Dokaz Jasno $X_{-\infty} \in \mathcal{F}_{-\infty}$. $X_n = E(X_0 | \mathcal{F}_n)$

$\rightarrow$ za $A \in \mathcal{F}_{-\infty} \subseteq \mathcal{F}_n$

$$\int_A X_n \, dP = \int_A X_0 \, dP$$

tm 37 + Lema 32 $\rightarrow$ $E(X_n; A) \to E(X_{-\infty}; A)$

$\rightarrow$ $\int_A X_{-\infty} \, dP = \int_A X_0 \, dP \quad \forall A \in \mathcal{F}_{-\infty}$

$\square$

TEOREM 39

Ako $\mathcal{F}_n \searrow \mathcal{F}_{-\infty}$ za $n \to -\infty$ tj. $\mathcal{F}_{-\infty} = \bigcap_n \mathcal{F}_n$

tada
$$E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_{-\infty}) \quad \text{g.s. \& u } L^1$$

za sve integrabilne Y.

Dokaz: $X_n = E(Y|\mathcal{F}_n)$ je martingal unatrag

tm-i 37, 38 $\Rightarrow$ $X_n \to X_{-\infty}$ u L^1 & g.s. za $n \to -\infty$

gdje
$$X_{-\infty} = E(X_0|\mathcal{F}_{-\infty}) = E(E(Y|\mathcal{F}_0)|\mathcal{F}_{-\infty}) = E(Y|\mathcal{F}_{-\infty})$$

$\square$

—

PRIMJENA (Teorem de Finettija)

Ako su $\vartheta, X_1, X_2, \ldots$ sluč. varijable t.d.
$\vartheta \in [0,1]$ i uvjetno na ϑ, $X_n \overset{n.j.d.}{\sim} \begin{pmatrix} 0 & 1 \\ 1-\vartheta & \vartheta \end{pmatrix}$

tj.
$$P(X_1 = u_1, \ldots, X_n = u_n \mid \vartheta) = \vartheta^s (1-\vartheta)^{n-s} \qquad (*)$$

$$s = u_1 + \cdots + u_n$$

Npr $Z_n \overset{n.j.d.}{\sim} U(0,1)$, $\vartheta \sim F$ nez. od (Z_n)

$$X_n = 1_{(Z_n \le \vartheta)}$$

$$(*) \Rightarrow P(X_1 = u_1, \ldots, X_n = u_n) = \mathbb{E}\left[\vartheta^s (1-\vartheta)^{n-s}\right]$$

Kažemo (X_n) je miješani Bernoullijev niz.

On je izmjenjiv u smislu

$$(X_1, \dots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \dots, X_{\pi(n)}) \qquad (\Delta)$$

$\forall n$: $\forall$ permutaciju π od $\{1, 2, \dots, n\}$

No interesantno je da je $\forall$ izmjenjiv niz sluč. varijabli s vrijednostima u $0-1$ miješani Bernoullijev niz.

$\mathcal{E}_n$ = σ-algebra događaja koji invarijantni u odn. na permutacije $(X_1, \dots, X_n)$

$\mathcal{E} = \bigcap \mathcal{E}_n$ = σ-algebra izmjenjivih događaja (v. pog 4)

TEOREM 40 (de Finetti)

Ako je $X_1, X_2, \dots$ izmjenjiv niz (v. (Δ)) tada uvjetno na $\mathcal{E}$, (X_n) čine n.jd. niz.

Dokaz: φ funkcija,

$(n)_k = n(n-1)\cdots(n-k+1)$

$$A_n(\varphi) = \frac{1}{(n)_k} \sum \varphi(X_{i_1}, \dots, X_{i_k})$$

po svim cijelim brojevima $1 \le i_1, \dots, i_k \le n$

koji su međusobno različiti

Ako je (X_n) izmjenjiv

$$A_n(\varphi) = E(A_n(\varphi) \mid \mathcal{E}_n) \leftarrow \text{-- naime } A_n(\varphi) \in \mathcal{E}_n$$

$$= \frac{1}{(n)_k} \sum E(\varphi(X_{i_1}, \dots, X_{i_k}) \mid \mathcal{E}_n)$$

$$= E(\varphi(X_1, \dots, X_k) \mid \mathcal{E}_n)$$

Tm 39
$\Rightarrow$

$$A_n(\varphi) \to E(\varphi(X_1, \dots, X_k) \mid \mathcal{E}) \qquad (**)$$

Ako su f, g ograničene na $\mathbb{R}^{k-1}$, odn $\mathbb{R}$

& $I_{n,k} = \{ i_1, \dots, i_k \text{ različiti}, 1 \leq i_1, \dots, i_k \leq n \} \Rightarrow$

-

$$(n)_{k-1} A_n(\varphi) \cdot n A_n(g) = \sum_{i \in I_{n,k-1}} f(X_{i_1}, \dots, X_{i_{k-1}}) \sum_m g(X_m)$$

$$= \sum_{i \in I_{n,k}} f(X_{i_1}, \dots, X_{i_{k-1}}) \, g(X_{i_k})$$

$$+ \sum_{i \in I_{n,k-1}} \sum_{j=1}^{k-1} f(X_{i_1}, \dots, X_{i_{k-1}}) \, g(X_{i_j})$$

Stavimo $\quad \varphi(x_1, \dots, x_k) = f(x_1, \dots, x_{k-1}) \, g(x_k)$

i.z. $\quad \dfrac{(n)_{k-1} \, n}{(n)_k} = \dfrac{n}{(n-k+1)} \quad \& \quad \dfrac{(n)_{k-1}}{(n)_k} = \dfrac{1}{(n-k+1)}$

$$\Rightarrow \quad A_n(\varphi) = \frac{n}{n-k+1} A_n(f) \, A_n(g) - \frac{1}{n-k+1} \sum_{j=1}^{k-1} A_n(\varphi_j)$$

gdzie $\varphi_j(x_1, \dots, x_{k-1}) = f(x_1, \dots, x_{k-1})\, g(x_j)$

(**) $\Rightarrow$
na f, g, φ

$$E(f(X_1, \dots, X_{k-1})\, g(X_k) \mid \mathcal{E}) =$$

$$E(f(X_1, \dots, X_{k-1}) \mid \mathcal{E}) \cdot E(g(X_k) \mid \mathcal{E})$$

Indukcyjnie $\Rightarrow$

$$E\left(\prod_{j=1}^{k} f_j(X_j) \mid \mathcal{E}\right) = \prod_{j=1}^{k} E(f_j(X_j) \mid \mathcal{E})$$

$\square$

TEOREMI O OPCIONALNOM ZAUSTAVLJANJU

TEOREM 41 (o opcionalnom zaustav.)

Ako su L, M, $L \leq M$ zaustavna vremena, $Y_{M \wedge n}$ unif. integr. submartingal $\Rightarrow$

$$E Y_L \leq E Y_M$$

i

$$Y_L \leq E(Y_M \mid \mathcal{F}_L)$$

v. odjeljak 5.7.