Za općenite slučajne šetnje se pokazuje:

(Sn) je povratna ako

- $d=1$ & $S_n/n \xrightarrow{P} 0$
- $d=2$ & $S_n/\sqrt{n} \xrightarrow{d}$ nedegener. normalna razd.

a prolazna ako

- $d \geq 3$ & (S_n) je istinski 3(ili više)-dimenzionalna

Naći ćemo i nužan i dovoljan uvjet za povratnost

LEMA 12

i) Ako $\sum_1^\infty P(\|S_n\| < \varepsilon) < \infty \Rightarrow P(\|S_n\| < \varepsilon \text{ b.č.}) = 0$

ii) Ako $\sum_1^\infty P(\|S_n\| < \varepsilon) = \infty \Rightarrow P(\|S_n\| < 2\varepsilon \text{ b.č.}) = 1$

Dokaz i) po 1. Borel-Cantelli lemi

ii) Neka $F_\varepsilon = \{\|S_n\| < \varepsilon \text{ b.č.}\}^c \Rightarrow$

$$P(F_\varepsilon) = \sum_0^\infty P(\|S_m\| < \varepsilon, \|S_n\| \geq \varepsilon \ \forall n \geq m+1)$$

$$\geq \sum_0^\infty P(\|S_m\| < \varepsilon, \|S_n - S_m\| \geq 2\varepsilon \ \forall n \geq m+1)$$

$$= \underbrace{\sum_0^\infty P(\|S_m\| < \varepsilon)}_{= +\infty} \cdot \underbrace{P(\|S_n\| \geq 2\varepsilon \ \forall n \geq 1)}_{=: \varrho_{2\varepsilon,1}}$$

$$\overset{\wedge}{1} \qquad \Rightarrow \varrho_{2\varepsilon,1} = 0$$

Kako $\sum P(\|S_m\| < \varepsilon) = +\infty \Rightarrow \varrho_{2\varepsilon,1} = 0$, slično

za $\varrho_{\delta,k} = P(\|S_n\| \geq \delta, \forall n \geq k)$, neka za fiksno k

$$A_m = \{\|S_m\| < \varepsilon, \|S_n\| \geq \varepsilon \quad \forall n \geq m+k\}$$

$\forall \omega$ je u najviše k skupova A_m, $m = 0, 1, \ldots \Rightarrow$

$$k \geq \sum_{m=0}^{\infty} P(A_m) \geq \sum_{0}^{\infty} P(\|S_m\| < \varepsilon)\, \varrho_{2\varepsilon,k}$$, opet $\Rightarrow$

$\underline{\varrho_{2\varepsilon,k} = 0 \quad \forall k}$, no

$$P(F_{2\varepsilon}) \leq \sum_{k=1}^{\infty} \underbrace{P(\|S_n\| \geq 2\varepsilon, \forall n \geq k)}_{=\varrho_{2\varepsilon,k}} = 0$$

$\square$

Norma u idućoj lemi je $\|x\| = \|x\|_\infty = \sup_i |x_i|$

LEMA 13 Za $m \geq 2$, $m \in \mathbb{N}$

$$\sum P(\|S_n\| < m\varepsilon) \leq (2m)^d \sum_{n=0}^{\infty} P(\|S_n\| < \varepsilon)$$

Dokaz $\sum_{0}^{\infty} P(\|S_n\| < m\varepsilon) \leq \sum_{n=0}^{\infty} \sum_{k} P(S_n \in k\varepsilon + [0,\varepsilon)^d)$

$k \in \{-m, \dots, (m-1)\}^d$

$T_k = \inf\{ l \geq 0 : S_l \in k\varepsilon + [0,\varepsilon)^d$

Fubini $\Rightarrow$

$$\sum_{n=0}^{\infty} P(S_n \in k\varepsilon + [0,\varepsilon)^d) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty} P(S_n \in k\varepsilon + [0,\varepsilon)^d, T_k = \ell)$$

$$\leq \sum_{\ell=0}^{\infty}\sum_{n=\ell}^{\infty} P(\underbrace{\|S_n - S_\ell\| < \varepsilon, T_k = \ell}_{\text{nezavisni}})$$

$$= \sum_{m=0}^{\infty} P(T_k = m) \sum_{j=0}^{\infty} P(\|S_j\| < \varepsilon) \leq \sum_{j=0}^{\infty} P(\|S_j\| < \varepsilon)$$

Kako je točno $(2m)^d$ k-ova u $\{-m, \dots, (m-1)\}^d$

$\Rightarrow$ tvrdnja

$\square$

Podsjetimo se: $x \in \mathbb{R}^d$ je povratna za S^d ako

$$P(\|S_n - x\| < \varepsilon \text{ b.č.}) = 1 \qquad \forall \varepsilon > 0$$

šetnja je <u>povratna</u> / <u>prolazna</u> ako je skup ovakvih x <u>neprazan</u> / <u>prazan</u> (on je uvijek zatvorena aditivna grupa). Leme 12 & 13 $\Rightarrow$

<u>TEOREM 14</u> Ako i za jedan $\varepsilon > 0$

$\sum_{0}^{\infty} P(\|S_n\| < \varepsilon)$ <u>$< \infty$</u> odn. <u>$= +\infty$</u> šetnja je <u>prolazna</u> odn. <u>povratna</u>

Za $d=1$, ako $EX_i = \mu \neq 0$ $\;$ J.Z.V.B. $\Rightarrow$

$$S_n/n \xrightarrow{g.s.} \mu \Rightarrow |S_n| \to \infty \text{ g.s.} \Rightarrow (S_n) \text{ prolazna}$$

<u>TEOREM 15</u> (Chung-Fuchs)

Za $d=1$, ako

$$S_n/n \xrightarrow{P} 0 \Rightarrow (S_n) \text{ povratna.}$$

Dokaz $\quad u_n(x) := P(|S_n| < x)$, $x > 0$, Lema 13 $\Rightarrow$

$$\sum_{n=0}^{\infty} u_n(1) \geq \frac{1}{2m} \sum_{0}^{\infty} u_n(m) \geq \frac{1}{2m} \sum_{n=0}^{Am} u_n(n/A) \quad \forall A > 0 = \dots$$

jer je u_n rastuća funkcija. Po pretp. $u_n(n/A) \to 1 \Rightarrow$

$$\dots \geq \frac{A}{2} \cdot \frac{1}{Am} \sum_{n=0}^{\infty} u_n(n/A) \xrightarrow[m\to\infty]{} A/2$$

<u>Césaro</u>, ali A je proizvoljan, tm 14 $\Rightarrow$ tvrdnja $\square$

TEOREM 16

Za $d=2$, ako

$S_n/\sqrt{n} \xrightarrow{d}$ nedegenerir. norm. razdioba $\Rightarrow (S_n)$ povratna.

NAP) Tvrdnja je istinita i ako je limes degeneriran npr 0 ili 1 dimenzionalna normalna razdioba, no onda je i šetnja najviše 1-dimenzionalna, pa se može primjeniti teorem 15.

Jelenković 16

$u(n,m) = P(\|S_n\| < m)$, lema 13 $\Rightarrow$

$$\sum_{n=0}^{\infty} u(n,1) \geq \frac{1}{4m^2} \sum_{n=0}^{\infty} u(n,m) \qquad (1)$$

ako $m/\sqrt{n} \to c \Rightarrow$

$$u(n,m) \to \int_{[-c,c]^2} n(x)\,dx =: \varrho(c)$$ gdje je n gustoća granične norm. distrib.

Za $n = \lfloor \vartheta m^2 \rfloor \Rightarrow u(\lfloor \vartheta m^2 \rfloor, m) \to \varrho(\vartheta^{-1/2})$. Kako možemo pisati

$$\frac{1}{m^2} \sum_{n=0}^{\infty} u(n,m) = \int_0^{\infty} u(\lfloor \vartheta m^2 \rfloor, m)\,d\vartheta$$

pustimo $m \to \infty$, Fatou $\Rightarrow$

$$\liminf_{m\to\infty} \frac{1}{4m^2} \sum_{n=0}^{\infty} u(n,m) \geq \frac{1}{4} \int_0^{\infty} \varrho(\vartheta^{-1/2})\,d\vartheta \qquad (1)$$

Kako je n neprekidna i pozitivna u 0

$$\rho(c) = \int_{[-c,c]^2} n(x)\,dx \sim n(0)(2c)^2 \quad \text{za } c \to 0$$

$\Rightarrow$ $\rho(v^{-1/2}) \sim 4\,n(0)/v$ za $v \to \infty$, pa integral

u (2) divergira $\Rightarrow$ prema (1)

$$\sum_{n=0}^{\infty} u(n,1) = +\infty \Rightarrow \text{povratnost}$$

$\square$

Neka je

$$\varphi(t) = E e^{i\langle t, X_1\rangle}$$

karakteristična funkcija koraka sl. šetnje

TEOREM 17

Za $\delta > 0$, (S_n) je povratna $\iff$

$$\int_{(-\delta,\delta)^d} \mathrm{Re} \frac{1}{1-\varphi(y)}\, dy = +\infty$$

TEOREM 18

Za $\delta > 0$, (S_n) je povratna $\iff$

$$\sup_{r<1} \int_{(-\delta,\delta)^d} \mathrm{Re} \frac{1}{1-r\varphi(y)}\, dy = +\infty$$

U knjizi je dokazan samo tm 18, nužnost u tm 1.7 ⟹ onda iz

$$0 \leq \operatorname{Re} \frac{1}{1-r\varphi(y)} \longrightarrow \operatorname{Re} \frac{1}{1-\varphi(y)} \quad \text{za } r \to 1$$

drugi smjer je vrlo netrivijalan. Tm 18 se pojavljuje u Chung-Fuchs (1951), a tm 17 tek dokazuju 1969. nezavisno Ornstein i Stone.

LEMA 19 (Parsevalova relacija)

Za vjerojat. mjere μ & ν na $\mathbb{R}^d$ s karakt. funkcijama φ & ψ

$$\int \psi(t)\,\mu(dt) = \int \varphi(x)\,\nu(dx)$$

Dem. $e^{i\langle t,x\rangle}$ je ogr. $\Rightarrow$ Fubini $\Rightarrow$

$$\int \psi(t)\mu(dt) = \iint e^{i\langle t,x\rangle}\nu(dx)\mu(dt) = \int \varphi(x)\,\nu(dx)$$

✓

LEMA 20 Za $|x| \leq \pi/3$

$$1 - \cos x \geq x^2/4$$

Dokaz tma 18

Gustoća $\frac{\delta-|x|}{\delta^2}$, $|x|\le\delta$ ima karakt fju $\frac{2(1-\cos\delta t)}{(\delta t)^2}$

$\mu_n(A) = \mathbb{P}(S_n \in A)$, leme 19 & 20 $\Rightarrow$

$$\mathbb{P}(\|S_n\| < 1/\delta) \le 4^d \int \prod \frac{1-\cos(\delta t_i)}{(\delta t_i)^2}\, \mu_n(\delta t)$$

$$= 2^\delta \int_{(-\delta,\delta)^d} \prod \frac{\delta-|x_i|}{\delta^2}\, \varphi^n(x)\,dx$$

$$\sum_{n=0}^{\infty} r^n \mathbb{P}(\|S_n\| < 1/\delta) = \left(\begin{matrix} 20 \\ r<1 \end{matrix}\right) \le \underbrace{2^d \int_{(-\delta,\delta)^d} \prod \frac{\delta-|x_i|}{\delta^2}\, \frac{1}{1-r\varphi(x)}\,dx}_{}$$

realni integral zbog simetrije kocke $(-\delta,\delta)^d$, dakle jednak je svom realnom dijelu

Pustimo $r \to 1$, iz $(\delta - |x|)/\delta \le 1 \Rightarrow$

$$\sum P(\|S_n\| < 1/\delta) \le \left(\frac{2}{\delta}\right)^d \sup_{r<1} \int_{(-\delta,\delta)^d} \mathrm{Re} \frac{1}{1 - r\varphi(x)} \, dx$$

tm 14 $\Rightarrow$ nužnost.

Kako gustoća $\dfrac{1-\cos(x/\delta)}{\pi x^2/\delta}$ ima kar. fju $1 - |\delta t|$, za $|t| \le 1/\delta$

Iz $1 \ge \prod_1^d (1 - |\delta x_i|)$ i leme 19 $\Rightarrow$

$$P(\|S_n\| < 1/\delta) \ge \int_{(-1/\delta, 1/\delta)^d} \prod_1^d (1 - |\delta x_i|) \, \mu_n(dx)$$

$$= \int \prod_1^d \frac{1 - \cos t_i/\delta}{\pi t_i^2/\delta} \varphi^n(t) \, dt$$

$\Rightarrow$ pomnožimo s r^n i sumiramo $n = 0, \dots, \infty$ $\Rightarrow$

$$\sum_{n=0}^{\infty} r^n P(\|S_n\| < 1/\delta) \geq \int_{(-\frac{1}{\delta},\frac{1}{\delta})^d} \underbrace{\prod_1^d \frac{1-\cos(t_i/\delta)}{\pi t_i^2/\delta} \frac{1}{1-r\varphi(t)}}_{\text{realni integral}} dt$$

po lemi 20

$$\geq \left(\frac{1}{4\pi\delta}\right)^d \int_{(-\delta,\delta)^d} \mathrm{Re}\, \frac{1}{1-r\varphi(t)} dt$$

tm 14 $\Rightarrow$ dovoljnost $\square$

Primjer (simetr. stabilno distrib. korak)

a) $d=1$, neka $\varphi(t) = e^{-|t|^\alpha}$

Jasno

$$1 - r\varphi(t) \to 1 - e^{-|t|^\alpha} \quad r \nearrow 1$$

$$1 - e^{-|t|^\alpha} \sim |t|^\alpha \qquad \text{za } t \to 0$$

$\Downarrow$

odgovarajuća sluč. šetnja je

prolazna za $\alpha < 1$

povratna za $\alpha \geq 1$

Povratnost za $\alpha > 1$ slijedi iz tma 15 (C.F.:

$\alpha = 1$ je nov rezultat za nas.

Po tm-u 2, za $\alpha < 1$

$$-\infty = \liminf S_n < \limsup S_n = \infty$$

ali $\quad P(|S_n| < M \text{ b.č.}) = 0 \quad \forall M < \infty$

NAP Shepp (1964) je pokazao da postoji šetnja proizvoljno teškog repa za koju

$P(|X_1| > x) \to 0$ proizvoljno polako

ali je šetnja povratna!!

Dakle iz stabilnog primjera ne slijedi da sve ovisi o težini repa.

b) $d = 2$, $\alpha < 2$ $\quad \varphi(t) = \exp(-\|t\|_2^\alpha)$

ova razdioba je inv. na rotacije i obje komponente imaju sim. stabilnu razdiobu

Ponovo

$$1 - r\varphi(t) \searrow 1 - e^{-\|t\|^\alpha} \quad \text{za } r \to 1$$

$$1 - e^{-\|t\|^\alpha} \sim \|t\|^\alpha \quad \text{za } t \to 0$$

promjenom na polarne koordinate iz

$$2\pi \int_0^\delta dx \cdot x \cdot x^{-\alpha} < \infty \quad \text{za} \quad 1 - \alpha > -1$$

$\Rightarrow$ za $\alpha < 2$ ove sl. šetnje su prolazne

c) $d \geq 3$

Integrale $\int_0^\delta dx\, x^{d-1} x^{-2} < \infty \Rightarrow$ za

$d \geq 3$ sl. šetnja je povratna ako kar. fja

pada k 1 brže od t^2, no ako

$\varphi(r) = 1 + o(r^2) \Rightarrow \varphi(r) \equiv 1$, Ex. 3.3.19 in Durett

Stoga ako isključimo sl. šetnje po ravnini

niti jedna 3-dim šetnja nije

povratna.

Sl. šetnja u $\mathbb{R}^3$ je istinski 3-dimenzionalna ako $P(\langle X_1, v\rangle \neq 0) > 0 \quad \forall v \neq 0$

<u>TEOREM</u> 21

Nijedna istinski 3-dimenzionalna šetnja nije povratna.

Dokaz: Iz tm-a 18.

NAP: Analiza je jednostavnija za sl. šetnje u $\mathbb{Z}^d$, naime tm. inverzije (pog 3) $\Rightarrow$

$$P(S_n = 0) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \varphi^n(t)\,dt$$

množenjem s r^n i sumiranjem $\Rightarrow$

$$\sum_{n=0}^{\infty} r^n P(S_n=0) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \frac{1}{1-r\varphi(t)}\,dt$$

Za jed. sl. šet. u $d=3$, $\varphi(t) = \frac{1}{3}\sum_{j=1}^{3} \cos t_j \in \mathbb{R}$

$$\frac{1}{1-r\varphi(t)} \nearrow \frac{1}{1-\varphi(t)} \qquad \text{za } \varphi(t)>0$$

$$0 \le \frac{1}{1-r\varphi(t)} \le 1 \qquad \text{za } \varphi(t) \le 0$$

Iz tma o mon. i dom. konv. $\Rightarrow$

$$\sum_{n=0}^{\infty} P(S_n=0) = \frac{1}{(2\pi)^3} \int \left(1 - \frac{1}{3}\sum_{1}^{3} \cos x_i\right)^{-1} dx = 1.516386\ldots$$

Watson 1939
Glasser & Zucker 1977

$\Rightarrow \quad \pi_3 = 0.34053\ldots \quad \leftarrow$ vj. povratka u 0

ZAKONI ARKUS SINUSa

"... combinatorial methods as an antidote to the analytic skulduggery above."

K.L. Chung.

SIM. JEDN. SLUČAJNA ŠETNJA

$$L_{2n} = \sup\{ m \le 2n : S_m = 0 \}$$

vrijeme zadnjeg posjeta u 0

<u>TEOREM 22</u> Za $0 < a < b < 1$

$$P\left(a \le \frac{L_{2n}}{2n} \le b\right) \longrightarrow \int_a^b \frac{1}{\pi\sqrt{x(1-x)}}\, dx \qquad (*)$$

Uočite $(*) = \int_{\sqrt{a}}^{\sqrt{b}} \frac{2}{\pi} (1-y^2)^{-1/2} = \frac{2}{\pi}\left(\arcsin\sqrt{b} - \arcsin\sqrt{a}\right)$

Činjenica da $L_{2n}/2n$ ima razdioba simetričnu oko $1/2$ je neintuitivna

$\Rightarrow$

$$P\left(\frac{L_{2n}}{2n} \le \frac{1}{2}\right) \longrightarrow \frac{1}{2}$$

OPĆENITE SLUČ. ŠETNJE

TEOREM 23 (Sparre-Andersen)

Neka je $V_n = \left| \{ k : 1 \le k \le n ; S_k > 0 \} \right| \Rightarrow$

i) $P(V_n = k) = P(V_n = k) \cdot P(V_{n-k} = 0)$

ii) Ako X_i imaju sim. razdiobu & $P(S_m = 0)$ za $m \ge 1 \Rightarrow$

$$P(V_n = k) = u_{2k} \cdot u_{2n-2k}$$

gdje $u_{2m} = 2^{-2m} \binom{2m}{m}$ = vj. da jed. sim. sl. šetnja bude u 0 u času $2m$

iii) uz uvjete u ii)

$$P\left(a \le \frac{V_n}{n} \le b\right) \to \int_a^b \frac{1}{\pi \sqrt{x(1-x)}}\, dx \qquad 0 < a < b < 1$$