TIME SERIES

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TIME SERIES ANALYSIS

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dates: 26.2., 12.3., 26.3., 16.4., 7.5., 21.5., [28.5]

aim: - introduce rigorously concepts & some results of classical & contemporary time series analysis

- give basic R-skills in practical t.s.a

Student assesment: (exams, take home exercises, ...)

COURSE OUTLINE

Ch 1 INTRODUCTION : BASIC PRINCIPLES , NOTIONS & MODELS

Ch 2 LINEAR & NONLINEAR PREDICTION : HILBERT SPACES & CONDITIONAL EXPECTATION

Ch 3 ESTIMATION OF PARAMETERS : LIMIT THEOREMS

Ch 4 ARMA MODELS

Ch 5 GARCH & RELATED MODELS . HEAVY TAILS

Ch 6 SELECTED TOPICS : SPECTRAL THEORY . UNIT ROOTS . COINTEGRATION .

TIME SERIES – sequence of observations indexed over time

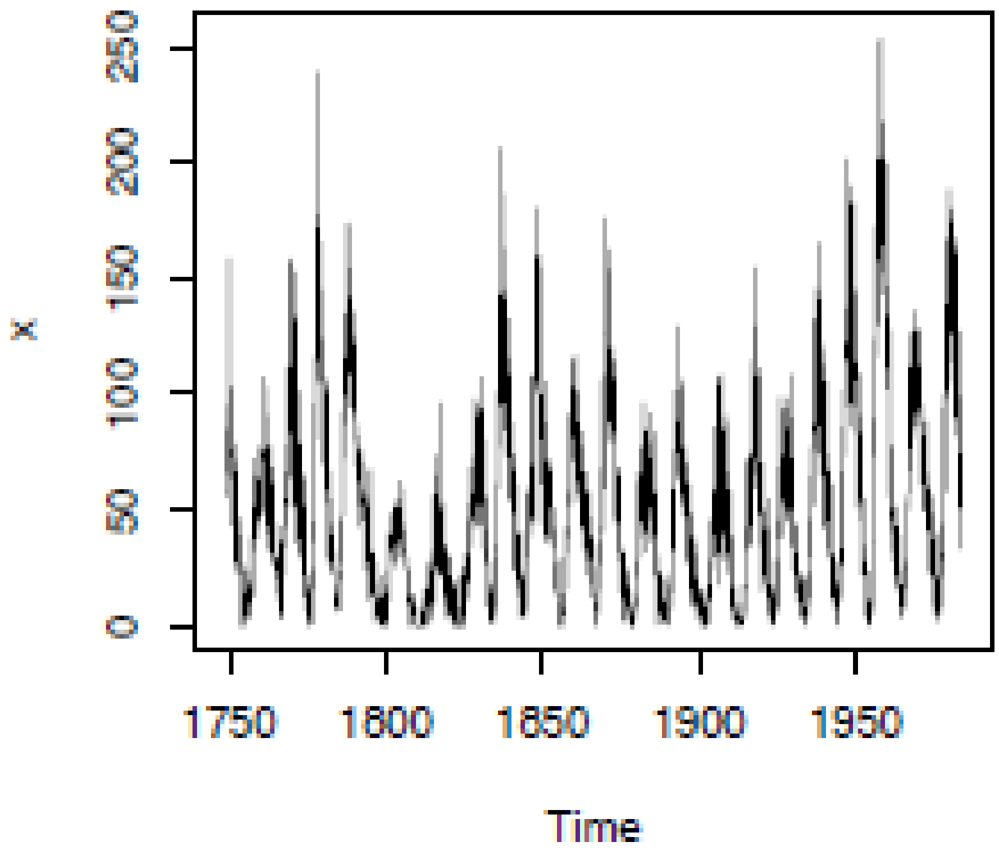
$(X_t)_{t \in I}$, where $I \subseteq \mathbb{Z}$ typically

(I can be interval → continuous t.s.)

TIME SERIES (MODEL) – a <u>stochastic process</u> indexed over time

$(X_t)_{t \in I}$, $X_t : \Omega \to \mathbb{R}$ random variables

SLIGHTY CONFUSING !?!


x
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100
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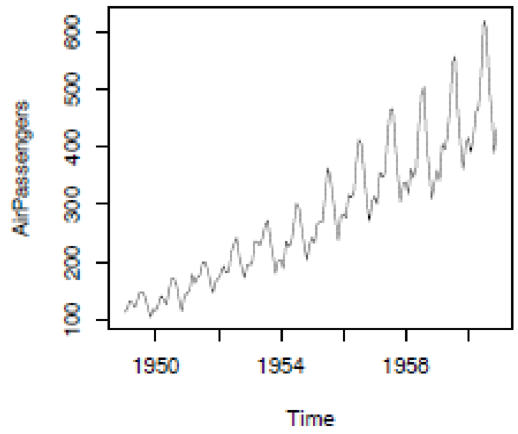
250
1750
1800
1850
1900
1950
Time


AirPassengers
100
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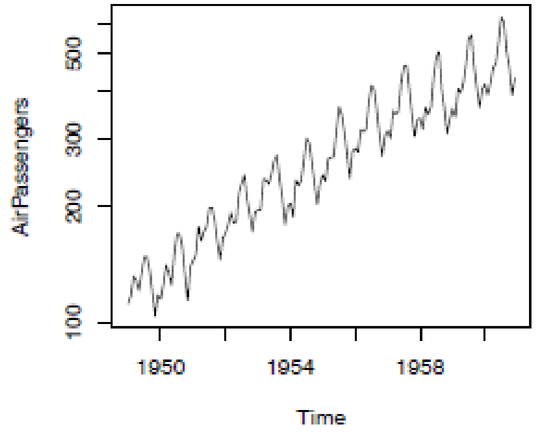
600
1950
1954
1958
Time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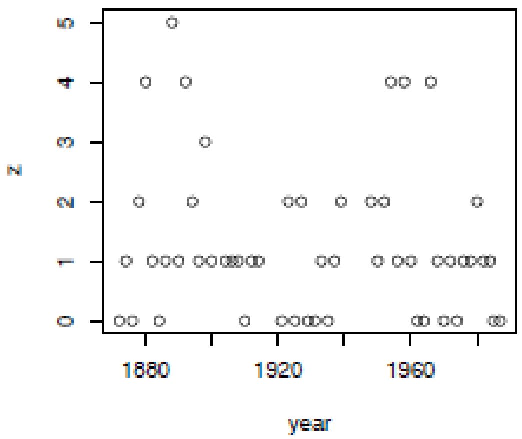
z
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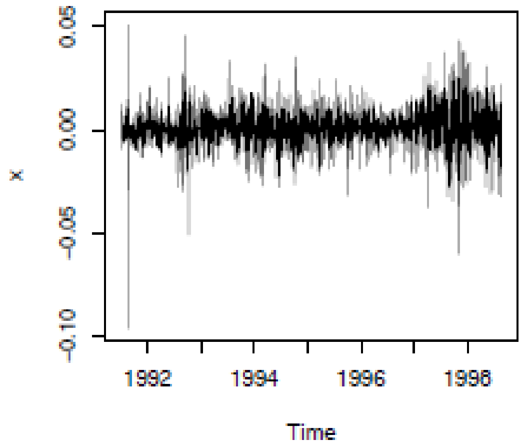
5
1880
1920
1960
year


x
−0.10
−0.05
0.00
0.05
1992
1994
1996
1998
Time

MORE PRECISE :

$(X_t)_{t\in\mathbb{Z}}$ = stochastic process in discrete time / $(X_t)_{t\in\mathbb{Z}}$ = its realization or path

DISTRIBUTION of stochastic process $(X_t)_{t\in\mathbb{Z}}$ is determined by all finite dimensional distribution of (X_t), i.e. distributions of all random vectors

$$(X_{t_1}, X_{t_2}, \dots, X_{t_k}), \quad t_1 < t_2 < \dots < t_k \in \mathbb{Z}$$

<u>EXAMPLE 1</u>

a) X_i iid. with distribution function F ($X_i \overset{iid}{\sim} F$), then $(X_i)_{i\in\mathbb{Z}}$ is time series model.

b) X_i as above, $S_0 = 0$, $S_n = S_{n-1} + X_n$ $n \geq 0$ then <u>random walk</u> $(S_n)_{n\in\mathbb{N}_0}$ represents time series again

DEF Time series $(X_t)_{t \in \mathbb{Z}}$ is (<u>weakly</u>) <u>stationary</u> if

i) $E|X_t|^2 < \infty \quad \forall t \in \mathbb{Z}$

ii) $EX_t = m \quad \forall t \in \mathbb{Z}$

iii) $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+r}, X_{t+r})$
$\forall s,t \in \mathbb{Z},\ r \in \mathbb{N}$

▷ stationary sequences have finite 2nd moment, constant expectation, & <u>linear dependence</u> between X_s & X_t depends on $t-s$ only.

$\boxed{\text{DEF}}$ Suppose stoch. process $(X_t)_{t\in\mathbb{Z}}$ satisfies $\operatorname{Var} X_t < \infty$ $\forall\, t \in \mathbb{Z}$, then <u>autocovariance function</u> $\gamma_X : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$ of the process (X_t) is defined by

$$\gamma_X(s,t) = \operatorname{Cov}(X_s, X_t) = E\left[(X_s - EX_s)(X_t - EX_t)\right]$$

$$s, t \in \mathbb{Z}$$

▷ For weakly stationary processes γ_X is essentially function of one variable only, since

$$\gamma_X(s,t) = \gamma_X(s-t, 0) = \gamma_X(s+r, t+r) \quad \forall s,t,r$$

We define then

$$\gamma_X(h) := \gamma_X(h, 0)$$

& call this function $\gamma_X : \mathbb{Z} \to \mathbb{R}$ <u>autocovariance function</u> of weakly stationary sequence $(X_t)_{t \in \mathbb{Z}}$.

Autocorrelation function is defined analogously:

for weakly stationary sequence (X_t) with $\operatorname{Var} X_t = \gamma_X(0) > 0$ autocorr. function $\rho_X : \mathbb{Z} \to [-1, 1]$ is defined by

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \operatorname{Corr}(X_{t+h}, X_t) \qquad t, h \in \mathbb{Z}$$

$\boxed{\text{DEF}}$ Time series $(X_t)_{t\in\mathbb{Z}}$ is <u>strongly</u> (<u>strictly</u>) stationary if the joint distribution of random vectors $(X_{t_1+h}, \dots, X_{t_k+h})$ do not depend on $h \in \mathbb{N}$ for all $k \in \mathbb{N}$, $t_1 < t_2 < \dots < t_k \in \mathbb{Z}$

$\boxed{\text{DEF}}$ Class of distribution functions $\{F_{\mathbf{t}}\}_{\mathbf{t}\in J}$ where $J = \{(t_1, \dots, t_k) : t_1 < \dots < t_k,\ t_i \in \mathbb{R},\ k \in \mathbb{N}\}$

$$F_{\mathbf{t}} = F_{t_1\dots,t_k}(x_1, \dots, x_k) = P(X_{t_1} \le x_1, \dots, X_{t_k} \le x_k) \quad (x_1, \dots, x_k) \in \mathbb{R}^k$$

determines all <u>finite-dimensional distributions</u> of a given stochastic process (X_t).

THEOREM 1 (Kolmogorov)

Class of distribution functions $\{F_{\mathbf{t}}\}_{\mathbf{t} \in \mathcal{T}}$ is a class of fidi's for some stochastic process $\iff$ $\forall k \in \mathbb{N}, \forall \mathbf{t} = (t_1, \ldots, t_k) \in \mathcal{T}$

$$\lim_{x_i \to \infty} F_{\mathbf{t}}(\mathbf{x}) = F_{\mathbf{t}(i)}(\mathbf{x}(i)) \quad \forall \mathbf{x}(i) \in \mathbb{R}^{k-1}$$

where x_i = i'th coordinate of the vector $\mathbf{x} \in \mathbb{R}^k$ on the l.h.s.

$\mathbf{t}(i), \mathbf{x}(i)$ = vectors $\mathbf{x}, \mathbf{t}$ with i'th coordinate omitted.

▷ These are so called consistency conditions, for proof see Billingsley or ...

Sometimes function γ_X & value m in the definition of stationary process (X_t) determine its fidi's.

$\boxed{\text{DEF}}$ Random vector $\mathbb{Y} = (Y_1, \ldots, Y_k)$ has multivariate normal distribution if there is $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ & matrix $B \in M_{n \times n}$ & random vector $\mathbb{X} = (X_1, \ldots, X_n)$ s.t. $X_i \overset{iid}{\sim} N(0,1)$

& $\mathbb{Y} = a + B\mathbb{X}$

In that case

$$E\mathbf{y} = (Ey_1, \ldots, Ey_n) = \mathbf{a}$$

& covariance matrix for $\mathbf{y}$ equals

$$\Sigma_{\mathbf{y}} = BB^{\tau}$$

we say $\mathbf{y}$ has multivariate normal (gaussian) distribution with parameters $\mathbf{a}$ & $\Sigma_{\mathbf{y}}$

If $\det \Sigma_{\mathbf{y}} > 0$, $\mathbf{y}$ has density function

$$f_{\mathbf{y}}(y) = (2\pi)^{-\frac{n}{2}} \frac{1}{\sqrt{\det \Sigma}} \, e^{-\frac{1}{2}\langle (y-a), \Sigma^{-1}(y-a)\rangle}$$

$\boxed{\text{DEF}}$ Stochastic process is called gaussian if its fidi's are all multivariate normal.

EXE. 1> For gaussian time series (X_t) with known constant expectation m; function γ_X determines its fidi's.

EXE. 2> If (X_t) is gaussian & weakly stationary, it is strongly stationary as well.

EXE. 3> If (X_t) is strongly stationary & $\mathrm{Var}\, X_0 < \infty$, then (X_t) is weakly stationary gaussianity + weak stationarity $\Rightarrow$ strong stationarity

strong stationarity + finite 2nd moment $\Rightarrow$ weak stationarity

EXAMPLE 2 (WHITE NOISE)

A weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ is called white noise if $EX_t = 0 \quad \forall t$ &

$$\gamma_X(h) = \begin{cases} \sigma^2 = \operatorname{Var} X_0 , & h = 0 \\ 0 , & \text{otherwise.} \end{cases}$$

We write $X_t \sim WN(0, \sigma^2)$.

- If $X_t \overset{iid}{\sim} F$, & $EX_0 = 0$, $\operatorname{Var} X_0 = \sigma^2 < \infty$

then $X_t \sim WN(0, \sigma^2)$ too;

we also write $X_t \sim IID(0, \sigma^2)$

EXE 4) Find an example of white noise sequence which is not iid.

EXAMPLE 3 (RANDOM WALK)

$(X_t) \sim IID(\mu, \sigma^2)$, then random walk

$S_0 = 0$, $S_t = X_1 + \dots + X_t$ $t \in \mathbb{N}$

is a stoch. process which is not stationary unless $\sigma^2 = 0$.

Really $\gamma_S(0,0) = 0$, $\gamma_S(h,h) = \operatorname{Var} S_h$

$= h \cdot \operatorname{Var} X_1 \to +\infty$

for $h \to \infty$

REMARK

Definitions of weak/strong stationarity and functions γ_X, ρ_X are straightforward to extend to time series indexed over $\mathbb{N}$ or $\mathbb{N}_0$.

Even continuous time processes $(X_t)_{t \in [0,T]}$ are sometimes considered.

EXAMPLE 3 (PERIODIC PROCESSES)

Suppose A & Θ are independent random variables, & $\Theta \sim \mathcal{U}[0, 2\pi]$

$$X_t = A \cos(\nu t + \Theta), \quad t \geq 0$$

where $\nu > 0$ is a real parameter.

Clearly, (X_t) has period $2\pi/\nu$,

its amplitude & phase are random

Note

$$X_t = A(\cos \nu t \cdot \cos \Theta - \sin \nu t \cdot \sin \Theta)$$
$$= A_1 \cos \nu t + A_2 \sin \nu t \qquad (*)$$

If $EA^2 < \infty \Rightarrow EA_1 = EA_2 = 0$, e.g.

$$EA_1 = EA \cdot E\cos\Theta = EA \int_0^{2\pi} \cos t \, dt = EA \sin t \Big|_0^{2\pi} = 0.$$

$$Cov(A_1, A_2) = EA_1A_2 = -EA^2 \cdot E\cos\Theta \sin\Theta$$
$$= -EA^2 E \frac{\sin\Theta \cdot 2}{2} = -\frac{EA^2}{2} \int_0^{2\pi} \sin 2t \, dt$$
$$= \frac{EA^2}{2} \frac{\cos 2t}{2} \Big|_0^{2\pi} = 0$$

But, if A_1, A_2 are uncorrelated with expectation 0, process

$$X_t = A_1 \cos \nu t + A_2 \sin \nu t$$

is weakly stationary if $EA_1^2 = EA_2^2 = \sigma^2$.

Clearly $EX_t^2 < \infty$, $EX_t = 0$

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\big[A_1 \cos\nu(t+h) + A_2 \sin\nu(t+h),\ A_1 \cos\nu t + A_2 \sin\nu t\big]$$

$$= EA_1^2 \cos\nu t \cos\nu(t+h) + EA_2^2 \sin\nu t \sin\nu(t+h)$$

$$= \sigma^2 \cdot \cos\nu h$$

$\Leftarrow$ depends only on h !!!

In discrete times $t = 0, 1, 2, \dots$ path of (X_t) sits on the graph of periodic function, but

- from one trajectory of the process it is not clear if the process is stationary / periodic / or even random at all !?!! !!!

EXE 5 > If (Y_t) is weakly stationary, define

$$X_t = \begin{cases} Y_t, & t \text{ even} \\ Y_{t+1}, & t \text{ odd} \end{cases}$$

Show $\mathrm{cov}(X_{t+h}, X_t)$ depends only on h, but (X_t) is not weak. stationary.

EXAMPLE 4 (MOVING AVERAGE / MA(1))

Suppose $(Z_t) \sim WN(0, \sigma^2)$, $\vartheta \in \mathbb{R}$,

define

$$X_t = Z_t + \vartheta Z_{t-1}, \qquad t \in \mathbb{Z}.$$

Process (X_t) is called MA(1) process,

Clearly $EX_t = 0$

$$\gamma_X(h) = \mathrm{Cov}(Z_{t+h} + \vartheta Z_{t+h-1}, Z_t + \vartheta Z_{t-1})$$

$$= \begin{cases} (1+\vartheta^2)\sigma^2, & h = 0 \\ \vartheta\sigma^2 & h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus X_s & X_t are uncorrelated if $|s-t| > 1$

↳ dependence in the sequence (X_t) is of finite (short) range.

Time series literature treats differently

- short range dependence
- long range dependence,

but these terms are somewhat vaguely defined.

EXE. 6> For $(Z_t) \sim IID$ in Ex 4. Show that (X_t) is strongly stationary.

<u>EXAMPLE</u> 5 (AUTOREGRESSIVE PROC. / AR(1))

For $(Z_t) \sim WN(0,\sigma^2)$ consider recursion

$$X_t = \varphi X_{t-1} + Z_t \quad , \quad t \in \mathbb{Z} \qquad (*)$$

for some $\varphi \in \mathbb{R}$.

In contrast to MA(1) process, this recursion does <u>not</u> define time series (X_t), since any X_0 gives different time series.

Q: Can we find stationary (X_t) s.t. $(*)$ holds.

It turns out that AR(1) equation (*) either

- has unique stationary solution, or
- has no stationary solution.

<u>Case 1</u> : $|\varphi| < 1$

Iterating (*) backwards leads

$$
\begin{aligned}
X_t &= \varphi(\varphi X_{t-2} + Z_{t-1}) + Z_t \\
&= \dots \\
&= \varphi^k X_{t-k} + \varphi^{k-1} Z_{t-k+1} + \dots + \varphi Z_{t-1} + Z_t
\end{aligned}
$$

Clearly $E|\varphi^k X_{t-k}|^2 = \varphi^{2k} E X_{t-k}^2 \longrightarrow 0, \quad k \to \infty$

if (X_t) is weakly stationary

This suggest solution in the form

$$X_t = \sum_{k=0}^{\infty} \varphi^k Z_{t-k},$$

We will show this series converges a.s. since $|\varphi| < 1$

$\Downarrow$

(X_t) is well defined in this way,

& by direct calculation it satisfies $(*)$

If we can interchange expectation & summation signs (to be shown) $\Rightarrow$

$$EX_t = \sum_{k=0}^{\infty} \rho^k E Z_{t-k} = 0$$

$$\gamma_X(h) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho^i \rho^j \underbrace{E Z_{t+h-i} Z_{t-j}}_{\gamma_Z(h-i+j)} = \sum_{j=0}^{\infty} \rho^{h+j} \cdot \rho^j \, \sigma^2$$

$$= \sigma^2 \cdot \frac{\rho^{|h|}}{1-\rho^2} \longrightarrow 0$$

exponentially fast

($\to$ short range dependence)

<u>Case 2</u> : $\varphi = 1$

$$X_t = X_{t-1} + Z_t = \dots = X_0 + Z_1 + \dots + Z_t$$

↳ (X_t) is random walk

$$\mathrm{Var}(X_t - X_0) = t \cdot \sigma^2 \longrightarrow +\infty$$

From triangle inequality

$$\mathrm{s.d.}(X_t - X_0) \leq \mathrm{s.d.}(X_t) + \mathrm{s.d.}(X_0)$$

↳ standard deviation of X_t can't be constant

$\Downarrow$

no stationary solution exist

EXE 6 Show (*) has no stationary solution for $\varphi = -1$.

EXE 7 Show (*) has unique stationary solution in the form of random series in the case $|\varphi| > 1$.

EXAMPLE 6 (ARCH / GARCH)

The simplest models of stock price movements assume that log returns

$$X_t = \log S_t / S_{t-1}$$

can be modeled as an iid series, say

$$X_t = \sigma Z_t \,, \qquad Z_t \sim \mathrm{IID}(0, \sigma^2)$$

Empirical data do not contradict the assumption of mean zero or $\rho_X(h) = 1$, $h \neq 0$.

However, there are many properties of empirical data which are not captured by such a model (stilized facts)

Engle (1982) suggested the following model:

$$Z_t \sim IID(0, \sigma^2)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$$

$$X_t = \sigma_t \cdot Z_t, \qquad t \in \mathbb{Z}, \mathbb{N}.$$

For $\alpha_1 \in (0,1)$ we can again find stationary solution by iterating backwards

$$\sigma_t^2 = \alpha_0 + \alpha_0 \sum_{j=1}^{\infty} \alpha_1 Z_{t-1}^2 \cdots \alpha_1 \cdot Z_{t-j}^2$$

Since (Z_t) iid $\Rightarrow$ $X_t = \sigma_t Z_t$ solves the recursive equations above

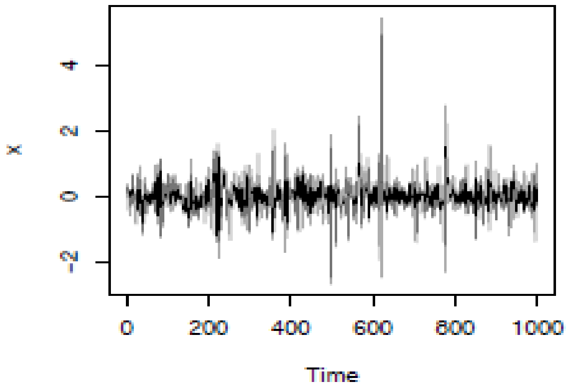
This (X_t) is strongly stationary as a fixed transformation of infinite sequence $(Z_t, Z_{t-1}, Z_{t-2}, \ldots)$

(X_t) is called ARCH(1) process

<u>autoregressive</u> <u>conditional</u> <u>heteroscadastic</u>

REM Generalization of this model introduced by Bollerslev (1986) – GARCH – is arguably the most popular model for practical analysis of financial time series.

AR(1), MA(1), ARCH(1), ... can be extended by introduction of additional parameters.



Path of ARCH(2) process

EXE 8 > For $\alpha_1 \in (0,1)$, $K = EZ_1^4 < \infty$ show that ARCH process (X_t) has finite 2nd & 4th moment, which can be found as

$$EX_0^2 = \frac{\alpha_0}{1-\alpha_1}$$

$$EX_0^4 = \frac{K\,\alpha_0}{(1-\alpha_1)^2}$$

EXAMPLE 7 (logistic map)

Take $x_0 \in (0,1)$ & consider

$$x_{t+1} = 4 x_t (1 - x_t), \qquad t = 1, 2, \ldots$$

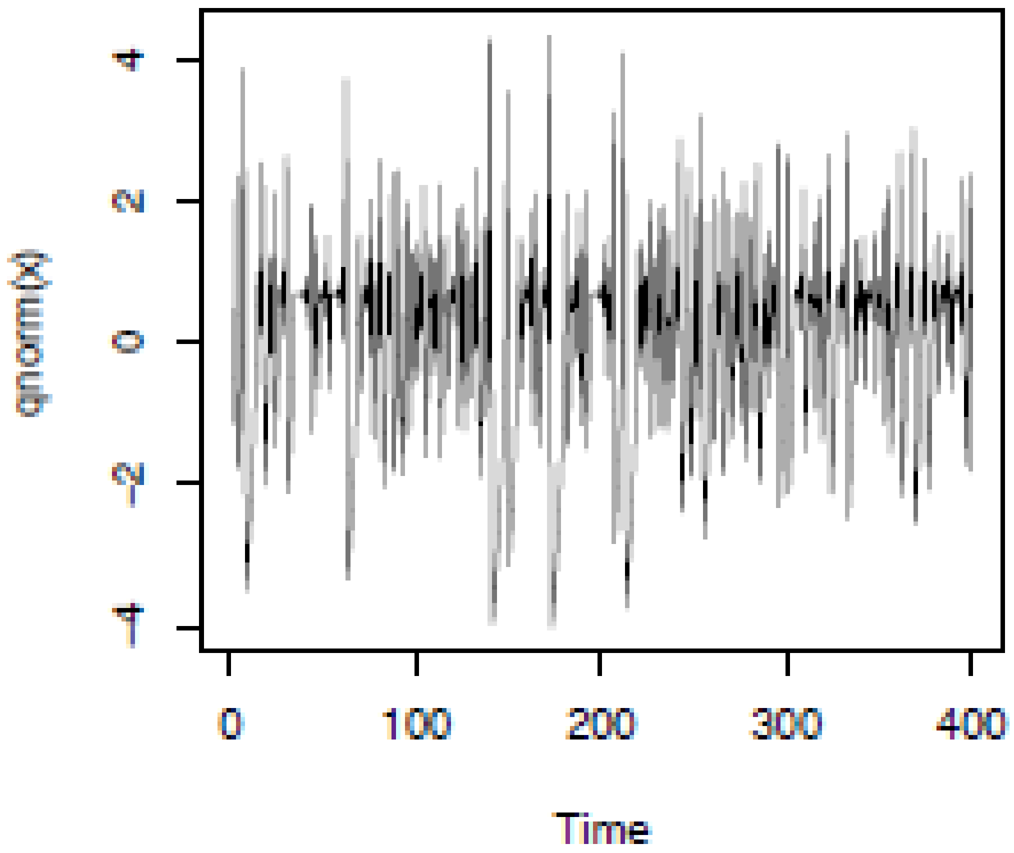
Its trajectory looks entirely random

Mapping $x \longmapsto \mu x (1-x)$ $\quad x \in (0,1)$

is important example from chaos theory.

For $X_0 = \sin^2 \frac{\pi U}{2}$ $\quad U \sim U(0,1)$

$X_t = 4 X_{t-1}(1 - X_{t-1})$ defines stationary sequence (which is completely predictable if we know X_0 of course).


gnorm(x)
4
2
0
-2
-4
0
100
200
300
400
Time

We have introduced autocovariance function

γ_X

$$\gamma_X(h) = \mathrm{Cov}(X_{t+h}, X_t), \quad h \in \mathbb{Z}$$

& autocorrelation function ρ_X

$$\rho_X(h) = \mathrm{Corr}(X_{t+h}, X_t) = \gamma(h)/\gamma(0), \quad h \in \mathbb{Z}$$

for a weakly stationary time series (X_t).

They characterize dependence completely for gaussian processes. They also contain sufficient information if we consider linear predictors

In practice they have to be estimated of course, together with the mean of X_t, standard estimators are

$$\overline{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j \qquad \text{sample mean}$$

$$\hat{\gamma}_X(h) = \frac{1}{n} \sum_{j=1}^{n-h} (X_{j+h} - \overline{X})(X_j - \overline{X}) \qquad 0 \le h < n$$

$$\hat{\gamma}_X(h) = \hat{\gamma}_X(-h), \quad -n < h \le 0$$

sample autocovariance function

$$\hat{\rho}_X(h) = \frac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)} \qquad |h| < 0$$

sample autocorrelation function

It is easy to show for any autocov. funct.

LEMA 2

i) $\gamma(0) \geq 0$

ii) $|\gamma(h)| \leq \gamma(0) \quad \forall h$

iii) γ is even function

iv) γ is positive semidefinite function

i.e. $\sum_{i,j=1}^{n} a_i \gamma(i-j) a_j \geq 0$

$\forall n \quad \forall (a_1, \dots, a_n) \in \mathbb{R}^n$

Property iv) $\Leftrightarrow$

$$\Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \ddots & & \\ & & \ddots & & \\ \gamma(n-1) & & & & \gamma(0) \end{bmatrix}$$ is positive semidefinite matrix

THEOREM 3

Function $\gamma : \mathbb{Z} \to \mathbb{R}$ is autocov. funct. of a stationary time series $\Leftrightarrow$ γ is even & positive semidefinite

Proof Lemma 2 + Thm 1

$\boxed{\text{REM}}$ Function ρ clearly has the same properties, plus $\rho(0) = 1$.

It is frequently easier to find time series with autocov. funct. γ than to check iii) & iv)

The end of the first part