# CENTRALNI GRANIČNI TEOREM

## TEOREM 19

Ako $c_n \to c \in \mathbb{C}$ tada $\left(1+\frac{c_n}{n}\right)^n \to e^c$ .

## TEOREM 20

Ako su $X_1, X_2, \ldots$ n.j.d. t.d. $EX_i = \mu$ i $\operatorname{Var} X_i = \sigma^2$

$\sigma^2 \in (0,\infty)$ , tada

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$$

$\mathcal{J}^{\text{dokaz}}$ B.s.o.m.p. $EX_i = 0$ (inače $X_i' = X_i - \mu$ ...)

$$\varphi(t) = Ee^{itX_1} = 1 - \frac{\sigma^2 t^2}{2} + o(t^2) \quad \text{v. tm. 17}$$

$\Downarrow$

$$E\, e^{itS_n/\sigma\sqrt{n}} = \left(1 - \frac{t^2}{2n} + o(1/n)\right)^n = \left(1 - \frac{\frac{t^2}{2} + o(1)}{n}\right)^n$$

$$\to e^{-t^2/2} \quad \checkmark \text{ po tm. 19} \qquad \square$$

Primjer (c.g.t. za sume do slučajnog indeksa)

Neka su $X_i$ njd., $EX_i = 0$, $\mathrm{Var}\, X_i = \sigma^2 \in (0,\infty)$

$(a_n)$ niz u $\mathbb{N}$ td. $a_n \to \infty$ i

$(N_n)$ niz sl. varijabli u $\mathbb{N}$ td. $N_n / a_n \xrightarrow{P} 1$

$$\Rightarrow \qquad S_{N_n} / \sigma\sqrt{a_n} \xrightarrow{d} N(0,1) \qquad (*)$$

Najjednostavniji argument je preko teorema

TEOREM 21 (Slutsky)

a) $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} c \Rightarrow (X_n, Y_n) \xrightarrow{d} (X, c)$

b) $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} c \Rightarrow X_n + Y_n \xrightarrow{d} X_n + c$

c) $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} c \Rightarrow X_n \cdot Y_n \xrightarrow{d} c \cdot X_n$

Teorem 21 ćemo diskutirati u kontekstu konvergencije sluč. vektora. U iskazu i uvjeti $(X_n), (Y_n)$ mogu biti višedimenzionalni, najvažnija (najčešća) primjena od b) je

d) $X_n \xrightarrow{d} X$, $Z_n - X_n \xrightarrow{P} 0 \Rightarrow Z_n \xrightarrow{d} X$

Dokaz formule (*)

Kako $a_n \to \infty$ $\qquad S_{a_n}/\sqrt{a_n} \xrightarrow{d} N(0,1)$

No $$|S_{N_n} - S_{a_n}| \leq \underbrace{\left|\sum_{a_n+1}^{N_n} X_i\right| 1_{N_n > a_n}}_{A_n} + \underbrace{\left|\sum_{N_n+1}^{a_n} X_i\right| 1_{a_n > N_n}}_{B_n}$$

Jasno je sad za $A_n$ vrijedi ocjena

$$P\left(\frac{\sum_{a_n+1}^{N_n}|X_i|}{\sqrt{a_n}} > \varepsilon\right) \le P\left(\frac{N_n-a_n}{a_n} > \varepsilon/k\right) \to 0$$

za $n \to \infty$ po pretpostavci i $\forall$ fiksni $k$

$$+ P\left(\frac{\sum_{a_n+1}^{a_n+\frac{\varepsilon}{k}a_n}|X_i|}{\sqrt{a_n}} > \varepsilon\right) \le C \cdot \frac{1}{k} = \frac{\operatorname{Var}|X_i|}{k}$$

Izaberemo li veliki $k$ ovaj izraz možemo učiniti proizvoljno malim $\Rightarrow$

$A_n/\sqrt{a_n} \xrightarrow{P} 0$ u prethodnoj formuli, a

slično i $B_n/\sqrt{a_n} \xrightarrow{P} 0$

$$\Rightarrow \quad \frac{S_{N_n}-S_{a_n}}{\sqrt{a_n}} \xrightarrow{P} 0 \quad \xRightarrow{\text{Tm } 2.1} \quad \frac{S_{N_n}}{\sigma\sqrt{a_n}} \xrightarrow{d} N(0,1)$$

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<u>Primjer</u> (c.g.t. za proces obnavljanja)

Neka su $(Y_i)$ njd. pozitivne s.d.

$$\mu = EY_i, \quad \sigma^2 = \mathrm{Var}\, Y_i \in (0,\infty)$$

$$S_n = Y_1 + \dots + Y_n$$

$$N(t) = \sup\{m : S_m \leq t\} \quad \text{tada}$$

$$\frac{N_t - t/\mu}{\sigma\sqrt{t\cdot\mu^3}} \xrightarrow{d} N(0,1)$$

prema prethodnom primjeru, koristeći $X_i = Y_i - \mu$

# LINDEBERG-FELLEROV TEOREM

## TEOREM 22 (Lindeberg-Feller)

Za svaki $n \in \mathbb{N}$, neka su $X_{n,m}$ $1 \le m \le n$ nezavisne sl. var. t.d. $EX_{n,m} = 0$. Pretpostavimo

i) $\displaystyle\sum_{m=1}^{n} E X_{n,m}^2 \longrightarrow \sigma^2 > 0$

ii) $\forall \varepsilon > 0 \quad \displaystyle\lim_{n\to\infty} \sum_{m=1}^{n} E\left(|X_{n,m}|^2 ; |X_{n,m}| > \varepsilon\right) = 0$

$\Rightarrow \quad S_n = X_{n,1} + \cdots + X_{n,n} \xrightarrow{d} N(0, \sigma^2)$

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NAP

Uočimo ii) $\Rightarrow$

iii) $\lim \sum_{1}^{n} P(|X_{n,m}| > \varepsilon) = 0$

Feller: ako pretpostavimo i) & iii) tada je ii)

i nužan uvjet za konvergenciju u tm.

Još

$$\varphi_{n,m} = E e^{itX_{n,m}} \qquad \sigma_{n,m}^2 = E X_{n,m}^2$$

po tmu neprekidnosti dovoljno je vidjeti

$$\prod_{m=1}^{n} \varphi_{n,m}(t) \longrightarrow e^{-t^2\sigma^2/2} \qquad \forall \text{ fiksni } t$$

Stavimo
$$z_{n,m} = \varphi_{n,m}(t)$$
$$w_{n,m} = 1 - \frac{t^2\sigma_{n,m}^2}{2}$$

Prema lemi 16 $\Rightarrow$

$$|z_{n,m} - w_{n,m}| = \left| E e^{itX_{n,m}} - \left(1 + \frac{E|itX_{n,m}|^2}{2}\right) \right| \leq$$

$$\leq E\left( \frac{|tX_{n,m}|^3}{3!} \wedge \frac{2|tX_{n,m}|^2}{2!} \right) \leq$$

$$\leq \quad E\left(\frac{|tX_{n,m}|^3}{6}; |X_{n,m}| \leq \varepsilon\right) + E\left(|tX_{n,m}|^2; |X_{n,m}| > \varepsilon\right)$$

$$\leq \quad \frac{\varepsilon t^3}{6} E\left(|X_{n,m}|^2; |X_{n,m}| \leq \varepsilon\right) + t^2 E\left(|X_{n,m}|^2; |X_{n,m}| > \varepsilon\right) \Rightarrow$$

$$\overline{\lim} \sum_{m=1}^{n} |z_{n,m} - w_{n,m}| \leq \frac{\varepsilon t^3}{6} \overline{\lim} \sum_{1}^{n} \underbrace{E(|X_{n,m}|^2; |X_{n,m}| \leq \varepsilon)}_{\to \sigma^2}$$

$$+ \; t^2 \overline{\lim} \sum_{1}^{m} \underbrace{E(|X_{n,m}|^2; |X_{n,m}| > \varepsilon)}_{\to 0} \leq \frac{\varepsilon t^3}{6} \sigma^2$$

zbog i) & ii)

No $\varepsilon > 0$ je proizvoljan $\Rightarrow$

$$\lim \sum_{1}^{n} |z_{n,m} - w_{n,m}| = 0$$

Nadalje:

$$|z_{n,m}| = |E e^{itX_{n,m}}| \le 1$$

$$\sigma_{n,m}^2 = E|X_{n,m}|^2 \le \varepsilon^2 + E(|X_{n,m}|^2; |X_{n,m}| > \varepsilon)$$

$$\text{ii)} \rightarrow \quad \overline{\lim_n} \sup_{1\le m\le n} \sigma_{n,m}^2 \le \varepsilon^2$$

$$\Rightarrow \quad \sup_{1\le m\le n} \sigma_{n,m}^2 \longrightarrow 0 \qquad \text{za } n \to \infty$$

jer je $\varepsilon$ proizvoljan

Pa za dovoljno velik $n$

$$1 - \frac{t^2 \sigma_{n,m}^2}{2} > -1 \qquad \forall m = 1, \dots, n$$

pa za takve $n$ i $\quad |w_{n,m}| \le 1 \qquad \forall m = 1, \dots, n$

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Sad je $\left|\prod_1^n e_{n,m}(t) - \prod_1^n \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right)\right| =$

$$\left|\prod_1^n z_{n,m} - \prod_1^n w_{n,m}\right| \leq \sum_{m=1}^n |z_{n,m} - w_{n,m}| \to 0 \quad n \to \infty$$

No

$$\prod_1^n \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right) \to e^{-t^2\sigma^2/2} \quad n \to \infty$$

po Lemi 1 b) uz

$$c_{m,n} = \frac{-t^2 \sigma_{n,m}^2}{2}$$

$\square$

<u>Primjer</u> (rekordi / sl. permutacije)

$X_1, X_2, \ldots$ neka su njd s neprekidnom razdiobom $F$

$$A_m = \{ X_m > \sup_{j<m} X_j \}, \quad R_n = \sum_1^n 1_{A_m} = \text{broj rekorda do } n$$

Pokazali smo

$$\mathbb{P}(A_m) = \frac{1}{m}, \text{ a } A_m \text{ čine niz nezavisnih}$$

događaja, te smo vidjeli i

$$\frac{R_n}{\log n} \xrightarrow{g.s.} 1$$

Neka su sada $Y_1, Y_2, \ldots$ nez. sl. varijable t.d.

$$P(Y_m = 1) = 1/m \qquad P(Y_m = 0) = 1 - 1/m$$

npr. $Y_m = 1_{A_m}$, $EY_m = 1/m$, $\mathrm{Var}\, Y_m = \frac{1}{m} - \frac{1}{m^2}$

neka je $S_n = Y_1 + \dots + Y_n \stackrel{d}{=} R_n$, tada

$ES_n \sim \log n$, $\mathrm{Var}\, S_n \sim \log n$ stavimo

$$X_{n,m} := \frac{Y_m - 1/m}{\sqrt{\log n}} \qquad m = 1, \dots, n \Rightarrow$$

$$EX_{n,m} = 0, \qquad \sum_{m=1}^{n} EX_{n,m}^2 = \frac{\mathrm{Var}\, S_n}{\log n} \longrightarrow 1 \qquad n \to \infty$$

te je i $\forall \varepsilon > 0$

$$\sum_{1}^{n} E\left(|X_{n,m}|^2 ; |X_{n,m}| > \varepsilon\right) \longrightarrow 0$$

Lindeberg - Feller $\Rightarrow$

$$\frac{1}{\sqrt{\log n}} \left( S_n - \sum_{m=1}^{n} \frac{1}{m} \right) \xrightarrow{d} N(0,1)$$

Kako $\displaystyle\sum_{1}^{n-1} \frac{1}{m} \geq \int_{1}^{n} \frac{1}{x}\,dx = \log n \geq \sum_{2}^{n} \frac{1}{m} \Rightarrow$

$$\left|\log n - \sum_{1}^{n} 1/m \right| \leq 1 \Rightarrow i$$

GONCHAROVljev TEOREM

$$\frac{1}{\sqrt{\log n}} \left( S_n - \log n \right) \xrightarrow{d} N(0,1)$$

(Slutsky)

Primjer (c.g.t. uz beskonačnu varijancu)

Neka su $X_1, X_2, \ldots$ n.j.d. simetrične sl. var.

t.d. $\mathbb{P}(|X_i| > x) = x^{-2}$, $x \geq 1$

$$E|X_1|^2 = \int_0^\infty 2x\,\mathbb{P}(|X_1| > x)\,dx = +\infty$$

Pokazat ćemo da $S_n$ dobro normalizirane

ipak konvergiraju ka $N(0,1)$ razdiobi.

Stavimo

$$Y_{n,m} = X_m \cdot \mathbb{1}_{(|X_m| \le \underbrace{\sqrt{n}\,\log\log n}_{c_n})}$$

Nivo rezanja je takav da

$$\sum_1^n P(Y_{n,m} \neq X_m) = \sum_1^n P(|X_m| > c_n)$$

$$= n \cdot P(|X_1| > c_n) = n\, c_n^{-2} = \frac{1}{(\log\log n)^2} \to 0 \qquad (**)$$

Pokažimo $E Y_{n,m}^2 \sim \log n$

Iz $P(|Y_{n,m}| > x) \le P(|X_m| > x) \Rightarrow$

$$E Y_{n,m}^2 \le \int_0^{c_n} 2y\, P(|X_1| > y)\, dy = 1 + \int_1^{c_n} \frac{2}{y}\, dy = 1 + 2\log c_n$$

No $1+2\log c_n \sim \log n \Rightarrow EY_{n,m}^2 \lesssim \log n$

No vrijedi i obrnuta nejednakost

$$P(|Y_{n,m}|>x) = P(|X_1|>x) - P(|X_1|>c_n)$$

$$= P(|X_1|>x)\cdot\left(1-\frac{P(|X_1|>c_n)}{P(|X_1|>x)}\right)$$

$$= P(|X_1|>x)\cdot\left(1-\frac{x^2}{n(\log\log n)^2}\right) \geq$$

$$\geq P(|X_1|>x)\cdot\left(1-\frac{1}{(\log\log n)^2}\right) \quad \text{za} \quad x\leq\sqrt{n}$$

$\Rightarrow$

$$EY_{n,m}^2 = \int_0^\infty 2x\, P(|Y_{n,m}| > x)\, dx$$

$$\geq \int_1^{\sqrt{n}} P(|X_1| > x) \cdot \left(1 - (\log\log n)^{-2}\right) dx$$

$$= \left(1 - (\log\log n)^{-2}\right) \cdot \int_1^{\sqrt{n}} \frac{2}{x}\, dx \sim \log n$$

Stavimo $\quad S_n' = Y_{n,1} + \dots + Y_{nn}$

$\Rightarrow \quad \operatorname{Var} S_n' \sim n \log n$,

$X_{n,m} = \dfrac{Y_{n,m}}{(n \log n)^{1/2}}$ zadovoljavaju

$$\sum_1^n EX_{n,m}^2 = \frac{1}{n\log n} \operatorname{Var} S_n' \longrightarrow 1 \qquad n \to \infty$$

$$\sum_1^n E\left(|X_{n,m}|^2 ; |X_{n,m}| > \varepsilon\right) = \sum_1^n E\left(|X_{n,m}|^2 \cdot \frac{|Y_{n,m}|}{\sqrt{n \log n}} > \varepsilon\right)$$

međutim $\quad Y_{n,m} \le \sqrt{n} \cdot \log\log n$

pa je za dani $\varepsilon > 0$

$|Y_{n,m}| \le \varepsilon \sqrt{n \log n}$ za sve dovoljno velike $n$

Stoga je gornja suma $0$.

$$\overset{\text{Lin. Fell.}}{\Longrightarrow} \quad \frac{S_n'}{\sqrt{n \log n}} \xrightarrow{d} N(0,1)$$

$$(**) \to P(S_n \ne S_n') \to 0 \Rightarrow \frac{S_n}{\sqrt{n \log n}} \xrightarrow{d} N(0,1).$$

TEOREM 23 (Lévy)

Neka su $X_1, X_2, \ldots$ njd $\quad S_n = X_1 + \cdots + X_n$

Postoje nizovi konstanti $a_n$ i $b_n > 0$ t.d.

$$\frac{S_n - a_n}{b_n} \xrightarrow{d} N(0,1)$$

$$\Leftrightarrow \quad \frac{y^2 P(|X_1| > y)}{E(|X_1|^2; |X_1| \le y)} \to 0 \qquad y \to \infty$$

Iz teorema 22 slijedi i dovoljnost Ljapunovljevog uvjeta za c.g.t.

- pretpostavite $E|X_{n,k}|^{2+\delta} < \infty$ za neki $\delta > 0$ i sve $n,k$

- (Lj.U.)

$$\lim_n \sum_{k=1}^{n} E|X_{n,k}|^{2+\delta} = 0$$

Sad Lindebergov uvjet (tm 22 ii) vrijedi jer

$$\sum_1^n E\left(|X_{n,m}|^2 ; |X_{n,m}| > \varepsilon\right) \leq \sum_1^n \frac{E|X_{n,m}|^{2+\delta}}{\varepsilon^\delta} \longrightarrow 0$$

# BRZINA KONVERGENCIJE

Ako su $X_i$ n.j.d. $EX_i = 0$, $\operatorname{Var} X_i = \sigma^2 \in (0, \infty)$

sad znamo $\qquad S_n / \sigma\sqrt{n} \xrightarrow{d} N(0,1)$

tj. f-ja distrib $F_n(x) := P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) \longrightarrow \Phi(x) \ \forall x$

no postavlja se pitanje brzine te konvergencije

TEOREM 24 (Berry - Esseen)

Ako $\quad E|X_i|^3 = \varrho < \infty \quad \Rightarrow$

$$|F_n(x) - \Phi(x)| \leq 3 \frac{\varrho}{\sigma^3\sqrt{n}}$$

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# LOKALNI GRANIČNI TEOREMI

$\boxed{\text{DEF}}$ Sl. var. $X$ ima razdiobu <u>koncentriranu</u>
<u>na rešetci</u> ako postoje $b \in \mathbb{R}$, $h > 0$ t.d.

$$\mathbb{P}(X \in b + h\mathbb{Z}) = 1$$

[engl. lattice distribution]

Mi znamo da prije de-Moivre Laplaceovog teorema
možemo nešto reći i o konvergenciji
(diskretnih) gustoća prema normalnoj
gustoći!

TEOREM 25

Za $\varphi(t) = Ee^{itX}$ uvijek vrijedi jedna od sljedeće tri mogućnosti:

i) $|\varphi(t)| < 1 \quad \forall t \neq 0$ $\rightarrow$ nonlattice

ii) $\exists \lambda > 0$ t.d. $|\varphi(\lambda)| = 1$ & $|\varphi(t)| < 1$ za $t \in (0, \lambda)$, tada je $X$ koncentrirana na rešetci širine $2\pi/\lambda$ $\rightarrow$ lattice

iii) $|\varphi(t)| = 1 \quad \forall t$, tada je $X = b$ g.s.

$\rightarrow$ degenerate case

Neka su $X_1, X_2, \ldots$ n.j.d. $EX_i = 0$

$0 < \mathrm{Var}\, X_i = \sigma^2 < \infty$ i imaju razdioba

koncentriranu na rešetci širine $h$

$$\rightarrow P(X_i \in b + h\mathbb{Z}) = 1 \Rightarrow P(S_n \in nb + h\mathbb{Z}) = 1$$

$$p_n(x) = P\left(\frac{S_n}{\sqrt{n}} = x\right) \qquad x \in \left\{ \frac{nb + hz}{\sqrt{n}}, \ z \in \mathbb{Z} \right\} =: \mathcal{L}_n$$

TEOREM 26

$$\sup_{x \in \mathcal{L}_n} \left| \frac{\sqrt{n}}{h} \cdot p_n(x) - \varphi_\sigma(x) \right| \longrightarrow 0$$

$$\varphi_\sigma(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

U "nerešetkastom slučaju", pretpostavimo

$X_1, X_2, \dots$ njd. $EX_i = 0$, $0 < \operatorname{Var} X_i = \sigma^2 < \infty$

te $|\varphi(t)| < 1$ $\forall t \neq 0$

TEOREM 27

Pretpostavite $x_n / \sqrt{n} \to x$, te $a < b$

$$\sqrt{n}\, P\left(S_n \in (x_n + a, x_n + b)\right) \longrightarrow (b-a)\, \varphi_\sigma(x)$$

$$\|$$

$$\sqrt{n}\, P\left(\frac{S_n}{\sqrt{n}} \in \left(\frac{x_n}{\sqrt{n}} + \frac{a}{\sqrt{n}}, \frac{x_n}{\sqrt{n}} + \frac{b}{\sqrt{n}}\right)\right)$$

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