POISSONOV PROCES

Tm o Poissonovoj aproksimaciji ima puno raznih generalizacija, no pojava Poissonove razdiobe u primjenama usko je vezana i uz pojam Poissonovog procesa.

Neka $(N(s,t])_{s\le t}$ predstavljaju familiju sl. varijabli koja modelira broj dolazaka / incidenata / šteta / ... u intervalu $(s,t]$; neka

i) # dolazaka u disj. intervalima budu nezavisne sl. var.

ii) razdioba $N(s,t]$ ovisi samo o $t-s$.

iii) $P(N(0,h]=1) = \lambda h + o(h)$

iv) $P(N(0,h] \geq 2) = o(h)$

TEOREM 32 Uz uvjete i) – iv), $N(0,t]$ imaju Poiss. razdiobu s parametrom λt.

Dokaz: $X_{n,m} = N\left(\frac{m-1}{n}t, \frac{m}{n}t\right]$ $m = 1, 2, \ldots, n$, iskoristi se teorem o Poissonovoj aproksimaciji, za

$$X'_{n,m} = X_{n,m} \wedge 1, \quad S'_n = X'_{n,1} + \cdots + X'_{n,n} \leq N(0,t]$$

No

$$N(0,t] = X_{n,1} + \cdots + X_{n,n} = S_n$$

$$P(S_n \neq S'_n) \to 0 \quad \text{po iv)}$$

$\square$

$\boxed{\text{DEF}}$ Familija sl. varijabli $N(t)$ koja zadovoljava

i) $\forall\ t_0 < t_1 < \dots < t_n$, $\left(N(t_k) - N(t_{k-1})\right)_{k=1,\dots,n}$ su nezavisne sl. var.

ii) $N(t) - N(s) \sim \mathcal{P}(\lambda(t-s))$ $\quad s \le t$

se naziva (homogeni) <u>Poissonov proces</u> s intenzitetom λ.

Dakle imamo dvije karakterizacije ovog procesa no postoji i treća, za

$$\xi_1, \xi_2, \dots \text{ n.j.d.} \sim \mathrm{Exp}(\lambda), \quad T_n = \xi_1 + \dots + \xi_n$$

$$N(t) = \sup\{n \ge 1 : T_n \le t\} \quad \text{je Poissonov proces}$$

Na prostoru s mjerom $(S, \mathcal{S}, \mu)$

opet možemo definirati Poissonov

proces kao slučajnu mjeru $(N(A))_{A \in \mathcal{S}}$ tdl.

i) za sve disjunktne $A_1, A_2, A_3, \ldots A_n$

$N(A_1), \ldots, N(A_n)$ su nezavisne sl. var.

ii) $N(A) \sim P(\mu(A))$.

Tipično kažemo da je N Poissonova slučajna

mjera sa srednjom mjerom μ

[PRM w. mean measure μ]

STABILNE RAZDIOBE

Za $X_1, X_2, \ldots$ n.j.d. t.d. $EX_i = \mu$ i $\mathrm{Var}\, X_i = \sigma^2 \in (0, \infty)$

$\Rightarrow$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$$

Pitanje: što ako $\mathrm{Var}\, X_i = +\infty$, tj. $EX_i^2 = +\infty$?

postoje li nizovi (b_n) (a_n) t.d.

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} Y$$

za neku nedegeneriranu sl. varijablu Y?

Primjer

Neka je X_i simetrična sl. var. t.d. za $\alpha \in (0,2)$

$$P(|X_i| > x) = x^{-\alpha}, \quad x \geq 1 \Rightarrow$$

$$P(X_i < -x) = P(X_i > x) = \frac{1}{2} x^{-\alpha}, \quad x \geq 1$$

Za $\varphi(t) = Ee^{itX}$ vrijedi

$$1 - \varphi(t) = \int_1^\infty (1 - e^{itx}) \frac{\alpha}{2|x|^{\alpha+1}} dx + \int_{-\infty}^{-1} (1 - e^{itx}) \frac{\alpha}{2|x|^{\alpha+1}} dx$$

$$= \alpha \int_1^\infty \frac{1 - \cos(tx)}{x^{\alpha+1}} dx = \begin{pmatrix} tx = u \\ dx = du/t \end{pmatrix}$$

$$= t^\alpha \cdot \alpha \cdot \int_t^\infty \frac{1 - \cos u}{u^{\alpha+1}} du$$

Za $u \to 0$, $1-\cos u \sim u^2/2 \Rightarrow \frac{1-\cos u}{u^{\alpha+1}} \sim u^{-\alpha+1}/2$

što je integrabilno jer $\alpha < 2 \Leftrightarrow -1 < -\alpha+1$

Ako je

$$C = \alpha \int_0^\infty \frac{1-\cos u}{u^{\alpha+1}}\, du < \infty$$

$\Rightarrow$

$$1-\varphi(t) \sim C \cdot |t|^\alpha \quad \text{za } t \to 0$$

(jer $\varphi(t) = \varphi(-t)$ zbog simetrije).

Za X_i njd. i $S_n = X_1 + \cdots + X_n \Rightarrow$

$$E e^{itS_n/n^{1/\alpha}} = \left(\varphi(t/n^{1/\alpha})\right)^n = \Big(1 - \underbrace{\left(1-\varphi(t/n^{1/\alpha})\right)}_{\to C|t|^\alpha}\Big)^n \Rightarrow$$

$$E e^{it S_n / n^{1/\alpha}} \longrightarrow e^{-C|t|^\alpha}$$

što je prema tm u 33.6. u knjizi karakteristična funkcija ili direktno po tmu o neprekidnosti

$\Rightarrow \exists$ nedegenirana Y, $E e^{itY} = e^{-C|t|^\alpha}$ td.

$$S_n / n^{1/\alpha} \xrightarrow{d} Y$$

Dokažimo to i alternativnim načinom

Za $0 < a < b$, t.d. $a n^{1/\alpha} > 1$

$$P\left(a n^{1/\alpha} < X_1 < b n^{1/\alpha}\right) = \frac{1}{2}\left(a^{-\alpha} - b^{-\alpha}\right) \cdot \frac{1}{n} \quad (1)$$

Po tm o Poiss. aproksimaciji

$$N_n(a,b) \equiv \#\left\{m \le n : X_m / n^{1/\alpha} \in (a,b)\right\} \xrightarrow{d} N(a,b)$$

gdje $\quad N(a,b) \sim P\left(\frac{1}{2}\left(a^{-\alpha} - b^{-\alpha}\right)\right)$

$(1) \Rightarrow \quad \forall A \in \mathbb{R} \setminus (-\delta, \delta)$, $\delta n^{1/\alpha} > 1$

$$P\left(X_1 / n^{1/\alpha} \in A\right) = \frac{1}{n} \int_A \frac{\alpha}{2|x|^{\alpha+1}} \, dx$$

$$\Rightarrow \quad N_n(A) \equiv \#\left\{m \le n : X_m / n^{1/\alpha} \in A\right\} \xrightarrow{d} N(A) \sim P(\mu(A))$$

gdje definiramo mjeru μ sa

$$\mu(A) = \int_A \frac{\alpha}{2|x|^{\alpha+1}}\, dx < \infty$$

Familija sluč. varijabli $(N(A))_A$ predstavlja Poissonov proces ili preciznije PRM sa srednjom mjerom μ.

Uočite

$$\mu(x,\infty) = \frac{1}{2} x^{-\alpha} \qquad \forall x > 0$$

$$\Rightarrow \quad N(x,\infty) < \infty \quad g.s.$$

Dakle slučajni skup

$$\mathcal{X}_n = \left\{ X_m / n^{1/\alpha} : 1 \le m \le n \right\}$$

kao "limes" ima PRM N.

Opišimo limes za $S_n / n^{1/\alpha}$ tj. sumirajmo točke skupa $\mathcal{X}_n$

Za $\varepsilon > 0$

$$I_n(\varepsilon) = \{ m \le n : |X_m| > \varepsilon n^{1/\alpha} \}$$

$$\hat{S}_n(\varepsilon) = \sum_{m \in I_n(\varepsilon)} X_m \qquad \overline{S}_n(\varepsilon) = S_n - \hat{S}_n(\varepsilon)$$

Pokazat ćemo da je doprinos od $\overline{S}_n$ relativno mali za male ε.

$$\overline{X}_m(\varepsilon) = X_m \cdot \mathbb{1}_{(|X_m| \le \varepsilon n^{1/\alpha})}, \Rightarrow E\overline{X}_m(\varepsilon) = 0$$ zbog simetrije

$$\Rightarrow \quad E(\overline{S}_n(\varepsilon)^2) = n\, E\overline{X}_1(\varepsilon)^2$$

$$E\overline{X}_1(\varepsilon)^2 = \int_0^\infty 2y\, P(|\overline{X}_1(\varepsilon)| > y)\, dy \le \int_0^1 2y\, dy +$$

$$+ \int_1^{\varepsilon n^{1/\alpha}} 2y\, y^{-\alpha}\, dy = 1 + \frac{2}{2-\alpha} \varepsilon^{2-\alpha} n^{2/\alpha - 1} - \frac{2}{2-\alpha} \le$$

$$\le \frac{2\varepsilon^{2-\alpha}}{2-\alpha} n^{2/\alpha - 1} \qquad (\text{jer } 0 < \alpha < 2)$$

$$\Rightarrow \quad E(\overline{S}_n(\varepsilon)/n^{1/\alpha})^2 \le \frac{2\varepsilon^{2-\alpha}}{2-\alpha} \quad (*)$$ što je proizvoljno malo za male ε.

Da bismo izračunali limes od $\hat{S}_n(\varepsilon)/n^{1/\alpha}$

uočimo

$$\text{card } I_n(\varepsilon) \sim \text{Bin}\left(n, \frac{\varepsilon^{-\alpha}}{n}\right)$$

ako $\text{card } I_n(\varepsilon) = m$, $\hat{S}_n^{(\varepsilon)}/n^{1/\alpha}$ je suma m nezavisnih

sl. varijabli s fjom. distrib. F_n^ε koja

je simetrična i

$$1 - F_n^\varepsilon(x) = P\left(X_1/n^{1/\alpha} > x \,\middle|\, |X_1|/n^{1/\alpha} > \varepsilon \right) = \frac{x^{-\alpha}}{2\varepsilon^{-\alpha}}, \; x \geq \varepsilon$$

to je distribucija jednaka onoj od εX_1, pa

ako $\varphi(t) = Ee^{itX_1}$, F_n^ε ima karak. fju

$$\varphi(\varepsilon \cdot t)$$

$\Downarrow$

$$E\left(\exp\left(it\hat{S}_n(\varepsilon)/n^{1/\alpha}\right)\right) = \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\varepsilon^{-\alpha}}{n}\right)^m \cdot \left(1-\frac{\varepsilon^{-\alpha}}{n}\right)^{n-m} \varphi(\varepsilon t)^m$$

Vrijedi $\binom{n}{m} \cdot \frac{1}{n^m} = \frac{1}{m!} \frac{n\cdot(n-1)\cdots(n-m+1)}{n^m} \leq \frac{1}{m!}$,

$$\left(1-\varepsilon^{-\alpha}/n\right)^n \leq e^{-\varepsilon^{-\alpha}}$$

tm o dom. konv $\Rightarrow$

$$E\left(\exp\left(it\hat{S}_n(\varepsilon)/n^{1/\alpha}\right)\right) \to \sum_{m=0}^{\infty} e^{-\varepsilon^{-\alpha}} (\varepsilon^{-\alpha})^m \varphi(\varepsilon t)^m/m!$$

$$= \exp\left[-\varepsilon^{-\alpha}\left(1-\varphi(\varepsilon t)\right)\right]$$

nije još u traženom obliku

LEMA 33 Ako $h_n(\varepsilon) \to g(\varepsilon)$ $\forall \varepsilon > 0$ i

$g(\varepsilon) \to g(0)$ za $\varepsilon \to 0$ $\Rightarrow$ $\exists$ niz (ε_n) tol. $\varepsilon_n \to 0$ i

$$h_n(\varepsilon_n) \to g(0)$$

Stavimo

$$h_n(\varepsilon) = E e^{it\hat{S}_n(\varepsilon)/n^{1/\alpha}}, \quad g(\varepsilon) = e^{-\varepsilon^{-\alpha}(1-\varphi(\varepsilon t))}$$

Kako $1-\varphi(t) \sim C|t|^\alpha$ za $t \to 0$ $\Rightarrow$

$$g(\varepsilon) \to e^{-C|t|^\alpha} \quad \text{za } \varepsilon \to 0$$

prema gornjoj lemi $\exists$ niz $\varepsilon_n \to 0$ tol.

$$h_n(\varepsilon_n) \to \exp(-C|t|^\alpha)$$

Ako Y ima kar. fju ovu $\Rightarrow$ $\hat{S}_n(\varepsilon_n)/n^{1/\alpha} \xrightarrow{d} Y$

S obzirom da $\varepsilon_n \searrow 0$ (*) $\Rightarrow$

$$\overline{S}_n(\varepsilon_n)/n^{1/\alpha} \xrightarrow{P} 0$$

$$\frac{S_n}{n^{1/\alpha}} = \underbrace{\hat{S}_n(\varepsilon_n)/n^{1/\alpha}}_{\downarrow d \atop Y} + \underbrace{\overline{S}_n(\varepsilon_n)/n^{1/\alpha}}_{\downarrow P \atop 0} \xrightarrow{d} Y$$

po Slutskyevoj lemi.

DEF Funkcija L je sporo promjenjiva / varirajuća ako

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1 \qquad \forall x>0$$

Primjer

- $\log t$, $\log\log t$, const, lin. kombinacije su sporo var.

- t^{ε} nije ako je $\varepsilon \neq 0$

DEF Funkcija G je regularno varirajuća s indeksom κ ako $G(x) = L(x)\cdot x^{\kappa}$

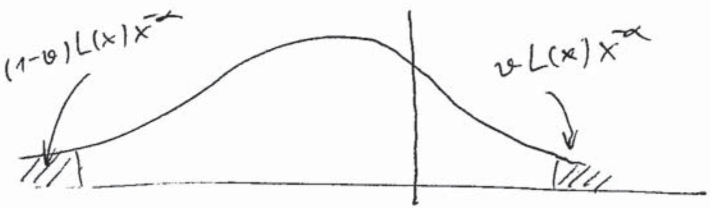
za

$\boxed{\text{DEF}}$ Rozdioba F je reg. varirajuća u $+\infty$ ako je $\overline{F}$ reg. varirajuća fja, a

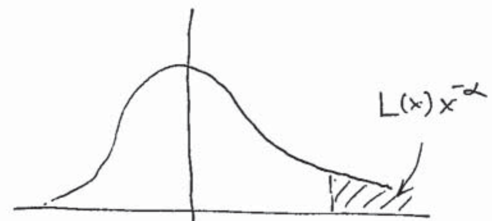
X je regularno varirajuća slučajna varijabla s indeksom α, ako je

i) $P(|X|>x)$ reg. var. fja s indeksom α

ii) $$\exists \lim_{x\to\infty} \frac{P(X>x)}{P(|X|>x)} = v \in [0,1]$$



reg. var.



reg. var. u $+\infty$

TEOREM 34

Ako su $X_1, X_2, \ldots$ n.j.d. regularno varirajuće s indeksom $\alpha \in (0,2)$, za

$$a_n = \inf\left\{x : P(|X_1|>x) \leq \frac{1}{n}\right\}, \quad b_n = n\,E(X_1 \cdot \mathbb{1}_{\{|X_1| \leq a_n\}})$$

vrijedi:
$$\frac{S_n - b_n}{a_n} \xrightarrow{d} Y$$

gdje: Y ima nedegeneriranu razdiobu

NAP Iako tm 34 ne izgleda kao značajna generalizacija primjera, njegov uvjet tj. regularna varijacija je ujedno i nužan uvjet da bi postojali (a_n), (b_n) i Y t.d.

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} Y$$

ako je $EX_i^2 = +\infty$.

(Gnedenko Kolmogorov (1954) ili Breiman (1968))

Dokaz tm 34

Po definic. niza (a_n)

$$n \cdot P(|X_1| > a_n) \to 1 \qquad (\Delta)$$

naime $\quad n P(|X_1| > a_n) \leq 1$, a za $\varepsilon > 0$

uzmemo li $\quad x = a_n/(1+\varepsilon)$ i $t = 1 + 2\varepsilon \Rightarrow$

$$(1+2\varepsilon)^{-\alpha} = \lim \frac{P(|X_1| > (1+2\varepsilon)a_n/(1+\varepsilon))}{P(|X_1| > a_n/(1+\varepsilon))} \leq$$

$$\leq \liminf \frac{P(|X_1| > a_n)}{1/n} \Rightarrow (\Delta)$$ jer je ε proizvoljan.

Zbog reg. varijacije

$$n P(X_1 > x a_n) \to p\, x^{-\alpha} \qquad \text{za } x > 0$$

$\Rightarrow I_n^+(x) = \mathrm{card}\ \{m \le n : X_m > x a_n\} \xrightarrow{d} \mathcal{P}(\vartheta x^{-\alpha})$

$\Rightarrow$ (kao i prije)

$\mathcal{X}_n = \{X_m / a_n : 1 \le m \le n\}$ konvergira ka PRM

sa srednjom mjerom μ td.

$$\mu(A) = \int_{A \cap (0,\infty)} \frac{\vartheta \alpha}{|x|^{\alpha+1}}\, dx + \int_{A \cap (-\infty,0)} \frac{(1-\vartheta)\alpha}{|x|^{\alpha+1}}\, dx .$$

Neka

$$\hat{\mu}(\varepsilon) = E X_m \mathbb{1}_{(\varepsilon a_n \le |X_m| \le a_n)},$$
$$I_n(\varepsilon) = \mathrm{card}\ \{m \le n : |X_m| > a_n \varepsilon)$$

$$\bar{\mu}(\varepsilon) = E X_m \mathbb{1}_{(|X_m| \le \varepsilon a_n)}, \quad \hat{S}_n(\varepsilon) = \sum_{m \in I_n(\varepsilon)} X_m$$

$$\overline{S}_n(\varepsilon) = (S_n - b_n) - (\hat{S}_n(\varepsilon) - n\hat{\mu}(\varepsilon))$$
$$= \sum_1^n \left(X_m 1_{(|X_m| \le \varepsilon a_n)} - \bar{\mu}(\varepsilon) \right)$$

Za $\overline{X}_m(\varepsilon) = X_m 1_{(|X_m| \le \varepsilon a_n)}$ je

$$E(\overline{S}_n(\varepsilon)/a_n)^2 = n \cdot Var(\overline{X}_1(\varepsilon)/a_n) \le n . E(\overline{X}_1(\varepsilon)/a_n)^2$$
$$E(\overline{X}_1(\varepsilon)/a_n)^2 \le \int_0^\varepsilon 2y\, P(|X_1| > y a_n)\, dy$$
$$= \underbrace{P(|X_1| > a_n)}_{\downarrow \, 0} \int_0^\varepsilon 2y \underbrace{\frac{P(|X_1| > y a_n)}{P(|X_1| > a_n)}}_{\downarrow \, y^{-\alpha} \text{ za } n \to \infty} dy$$

no da bismo zamijenili limes i integral trebamo

$\to 0$ tada

LEMA 35 Za sve $\delta > 0$ $\exists$ C t.d.

$\forall$ $t \geq t_0$ i $y \leq 1$

$$P(|X_1| > yt) / P(|X_1| > t) \leq C y^{-\alpha-\delta}$$

Dokaz Po reg varijaciji $\Rightarrow$

$$\frac{P(|X_1| > t/2)}{P(|X_1| > t)} \to \left(\frac{1}{2}\right)^{-\alpha} = 2^{\alpha}$$

za $t \geq t_0$ imamo

$$P(|X_1| > t/2) / P(|X_1| > t) \leq 2^{\alpha+\delta}$$

Ponovimo to i stanemo kada

$$t/2^m < t_0 \quad i \quad \forall n \geq 1$$

$$P(|X_1| > t/2^n) \,/\, P(|X_1| > t) \leq C\, 2^{(\alpha+\delta)n}$$

gdje je $\quad C = 1 / P(|X_1| > t_0)$

Primjenjujući to na prvi n tdl. $1/2^n < y$

iz $\quad y \leq 1/2^{n-1} \quad \Longrightarrow$

$$P(|X_1| > yt) \,/\, P(|X_1| > t) \leq C \cdot 2^{\alpha+\delta}\, y^{-\alpha-\delta}$$

što dokazuje lemu $\quad \square$