POISSONOVA APROKSIMACIJA

Neka je S prebrojiv, a μ i ν dvije razdiobe na S udaljenost totalne / potpune varijacije između μ i ν definiramo kao

$$\|\mu-\nu\| = \|\mu-\nu\|_{TV} \equiv \frac{1}{2}\sum_{z}|\mu(z)-\nu(z)| \overset{*}{=} \sup_{A\subseteq S} |\mu(A)-\nu(A)|$$

$*$ je posljedica od

$$\sum|\mu(z)-\nu(z)| \geq |\mu(A)-\nu(A)| + |\mu(A^c)-\nu(A^c)| \quad \forall A \in S$$

$$= 2\,|\mu(A)-\nu(A)|$$

a za $A = \{z : \mu(z) \geq \nu(z)\}$ imamo i jednakost

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TEOREM 28 (zakon rijetkih događaja / slabi zakon malih brojeva)

Neka su $X_{n,m}$ $m = 1, \dots, n$ nezavisne za svaki n i t.d. $X_{n,m} \sim \begin{pmatrix} 0 & 1 \\ 1-p_{n,m} & p_{n,m} \end{pmatrix}$. Pretpostavimo

i) $\sum_{m=1}^{n} p_{nm} \to \lambda \in (0, \infty)$

ii) $\max_{1 \leq m \leq n} p_{n,m} \to 0$

Za $S_n = X_{n,1} + \dots + X_{n,n} \Rightarrow S_n \xrightarrow{d} Z$ $Z \sim \mathrm{Pois}(\lambda)$.

LEMA 29 Ako je $\mu_1 \times \mu_2$ produktna mjera na $\mathbb{Z} \times \mathbb{Z}$ t.d. $(\mu_1 \times \mu_2)(x,y) = \mu_1(x) \cdot \mu_2(y)$ tada

$$\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\| \leq \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\|$$

Dokaz

$$2\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\| = \sum_{x,y} |\mu_1(x)\mu_2(y) - \nu_1(x)\nu_2(y)|$$

$$\leq \sum_{x,y} |\mu_1(x)\mu_2(y) - \nu_1(x)\mu_2(y)| + \sum_{x,y} |\nu_1(x)\mu_2(y) - \nu_1(x)\nu_2(y)|$$

$$\leq \sum_y \mu_2(y) \cdot 2\|\mu_1 - \nu_1\| + \sum_x \nu_1(x) \cdot 2\|\mu_2 - \nu_2\|$$

$\square$

LEMA 30 Ako je $\mu_1 * \mu_2$ konvolucija razdioba μ_1 i μ_2 tj.

$$\mu_1 * \mu_2(x) = \sum \mu_1(x-y)\mu_2(y)$$

tada $\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| \leq \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|$

Dok/ $2\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| = \sum_x \left| \sum_y \mu_1(x-y)\mu_2(y) - \sum_y \nu_1(x-y)\nu_2(y) \right|$

$$\leq \sum_x \sum_y \left| \mu_1(x-y)\mu_2(y) - \nu_1(x-y)\nu_2(y) \right|$$

$$= 2\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|$$

$\square$

LEMA 31 Ako je μ vj. mjera t.d.

$\mu(1)=p$, $\mu(0)=1-p$, a ν Poissonova razdioba

s parametrom p tada

$$\|\mu-\nu\| \leq p^2$$

Dokaz

$$2\|\mu-\nu\| = |\mu(0)-\nu(0)| + |\mu(1)-\nu(1)| + \sum_{n\geq 2} \nu(n)$$

$$=|(1-p-e^{-p})| + |p-pe^{-p}| + 1-e^{-p}(1+p)$$

Iz $1-x \leq e^{-x} \leq 1$ za $x \geq 0$ $\Rightarrow$

$$= e^{-p}-1+p+p(1-e^{-p})+1-e^{-p}-pe^{-p}$$
$$= 2p(1-e^{-p}) \leq 2p^2$$

$\square$

Dokaz tm 28

Neka je $\mu_{n,m}$ razdioba od $X_{n,m}$

μ_n —"— S_n

$\nu_{n,m}$ Poiss. razdioba s param. $p_{n,m}$

ν_n —"— $\lambda_n = \sum_1^n p_{n,m}$

kako je $\mu_n = \mu_{n,1} * \cdots * \mu_{nn}$, $\nu_n = \nu_{n,1} * \cdots * \nu_{n,n}$

Leme 29-31 $\Rightarrow$

$$\|\mu_n - \nu_n\| \le \sum_1^n \|\mu_{n,m} - \nu_{n,m}\| \le 2 \cdot \sum_{m=1}^n p_{n,m}^2 \quad (A)$$

$$\le 2 \cdot \max p_{n,m} \sum_1^n p_{n,m} \to 0$$

Kako $\nu_n \xrightarrow{w} \nu \sim \mathrm{Pois}(\lambda)$

traženi rezultat slijedi.

POISSONOVSKA APROKSIMACIJA ZA ZAVISNE DOGAĐAJE

<u>Primjer</u>

r kuglica slučajno stavljamo u n kutija

(svih n^r rasporeda su jednako vjerojatni)

$A_i = \{ i\text{-ta kutija prazna} \}$

$N_n = \#\ \text{praznih kutija} = \sum_{1}^{n} 1_{A_i}$

Jasno $P(A_i) = \left(1 - \frac{1}{n}\right)^r$, $E N_n = n \cdot \left(1 - \frac{1}{n}\right)^r$

očito A_i nisu nezavisni događaji

a) pretpostavimo

$$\frac{r}{n} \to c, \text{ tada } \frac{EN_n}{n} = \left(1-\frac{1}{n}\right)^r$$

$$= \left[\left(1-\frac{1}{n}\right)^n\right]^{r/n} \to e^{-c}$$

Može se pokazati

$$\frac{N_n}{n} \xrightarrow{P} e^{-c}$$

$$E(N_n^2) = E\left(\sum_1^n 1_{A_m}\right)^2 = \sum_{1\le k,m\le n} P(A_k \cap A_m) =$$

$$= \sum_1^n P(A_m) + \sum_{k\neq m} P(A_k\cap A_m) = n\left(1-\frac{1}{n}\right)^r + n(n-1)\left(1-\frac{2}{n}\right)^r$$

$$\mathrm{Var}(N_n) = EN_n^2 - (EN_n)^2$$

$$= \cdots = n(n-1)\left[\left(1-\frac{2}{n}\right)^r - \left(1-\frac{1}{n}\right)^{2r}\right] + n\left[\left(1-\frac{1}{n}\right)^r - \left(1-\frac{1}{n}\right)^{2r}\right]$$

$\Downarrow$

$$\mathrm{Var}\,\frac{N_n}{n} = \frac{1}{n^2}\mathrm{Var}\,N_n \longrightarrow e^{-2c} - (e^{-c})^2 + 0 = 0$$

$\Downarrow$

$$P\left(\left|\frac{N_n - EN_n}{n}\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2}\,\mathrm{Var}\,N_n/n \longrightarrow 0 \implies$$

$$N_n/n \xrightarrow{P} e^{-c}$$

Fiksirajmo sada praznu kutiju.

$$X_j = \begin{cases} 1 & \text{ako } j\text{-ta kuglica u toj kutiji} \\ 0 & \text{inače} \end{cases}$$

Uočite $\quad X_j \overset{njd}{\sim} \mathrm{Ber}\left(\frac{1}{n}\right)$

$$\sum_{j=1}^{r} EX_j = r/n \to c$$

Teorem 28 $\Rightarrow$ $\quad \sum^{r} X_j \xrightarrow{d} \mathrm{Poi}(c)$

Spec. $\quad P(\text{kutija je prazna}) \to e^{-c}$

tj.

b) neka sada $\quad n e^{-r/n} \to \lambda \in (0, \infty)$

$$\Rightarrow \quad \log n - r/n \to \log \lambda$$

tj. $\quad r/n - \log n + \log \lambda \to 0$

$$\Rightarrow \quad r = n \log n - n \log \lambda + o(n)$$

Tvrdimo: broj praznih kutija konvergira Poissonovoj razdiobi s parametrom λ

Intuitivno: i-ta kutija je prazna s vjerojatnošću $e^{-r/n} \approx \lambda/n$, kad bi ti događaji bili nezavisni rezultat bi slijedio po tm 28

Dakle

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = \left(1 - \frac{k}{n}\right)^r = \left(\frac{n-k}{n}\right)^r$$

$p_m(r,n) = P(\text{točno } m \text{ kutija prazno})$

$$p_0(r,n) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \sum P(A_i) + \sum_{i<j} P(A_i \cap A_j) - \cdots$$

$$= 1 - n\left(1 - \frac{1}{n}\right)^r + \binom{n}{2}\left(1 - \frac{2}{n}\right)^r + \cdots + (-1)^{n+1}\left(\frac{1}{n}\right)^r$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}\left(1 - \frac{k}{n}\right)^r \Rightarrow$$

$\Rightarrow$

$$p_m(r,n) = \binom{n}{m} \left(1 - \frac{m}{n}\right)^r p_0(r, n-m)$$

broj # izbora m praznih kutija

vj. da je tih m kutija prazno

vjerojj. da niti jedna od ostalih nije prazna

Ako $n \cdot e^{-r/n} \to \lambda \Rightarrow$

$$\binom{n}{m} \cdot \left(1 - \frac{m}{n}\right)^r \longrightarrow \frac{\lambda^m}{m!}$$

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Ako u

$$p_0(r,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1-\frac{k}{n}\right)^r$$

pustmo $n \to \infty$ td. $n e^{-r/n} \to \lambda$

po tm. o dom. konv.

$$\lim p_0(r,n) = \sum_{0}^{\infty} (-1)^k \frac{\lambda^k}{k!} = e^{-\lambda}$$

za fiksni m

$$(n-m)\, e^{-r(n-m)} \to \lambda$$

t.d.

$$\phi_m(r,n) = \binom{n}{m}\left(1-\frac{m}{n}\right)^r p_0(r,n-m) \to \frac{\lambda^m}{m!} e^{-\lambda} \quad m \geq 1.$$